Probabilistic Inference in Credal Networks: New Complexity Results

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Abstract

Credal networks are graph-based statistical models whose parameters take values in a set, instead of being sharply specified as in traditional statistical models (e.g., Bayesian networks). The computational complexity of inferences on such models depends on the irrelevance/independence concept adopted. In this paper, we study inferential complexity under the concepts of epistemic irrelevance and strong independence. We show that inferences under strong independence are NP-hard even in trees with binary variables except for a single ternary one. We prove that under epistemic irrelevance the polynomial-time complexity of inferences in credal trees is not likely to extend to more general models (e.g., singly connected topologies). These results clearly distinguish networks that admit efficient inferences and those where inferences are most likely hard, and settle several open questions regarding their computational complexity. We show that these results remain valid even if we disallow the use of zero probabilities. We also show that the computation of bounds on the probability of the future state in a hidden Markov model is the same whether we assume epistemic irrelevance or strong independence, and we prove an analogous result for inference in Naive Bayes structures. These inferential equivalences are important for practitioners, as hidden Markov models and Naive Bayes networks are used in real applications of imprecise probability.

1 Introduction

Bayesian networks are multivariate probabilistic models where stochastic independence assessments are compactly represented by an acyclic directed graph whose nodes are identified with variables [42]. In addition to its acyclic directed graph, the specification of a Bayesian network requires the specification of a conditional probability distribution for every variable and every assignment of its parents. When information is costly to acquire, specifying these conditional probabilities can be a daunting task, whether they are estimated from data or elicited from experts. This causes the inferences drawn with the model to contain imprecisions and arbitrarinesses [31].

An arguably more principled approach to coping with the imprecision in the numerical parameters is by incorporating it into the formalism. One way of doing so is by means of closed and convex sets of probability distributions, which are called *credal sets* [33].[1] Bayesian networks whose numerical parameters are specified by conditional credal sets are known as *credal networks* [9, 12, 13]. Credal networks have been successfully applied to robust pattern recognition,[2] and to knowledge-based systems, where it has been argued that allowing parameters to be imprecisely specified facilitates elicitation from experts.[3]

A Bayesian network provides a concise representation of the (single) joint probability distribution that is consistent with the network parameters and satisfies (at least) the set of stochastic independences encoded in its underlying graph. Analogously, a credal network provides a concise representation of the credal set of joint distributions that are consistent with the local credal sets and satisfy (at least) the irrelevances encoded in its underlying graph. The precise characterization of that joint credal set depends however on the concept of irrelevance adopted.

The two most commonly used irrelevance concepts in the literature are *strong independence* and *epistemic irrelevance*. Two variables X and Y are *strongly independent* if their joint credal set can be regarded as originating from a number of precise probability distributions under each of which the two variables are stochastically independent. Strong independence is thus closely related to the sensitivity analysis interpretation of credal sets, which regards an imprecisely specified model as arising out of partial ignorance of an ideal precisely specified one [49, 31, 5, 56]. If a piece of uncertain knowledge is considered poorly represented by any precise probability distribution, then any irrelevance concept that is based on precise probability models might not be very suited for the task. Arguably, a proper notion

[1] Other approaches include random sets [29], evidence theory [46, 47], possibility theory [52], conditional plausibility measures [27], and coherent lower previsions [49, 23], the last one being largely equivalent to credal sets (there is an one-to-one correspondence between credal sets and coherent lower previsions).

[2] See Refs. [57, 54, 20, 1, 10, 3, 16].

[3] See Refs. [50, 6, 45, 19, 1, 43, 4].

of irrelevance between events in such a case should be a property of the conditional credal sets. Epistemic irrelevance is one such possible notion. A variable X is *epistemically irrelevant* to a variable Y if the marginal credal set of Y is the same whether we observe the value of X or not [49]. Intuitively, variable X is epistemically irrelevant to Y if our belief about the value of the latter is unaltered by the disclosure of the value of the former. Unlike strong independence, epistemic irrelevance is an asymmetric concept and cannot in general be characterized by the properties of the elements of the credal set alone [22]. Moreover, strong independence implies epistemic irrelevance, whereas the converse does not necessarily hold.

If on the one hand the flexibility provided by credal sets arguably facilitates model building, on the other, it imposes a great burden on the computation of inferences. For example, whereas computing the posterior probability of a variable is polynomial-time computable in polytree-shaped Bayesian networks, the analogous task of computing upper and lower bounds on the posterior probability of a given variable in a polytree-shaped credal networks is an NP-hard task [17]. There are however exceptional cases, such as the case of inference in polytree-shaped credal networks with binary variables, which can be solved in polynomial time under strong independence [25]. Like in Bayesian networks, the theoretical and practical tractability of inferences in credal networks depends strongly on the network topology and the cardinality of the variable domains. Credal networks however include another dimension in the parametrized complexity of inference, given by the type of irrelevance concept adopted, which in the Bayesian case is usually fixed. For instance, computing probability bounds in tree-shaped credal networks under the concept of epistemic irrelevance can be performed in polynomial time [22], whereas we show here that the same task is NP-hard under strong independence.

In the rest of this paper, we properly define credal networks and the inference problem we address (Section 2), and investigate the parametrized theoretical computational complexity of inferences in credal networks (Section 3), both under strong independence and epistemic irrelevance. We show that a particular type of inference in imprecise hidden Markov models (i.e., hidden Markov models with uncertainty quantified by local credal sets) is invariant to the choice of either irrelevance concept, being thus polynomial-time computable (as this is known to be the case under epistemic irrelevance). We obtain as corollaries of that result that inferences under strong independence and epistemic irrelevance coincide also in tree-shaped networks if no evidence is given, and in Naive structures. We also show that even in tree-shaped credal networks inferences are NP-hard if we assume strong independence, and that this is the same complexity of inference in polytree-shaped credal networks for both irrelevance concepts, even if we assume that all variables are (at most) ternary. We prove that the so-called precise-vacuous models, that is, credal networks that have vacuous root nodes and precise non-root nodes, lead to the same inferences whether we assume epistemic irrelevance or strong independence, and that the same is true (apart from an arbitrarily small error) when vacuous nodes are replaced by near-vacuous ones, avoiding the problematic case of zero probabilities. This last fact will be used to prove that our hardness results hold true even in cases where the lower probability of any possible event is strictly positive.

Part of the material presented here appeared in a different and simplified form in References [35], [38] and [34].

2 Updating Credal Networks

In this section, we review the necessary concepts and definitions, and formalize the problem of inference in credal networks.

2.1 Bayesian Networks

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ for some natural number n. For $N \subseteq [n]$, let $\mathbf{X} = \{X_i : i \in N\}$ be a finite set of categorical variables, and $\mathbf{Z} \subseteq \mathbf{X}$ be some set of variables. A *probability distribution* p of $\mathbf{Z}$ is a non-negative real-valued function on the space of assignments of $\mathbf{Z}$ such that $\sum_{\mathbf{z} \sim \mathbf{Z}} p(\mathbf{z}) = 1$, where the notation $\mathbf{z} \sim \mathbf{Z}$ entails that $\mathbf{z}$ is an arbitrary (joint) assignment or configuration of the variables in $\mathbf{Z}$. Any joint probability distribution p induces a probability measure P_p on the sigma-field of all subsets of assignments of $\mathbf{Z}$.

Let G be an acyclic directed graph (ADG) with nodes N. We denote the *parents* of a node i in G by $\mathrm{Pa}(i)$. The set of *non-descendants* of i, written $\mathrm{Nd}(i)$, contains the nodes not reachable from i by a directed path. Note that $\mathrm{Pa}(i) \subseteq \mathrm{Nd}(i)$. Fix a probability measure P on the sigma-field of subsets of $\mathbf{X}$ and associate every node i with a variable X_i. The ADG G represents the following set of stochastic independence assessments known as *local Markov conditions*:

$$\mathrm{P}(X_i = x_i | \mathbf{X}_{\mathrm{Nd}(i)} = \mathbf{x}_{\mathrm{Nd}(i)}) = \mathrm{P}(X_i = x_i | \mathbf{X}_{\mathrm{Pa}(i)} = \mathbf{x}_{\mathrm{Pa}(i)}) , \tag{1}$$

for all $i \in N$, and $\mathbf{x} \sim \mathbf{X}$. In words, every variable is stochastically independent from its non-descendant non-parents given its parents under a suitable measure P.

A Bayesian network is a triple $(\mathbf{X}, G, Q)$, where Q is the set of conditional probability assessments

$$\mathrm{P}(X_i = x_i | \mathbf{X}_{\mathrm{Pa}(i)} = \mathbf{x}_{\mathrm{Pa}(i)}) = q(x_i | \mathbf{x}_{\mathrm{Pa}(i)})\,, \tag{2}$$

for all $i \in N$, $x_i \sim X_i$ and $\mathbf{x}_{\mathrm{Pa}(i)} \sim \mathbf{X}_{\mathrm{Pa}(i)}$, where $q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})$ is a probability distribution of X_i. By assumption, a Bayesian network defines a joint probability distribution p of $\mathbf{X}$ by

$$p(\mathbf{x}) = \prod_{i \in N} q(x_i | \mathbf{x}_{\mathrm{Pa}(i)})\,, \tag{3}$$

for all $\mathbf{x} \sim \mathbf{X}$. It is not difficult to see that (1) and (2) imply (3) by the Chain Rule using a topological ordering of variables. That (3) and (2) imply (1) is a bit more intricate to see but also true [11]. Thus, these two seemingly different approaches to specifying a probability measure are virtually equivalent. To be more explicit: given a Bayesian network, the probability measure that satisfies (1) and (2) is the same probability measure that satisfies (3) and (2), and we can choose any pair of assumptions to define a (single) measure for a network. As we shall see, an analogous equivalence is not observed when probabilities are imprecisely specified, which leads to different definitions of credal networks with different computational complexity.

2.1.1 Probabilistic Inference In Bayesian Networks

An essential task in many applications of probabilistic modelling is to compute a certain probability value implied by a Bayesian network. We call such a computational task the BN-INF problem, and define it as follows.

> BN-INF
> *Input*: A Bayesian network $(\mathbf{X}, G, Q)$, a *target* node t, a target value x_t of X_t, a (possibly empty) set of evidence nodes O, and an assignment $\mathbf{x}_O$ to $\mathbf{X}_O$.
> *Output*: The conditional probability $\mathrm{P}(X_t = x_t | \mathbf{X}_O = \mathbf{x}_O)$, where P is the probability measure specified by the network.

In the problem above we assume that if $\mathrm{P}(\mathbf{X}_O = \mathbf{x}_O) = 0$ then any number between zero and one is a solution.

Let f be a real-valued function and p be a joint probability distribution, both defined on $\mathbf{x} \sim \mathbf{X}$. Define $\mathrm{E}_p(f)$ the expectation of f under p, that is,

$$\mathrm{E}_p(f) \stackrel{\text{def}}{=} \sum_{\mathbf{x} \sim \mathbf{X}} p(\mathbf{x}) f(\mathbf{x})\,. \tag{4}$$

Let also $\delta_{\mathbf{z}}$ denote the *Kronecker's delta function* at $\mathbf{z}$, which returns one at $\mathbf{z}$ and zero elsewhere. Since there is an one-to-one map between expectation and probability, we can state the problem above in a slightly different but equivalent way.

> BN-INF2
> *Input*: A Bayesian network $(\mathbf{X}, G, Q)$, a *target* node t, a target value x_t of X_t, a (possibly empty) set of evidence nodes O, and an assignment $\mathbf{x}_O$ to $\mathbf{X}_O$.
> *Output*: Solve $\mathrm{E}_p([\mu - \delta_{x_t}]\delta_{\mathbf{x}_O}) = 0$ for μ, where p is the joint distribution specified by the network.

It should be clear that BN-INF and BN-INF2 are the same problem.

[44] showed that BN-INF is #P-hard, defining a lower bound to the complexity of the problem. All known (exact) algorithms take time at least exponential in the treewidth of the network in the worst case. The treewidth is a measure of the resemblance of a network to a tree; small treewidth suggests a tree-like structure and the treewidth of a tree is one and is minimal [30]. Recently, [32] proved that contingent on the hypothesis that satisfiability of Boolean formulas takes exponential time in the worst-case (a widely believed hypothesis) this is the best performance an algorithm for BN-INF can achieve. As we shall see in the next section, Bayesian networks are particular instances of credal networks. As such, these complexity results set lower bounds on the complexity of inference in credal networks.

An ADG is said to be *singly connected* if there is at most one *undirected* path connecting any two nodes in the graph; it is a *tree* if additionally each node has at most one parent. If a graph is not singly connected, we say it is *multiply connected*. Singly connected directed graphs are also called *polytrees*. Pearl's belief propagation algorithm [42] computes BN-INF in polynomial time in singly connected Bayesian networks. More generally, the junction tree propagation algorithm [11] solves BN-INF in polynomial time in any network of bounded treewidth, which includes singly connected networks of bounded in-degree (i.e., maximum number of parents).

2.2 Credal Networks

In this section we describe credal sets, irrelevance concepts, credal networks and probabilistic inference in credal networks. We also discuss the previously known results about the complexity of inference in credal networks.

2.2.1 Credal Sets

Let $\mathbf{Z} \subseteq \mathbf{X}$. A *credal set* M is a closed and convex set of joint probability distributions on the same domain, say $\mathbf{z} \sim \mathbf{Z}$ [33]. The *vacuous credal set* of $\mathbf{Z}$ is the largest credal set on that domain, and is denoted by $V(\mathbf{Z})$. An *extreme distribution* of a credal set is an element of the set that cannot be written as a convex combination of other elements in the same set. We denote the set of extreme distributions of a credal set M by $\operatorname{ext} M$. A credal set is *finitely generated* if it contains a finite number of extreme distributions. A finite representation of a finitely generated credal set by means of its extreme distributions is called *vertex-based*. Any finitely generated credal set of $\mathbf{Z}$ defines a (bounded) polytope in the probability simplex of distributions of $\mathbf{Z}$, and can be specified through a finite set of linear inequalities of the form

$$\mathrm{E}_p(f_l) \stackrel{\text{def}}{=} \sum_{\mathbf{z} \sim \mathbf{Z}} f_l(\mathbf{z}) p(\mathbf{z}) \leq 0\,, \tag{5}$$

where $\{f_l\}$ is a finite collection of real-valued functions of $\mathbf{Z}$ [12]. The converse is also true: any finite set of linear inequalities of the form above determines a (bounded) polytope in the probability simplex [8, Chapter 2], and hence a finitely generated credal set. Thus, an alternative finite representation of a credal set is by means of a finite set of functions defining linear inequalities of the type above. Such a representation is called *constraint-based*. Vertex- and constraint-based representations of the same credal set can have very different sizes. To see this, consider a single variable X taking values in $[m]$, and let $M = \{p \in V(X) : p(k) \leq 1/(m+1), k = 1, \dots, m\}$. The set M is isomorphic to an m-dimensional hypercube, and therefore has 2^m extreme distributions,[4] whereas the same set can be represented in constraint-based form by m degenerate functions of X translated by $1/(m+1)$. Moving from a vertex-based to a constraint-based representation can also result in an exponential increase in the size of the input. Consider a variable X taking values in $[m]$ and let $M = \operatorname{co}\{e_1, \dots, e_m, 1 - e_1, \dots, 1 - e_m\}$, where co denotes the convex hull operator, and e_k is the degenerate distribution placing all mass at $X = k$. It can be shown that M is affinely equivalent to the m-dimensional cross-polytope $\{f(X) : \sum_x |f(x)| \leq 1\}$, which is the dual

[4] For any nonnegative integer k not greater than m and any (potentially empty) subset S of $[m]$ of cardinality k, any distribution that assigns mass $(m+1-k)/(m+1)$ to $p(0)$, mass $1/(m+1)$ to $p(j)$ such that j is in S, and zero mass elsewhere, is in M, since it satisfies all the constraints in M and is a valid distribution. There are 2^m such distributions, and each one cannot be written as a convex combination of any other distribution in the set, since each "touches" an inequality in at least one dimension.

of the m-dimensional hypercube and whose constraint-based representation requires 2^m inequalities, whereas its vertex-based representation needs only $2m$ distributions [28, page 11]. [48] and [21] studied the representation of credal sets defined by linear constraints of the form $l_x \leq p(x) \leq u_x$, where l_x and u_x are real numbers, and showed that these credal sets can have exponentially many extreme distributions in the number of constraints. [51] proved an attainable upper bound of $m!$ extreme distributions on credal sets more generally defined by a coherent lower probability function of an m-ary variable. More recently, [40] investigated the number of extreme distributions in credal sets defined by linear constraints of the form $p(x) \leq p(x')$ for $x \neq x'$, and proved an attainable upper bound of 2^{m-1} extreme distributions for the case of an m-ary variable. Importantly, both vacuous credal sets (of variables of any cardinality) and credal sets of binary variables can be succinctly represented in either vertex- or constraint-based form. Some of the complexity results we obtain later on use vacuous credal sets and/or binary variables and are thus representation independent. The next example is supposed to clarify the terminology and concepts about credal sets.

Example 1. *Consider* $\mathbf{X} = \{X_1, X_2\}$*, where* X_1 *takes values in* $\{0, 1, 2\}$ *and* X_2 *takes values in* $\{0, 1\}$*. The vacuous set of* X_1 *is the probability simplex on the plane, drawn as a triangle with vertices* $p(1)$*,* $p(2)$ *and* $p(3)$ *in Figure 1. Let*

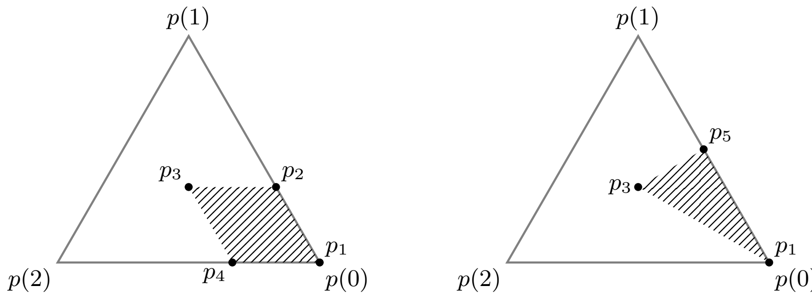
$$M(X_1|X_2\!=\!0) = \{p \in V(X_1) : p(k) \leq 1/3, k = 1, 2\}$$

and

$$M(X_1|X_2\!=\!1) = \{p \in V(X_1) : p(0) \leq p(1) \leq p(2)\}$$

be conditional credal sets for X_1 *given* X_2*, and* $M(X_2)$ *be the singleton containing the distribution* p *of* X_2 *such that* $p(0) = p(1) = 1/2$*. The first two sets are depicted in Figure 1. Let us represent a generic function* f *on* $\{0, \ldots, m\}$ *by the* m*-tuple* $(f(0), \ldots, f(m))$*, and define*

$$\begin{aligned}
p_1 &= (1, 0, 0)\,, & p_2 &= (2/3, 1/3, 0)\,,\\
p_3 &= (1/3, 1/3, 1/3)\,, & p_4 &= (2/3, 0, 1/3)\,,\\
p_5 &= (1/2, 1/2, 0)\,, & f_1 &= (-1, 2, -1)\,,\\
f_2 &= (-1, -1, 2)\,, & f_3 &= (1, -1, 0)\,,\\
f_4 &= (0, 1, -1)\,, & f_5 &= (1, -1)\,.
\end{aligned}$$



(a) $M(X_1|X_2\!=\!0)$ (b) $M(X_1|X_2\!=\!1)$

Figure 1: Barycentric coordinate-system visualization of the conditional credal sets in Example 1 (hatched regions) and their corresponding extreme distributions (black circles).

Then the set $M(X_1|X_2 = 0)$ can be represented in vertex- and constraint-based form, respectively, as $M(X_1|X_2\!=\!0) = \text{co}\{p_1, p_2, p_3, p_4\}$ and $M(X_1|X_2\!=\!0) = \{p \in V(X_1) : \mathrm{E}_p(f_1) \leq 0, \mathrm{E}_p(f_2) \leq 0\}$, while the set $M(X_1|X_2 = 1)$ is represented in vertex- and constraint-based forms as $M(X_1|X_2 = 1) = \text{co}\{p_1, p_3, p_5\}$ and $M(X_1|X_2\!=\!1) = \{p \in V(X_1) : \mathrm{E}_p(f_3) \leq 0, \mathrm{E}_p(f_4) \leq 0\}$, respectively. Similarly, $M(X_2)$ can be represented as $M(X_2) = \{(1/2, 1/2)\}$ in vertex-based form, and as $M(X_2) = \{p \in V(X_2) : \mathrm{E}_p(f_5) \leq 0, \mathrm{E}_p(-f_5) \leq 0\}$ in constraint-based form. □

2.2.2 Graph-Based Representation

So far we have only considered the explicit representation of a finitely generated credal set by a finite number of functions representing either the vertices of the set or a set of linear inequalities. Our final goal is however to be able to specify credal sets on large domains $\mathbf{x} \sim \mathbf{X}$. For a large set $\mathbf{X}$, such an explicitly representation is both too difficult to obtain and too large to manipulate in a computer. Thus, analogously to the more efficient graph-based representation of a large probability distribution given by a Bayesian network, a large joint credal set is usually more efficiently represented implicitly as the credal set that satisfies all irrelevances encoded in a given graph while agreeing on its projection with all local credal sets, where the latter are credal sets that can be efficiently represented (either in vertex- or constraint-based form) explicitly by functions of only small subsets of $\mathbf{X}$.

A separately specified *credal network* $\mathcal{N}$ is a triple $(\mathbf{X}, G, Q)$, where G is

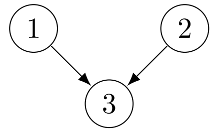


Figure 2: ADG of the credal network in Example 2.

an ADG with nodes N, and Q is a set of imprecise probability assessments

$$\forall f : \sum_{x_i \sim X_i} f(x_i) \mathrm{P}(X_i = x_i | \mathbf{X}_{\mathrm{Pa}(i)} = \mathbf{x}_{\mathrm{Pa}(i)}) \geq \min_{q \in Q(X_i | \mathbf{x}_{\mathrm{Pa}(i)})} \sum_{x_i \sim X_i} f(x_i) q(x_i) , \tag{6}$$

one for every $i \in N$ and $\mathbf{x}_{\mathrm{Pa}(i)} \sim \mathbf{X}_{\mathrm{Pa}(i)}$, where each $Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})$ is a credal set of X_i and f is an arbitrary real-valued function of X_i. Note that we left unspecified how these credal sets are represented.

A node i in the network and its associated variable X_i are said to be *precise* if the corresponding conditional credal sets $Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})$ are all singletons, otherwise they are said to be *imprecise*. If all of its local credal sets are vacuous, the node is said to be vacuous. A Bayesian network is simply a credal network with all nodes precise. The following example contains a simple separately specified credal network.

Example 2. *Consider the credal network $\mathcal{N}$ over variables X_1, X_2 and X_3 that take values in $\{0, 1\}$, and with graph structure as shown in Figure 2. The local credal sets are*

$$Q(X_1) = \{p \in V(X_1) : 0.5 \leq p(1) \leq 0.6\} = \mathrm{co}\{(0.4, 0.6), (0.5, 0.5)\} ,$$
$$Q(X_2) = \{p \in V(X_2) : 0.5 \leq p(1) \leq 0.6\} = \mathrm{co}\{(0.4, 0.6), (0.5, 0.5)\} ,$$

and $Q(X_3|X_1 = i, X_2 = j) = \{p^{ij}\}$ for any i and j, where p^{ij} is the probability distribution on $\{0, 1\}$ such that $p^{ij}(1) = 0$ if $i = j$ and $p^{ij}(1) = 1$ otherwise. □

The ADG G of a credal network specifies a set of conditional irrelevances between sets of variables which generalize the Markov condition in Bayesian networks. More specifically, for any node i in G, the set $\mathbf{X}_{\mathrm{Nd}(i) \setminus \mathrm{Pa}(i)}$ of non-descendant non-parent variables of X_i is assumed irrelevant to X_i conditional on its parent variables $\mathbf{X}_{\mathrm{Pa}(i)}$. The precise definition of this statement requires the definition of an irrelevance concept. For instance, if stochastic independence is adopted as irrelevance concept, then the ADG G describes a set of Markov conditions as a Bayesian network (stochastic irrelevance implies stochastic independence). In the credal network formalism, the two most common irrelevance concepts used are *strong independence* and *epistemic irrelevance*.

Fix a joint credal set M of probability distributions of $\mathbf{X}$, and consider subsets $\mathbf{Y}$, $\mathbf{Z}$ and $\mathbf{W}$ of $\mathbf{X}$. We say that a set of variables $\mathbf{Y}$ is *strongly independent* of a set of variables $\mathbf{Z}$ given variables $\mathbf{W}$ if $\mathbf{Y}$ and $\mathbf{Z}$ are stochastically independent conditional on $\mathbf{W}$ under every extreme distribution $p \in \operatorname{ext} M(\mathbf{X})$, which implies for every $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{w}$ that $\mathrm{P}_p(\mathbf{Y}=\mathbf{y}|\mathbf{Z}=\mathbf{z}, \mathbf{W}=\mathbf{w}) = \mathrm{P}_p(\mathbf{Y}=\mathbf{y}|\mathbf{W}=\mathbf{w})$. We say that a set of variables $\mathbf{Z}$ is *epistemically irrelevant* to a set of variables $\mathbf{Y}$ conditional on variables $\mathbf{W}$ if for every function f on $\mathbf{y} \sim \mathbf{Y}$ and assignments $\mathbf{z}$ and $\mathbf{w}$ it follows that

$$\min_{p \in M} \sum_{\mathbf{y} \sim \mathbf{Y}} f(\mathbf{y}) \mathrm{P}_p(\mathbf{Y}=\mathbf{y}|\mathbf{Z}=\mathbf{z}, \mathbf{W}=\mathbf{w}) = \min_{p \in M} \sum_{\mathbf{y} \sim \mathbf{Y}} f(\mathbf{y}) \mathrm{P}_p(\mathbf{Y}=\mathbf{y}|\mathbf{W}=\mathbf{w}) \,, \tag{7}$$

which is equivalent to say that the projection of M on $\mathbf{Y}$ conditioned on $\mathbf{W}=\mathbf{w}$ and $\mathbf{Z}=\mathbf{z}$ equals the projection of M on $\mathbf{Y}$ conditioned only on $\mathbf{W}=\mathbf{w}$. It is an immediate conclusion that strong independence implies epistemic irrelevance (and the converse is not necessarily true) [12, 24]. Variables $\mathbf{Y}$ and $\mathbf{Z}$ are *epistemically independent* conditional on $\mathbf{W}$ if, given any assignment $\mathbf{w}$, $\mathbf{Y}$ and $\mathbf{Z}$ are epistemically irrelevant to each other [49, Ch. 9].

The *strong extension* of a credal network $\mathcal{N} = (\mathbf{X}, G, Q)$ is the largest credal set K_S of distributions of $\mathbf{X}$ that satisfies the strong independence assessments in G (viz. that every variable is strongly independent of its non-descendant non-parents given its parents), and whose projections on local domains lie inside the local credal sets specified in Q, that is, K_S is the set of distributions of $\mathbf{X}$ whose induced measure satisfies (1) and (6). One can show that the strong extension can be equivalently defined as [7, 2]

$$K_S \overset{\text{def}}{=} \operatorname{co}\left\{ p \in V(\mathbf{X}) : p(\mathbf{x}) = \prod_{i \in N} q_i^{\mathbf{x}_{\mathrm{Pa}(i)}}(x_i),\; q_i^{\mathbf{x}_{\mathrm{Pa}(i)}} \in \operatorname{ext} Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)}) \right\} . \tag{8}$$

The following result connects vertex- and constraint-based credal networks.

Proposition 1. *Any vertex-based (separately specified) credal network can be efficiently reduced to a constraint-based network that induces the same strong extension.*

Proof. Let X_i be a variable whose local credal set $Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})$ is specified by the extreme distributions $p_1, \dots, p_m$, for a given assignment of the parents. Insert a new vacuous variable X_α taking values in $[m]$, and with X_i as its child and $\mathbf{X}_{\mathrm{Pa}(i)}$ as its parents, and redefine $Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})$ as the singleton that contains the conditional distribution $q(x_i|\mathbf{x}_{\mathrm{Pa}(i)}, x_\alpha\!=\!k) = p_k(x_i)$. One can verify that the strong extension of the new network after marginalizing X_α coincides with the original strong extension. □

The result above cannot be applied to derive complexity of singly connected networks since the reduction used in the proof inserts (undirected) cycles in the network. Thus, it is not true that hardness results obtained for vertex-based singly connected networks immediately extend to constraint-based singly connected networks, even though this is always the case in the results we present here (for instance, we only use credal sets that are easily translated from one representation to the other in our hardness results). Conversely, tractability of constraint-based singly connected networks does not immediately extend to vertex-based singly connected networks. It is unclear whether constraint-based networks can be efficiently reduced to vertex-based form by inserting new variables, but we conjecture that this is true.

The *epistemic extension* of a credal network will denote the largest joint credal set K_E of $\mathbf{X}$ that satisfies the epistemic irrelevance assessments in G (viz. the non-descendant non-parents are irrelevant to a variable given its parents), and the assessments in Q. Equivalently, the epistemic extension is the credal set K_E defined by the set of probability distributions p of $\mathbf{X}$ such that

$$\sum_{x_i} f(x_i)\mathrm{P}_p(x_i|\mathbf{x}_{\mathrm{Nd}(i)}) \geq \min_{q \in Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})} \sum_{x_i} f(x_i) q(x_i)\,, \tag{9}$$

for all functions f of X_i, and assignment $\mathbf{x}_{\mathrm{Nd}(i)}$. Note that these inequalities can be turned into linear inequalities of the form (5) by multiplying both sides by $\mathrm{P}_p(\mathbf{x}_{\mathrm{Nd}(i)})$ and rearranging terms.

Example 3. *Consider the network in Example 2, and represent a function f of a binary variable as the pair $(f(0), f(1))$. The strong extension K_S is the credal set whose extreme distributions are the four joint probability distributions $p \in V(X_1, X_2, X_3)$ such that*

$$p(x_1, x_2, x_3) = p_1(x_1)p_2(x_2)p_3^{x_1 x_2}(x_3) \quad \textit{for } x_1, x_2, x_3 \in \{0, 1\}\,,$$

where

$$p_1 \in \{(0.4, 0.6), (0.5, 0.5)\}\,, \qquad p_2 \in \{(0.4, 0.6), (0.5, 0.5)\}\,,$$
$$p_3^{00} = p_3^{11} = (1, 0)\,, \qquad p_3^{01} = p_3^{10} = (0, 1)\,.$$

Note that the strong extension contain four extreme distributions. The epistemic extension K_E is the set of joint probability distributions $p \in V(X_1, X_2, X_3)$

that satisfies the system of linear inequalities

$$0.5 = \min_{q \in Q(X_1)} q(1) \leq \mathrm{P}_p(X_1 = 1 | x_2) \leq \max_{q \in Q(X_1)} q(1) = 0.6 \qquad [x_2 = 0, 1],$$
$$0.5 = \min_{q \in Q(X_2)} q(1) \leq \mathrm{P}_p(X_2 = 1 | x_1) \leq \max_{q \in Q(X_2)} q(1) = 0.6 \qquad [x_1 = 0, 1],$$
$$\mathrm{P}_p(X_3 = 1 | X_1 = x, X_2 = x) = 0 \qquad [x = 0, 1],$$
$$\mathrm{P}_p(X_3 = 1 | X_1 = 0, X_2 = 1) = \mathrm{P}_p(X_3 = 1 | X_1 = 1, X_2 = 0) = 1 .$$

One can verify that the set K_E has the following six extreme distributions:

$$\begin{aligned}
p1 &= (0.25, 0, 0, 0.25, 0, 0.25, 0.25, 0) ,\\
p2 &= (0.16, 0, 0, 0.36, 0, 0.24, 0.24, 0) ,\\
p3 &= (0.2, 0, 0, 0.3, 0, 0.2, 0.3, 0) ,\\
p4 &= (0.2, 0, 0, 0.3, 0, 0.3, 0.2, 0) ,\\
p5 &= (2/9, 0, 0, 3/9, 0, 2/9, 2/9, 0) ,\\
p6 &= (2/11, 0, 0, 3/11, 0, 3/11, 3/11, 0) ,
\end{aligned}$$

where the tuples on the right represent distributions $p(x_1, x_2, x_3)$ by

$$(p(0,0,0), p(1,0,0), p(0,1,0), p(1,1,0), p(0,0,1), p(1,0,1), p(0,1,1), p(1,1,1)) .$$

We observe that distributions p_1 to p_4 are extreme distributions of the strong extension, whereas p_5 and p_6 are not in the strong extension. □

The example above shows an interesting and well-known relation between epistemic and strong extensions, namely, that the latter is always contained in the former, and thus produces more precise results [49, Chapter 9.2].

2.2.3 Probabilistic Inference

Similarly to Bayesian networks, a primary use of credal networks is in deriving bounds for probabilities implied by the model. The precise characterization depends on the choice of an irrelevance concept. We thus define the inference problem under strong independence as follows.

> STRONG-INF
> *Input:* A credal network $(\mathbf{X}, G, Q)$, a target node t, an assignment of X_t, a (possibly empty) set of evidence nodes O, and an

assignment $\mathbf{x}_O$ of $\mathbf{X}_O$.
Output: The numbers

$$\min_{p \in K_S} \mathrm{P}_p(X_t = x_t | \mathbf{X}_O = \mathbf{x}_O) \text{ and } \max_{p \in K_S} \mathrm{P}_p(X_t = x_t | \mathbf{X}_O = \mathbf{x}_O)\,,$$

where K_S is the strong extension of the network.

An analogous inference problem can be defined for epistemic irrelevance, simply by replacing the strong extension in the output of the problem above by the epistemic extension:

EPISTEMIC-INF
Input: A credal network $(\mathbf{X}, G, Q)$, a target node t, an assignment of X_t, a (possibly empty) set of evidence nodes O, and an assignment $\mathbf{x}_O$ of $\mathbf{X}_O$.
Output: The numbers

$$\min_{p \in K_E} \mathrm{P}_p(X_t = x_t | \mathbf{X}_O = \mathbf{x}_O) \text{ and } \max_{p \in K_E} \mathrm{P}_p(X_t = x_t | \mathbf{X}_O = \mathbf{x}_O)\,,$$

where K_E is the epistemic extension of the network.

We assume in both problems that when the lower probability of the evidence is zero (i.e., $\min_p \mathrm{P}_p(\mathbf{X}_O = \mathbf{x}_O) = 0$), then any value is a solution (that is, the minimization may achieve zero and the maximization one).[5] For a recent treatment of the zero probability case, see [15]. We emphasize that our complexity results hold true regardless of how zero probabilities are treated, because we take the appropriate care to avoid any event with zero probability in our reductions (as it will become clear later on).

Example 4. *Consider again the network in Example 2, and assume that the target node is $t = 3$, $x_t = 0$ and $\mathbf{X}_O$ is the empty set. The strong extension K_S has been defined in Example 3. The outcome of* STRONG-INF *is*

$$\begin{aligned}\min_{p \in K_S} \mathrm{P}(X_3 = 0) &= \min \sum_{x_1, x_2} p_1(x_1) p_2(x_2) p_3^{x_1 x_2}(0) \\ &= 1 + \min\{2p_1(0)p_2(0) - p_1(0) - p_2(0)\} \\ &= 1 - (2 \cdot 1/2 \cdot 1/2 - 1/2 - 1/2) = 1/2\,,\end{aligned}$$

[5] If the upper probability of the evidence is positive, the regular extension can be used to compute non-vacuous inferences, see [49, Appendix J].

where the minimizations on the right are performed over p_1 and p_2, and

$$\begin{aligned}\max_{p\in K_S} \mathrm{P}(X_3\!=\!0) &= \max \sum_{x_1,x_2} p_1(x_1)p_2(x_2)p_3^{x_1x_2}(0) \\ &= 1 + \max\{2p_1(0)p_2(0) - p_1(0) - p_2(0)\} \\ &= 1 - (2\cdot 0.4\cdot 0.4 - 0.4 - 0.4) = 0.52\,.\end{aligned}$$

The outcome of EPISTEMIC-INF *are the values of the solutions of the linear programs*

$$\min\{p(0,0,0) + p(1,1,0) : p \in \mathrm{K}_E\} = 5/11 < 1/2$$

and

$$\max\{p(0,0,0) + p(1,1,0) : p \in \mathrm{K}_E\} = 5/9 > 0.52\,,$$

where K_E is the epistemic extension defined in Example 3. □

The fact that the lower bound (resp., upper bound) of EPISTEMIC-INF in the example above is smaller (resp., greater) than the lower bound (upper bound) of STRONG-INF is a direct consequence of the fact that the strong extension is contained in the epistemic extension.

Analogously to BN-INF2, we can define equivalent root-finding forms for the inferential problems just presented.

> STRONG-INF2
> *Input:* A credal network $(\mathbf{X}, G, Q)$, a target node t, an assignment of X_t, a (possibly empty) set of evidence nodes O, and an assignment $\mathbf{x}_O$ of $\mathbf{X}_O$.
> *Output*: The solution μ of $\mathrm{opt}_{p\in K_S} \mathrm{E}_p([\mu - \delta_{x_t}]\delta_{\mathbf{x}_O}) = 0$, where K_S is the strong extension of the network, with opt $\in \{\min, \max\}$.

> EPISTEMIC-INF2
> *Input:* A credal network $(\mathbf{X}, G, Q)$, a target node t, an assignment of X_t, a (possibly empty) set of evidence nodes O, and an assignment $\mathbf{x}_O$ of $\mathbf{X}_O$.
> *Output*: The solution μ of $\mathrm{opt}_{p\in K_E} \mathrm{E}_p([\mu - \delta_{x_t}]\delta_{\mathbf{x}_O}) = 0$, where K_E is the epistemic extension of the network, and opt $\in \{\min, \max\}$.

The main advantage of these reformulations is that a linear-fractional programming problem is transformed into a linear programming problem, which facilitates obtaining some results. We will refer to both reformulations interchangeably, having in mind their equivalence.

3 Complexity Results

In this section, we study the complexity of inference in credal networks with respect to the irrelevance concept adopted, the network topology and the variable domain cardinality.

3.1 Summary of The Inferential Complexity

Computing the STRONG-INF is notoriously a hard task, whose complexity strongly depends on the topology of the ADG and the cardinality of the variable domains. [14] proved this problem to be NP^{PP}-hard. [17] studied the parametrized complexity and concluded that the problem is NP-hard even on singly connected networks of bounded treewidth. We shall see in the next section that the problem remains NP-hard in singly connected credal networks if we constraint variables to take on at most three values; in fact, we show that it is NP-hard already in credal trees. A long-known positive result is the 2U algorithm of [25], which solves the problem in polynomial time if the underlying graph is a polytree and all variables are binary. Up to now, this was the only positive result regarding the complexity of STRONG-INF (when the model is not completely precise, in which case the positive results from BN-INF apply). We show in the next section a new positive result in credal hidden Markov models (HMMs), which are particular types of tree-shaped credal networks commonly used to represent time-dependent processes. Specifically, we show here that if the target node is the last node of the longest directed path of the HMM, inference can be performed in polynomial time irrespectively of the variable cardinalities. It remains open the complexity of more general inferences in HMMs. As a corollary, we prove that STRONG-INF and EPISTEMIC-INF in Naive Bayes structures coincide, and the same is true for marginal inferences in trees, that is, when there is no evidence. These results have been previously obtained independently for each irrelevance concept and their tractability was thought to be coincidental.

Much fewer is known about the complexity of EPISTEMIC-INF. A positive result was given by [22], who developed a polynomial-time algorithm for computations in credal trees. We show later on that a polynomial-time algorithm for singly connected networks implies that P equals NP, and is therefore unlikely. We also show that the problem is NP^{PP}-hard in general networks, and NP-hard in networks of bounded treewidth. These results are summarized in Table 1. The star indicates models in which inferences under both irrelevance concepts coincide.

Table 1: Parametrized complexity of the inference in credal networks.

MODEL	STRONG-INF	EPISTEMIC-INF
*Naive Bayes	P	P
*Imprecise HMM (query on last node)	P	P
Imprecise HMMs	Unknown	P
*Credal trees (no evidence)	P	P
Credal trees	NP-hard	P
Credal polytrees with binary variables	P	Unknown
Credal polytrees with ternary variables	NP-hard	NP-hard
Bounded treewidth networks	NP-hard	NP-hard
Credal networks	NP^{PP}-hard	NP^{PP}-hard
*Precise-vacuous	NP^{PP}-hard	NP^{PP}-hard

3.2 Networks Specified With Computable Numbers

Before presenting the complexity results, we need to introduce the concept of *polynomial-time computable numbers*, and to discuss some properties of networks specified with such numbers. This will be used within later proofs in an essential way, including (but not only) to show that the hardness results hold true even if we assume that any possible event has positive probability.

A number r is polynomial-time computable if there exists a transducer Turing machine M_r that, for an integer input b (represented through a binary string), runs in time at most poly(b) (the notation poly(b) denotes an arbitrary polynomial function of b, and might indicate a different polynomial at each time it is used) and outputs a rational number r' (represented through its numerator and denominator in binary strings) such that $|r - r'| < 2^{-b}$. Of special relevance to us are numbers of the form $(2^{-v_1} - 2^{-v_2})/(1 + 2^{-v_3})$, with v_1, v_2 and v_3 being nonnegative rationals no greater than two. For any rational v between zero and two we can build a machine that outputs a rational r' that approximates 2^{-v} with precision b in time poly(b) by computing the Taylor expansions of 2^{-v} around zero with sufficiently many terms (depending on the value of b) similar to the proof of Lemma 4 in Ref. [39]. The desired numbers can then be obtained by the corresponding fractional expression. The following lemmas ensure that the outcome of STRONG-INF on networks specified with polynomial-time computable numbers can be approximated arbitrarily well using a network specified only with positive rational numbers. It allows us to specify the desired precision and for which nodes of the network the numerical parameters will be “approximated” by positive rational numbers.

Lemma 1. *Consider a vertex-based credal network $\mathcal{N}$ whose numerical parameters are specified with polynomial-time computable numbers encoded by their respective machines (or directly given as rational numbers), and let b be the size of the encoding of $\mathcal{N}$. Given a subset of the nodes $N' \subseteq N$ of $\mathcal{N}$ and a rational number $1 \geq \varepsilon \geq 2^{-\mathrm{poly}(b)}$, we can construct in time* $\mathrm{poly}(b)$ *a vertex-based credal network $\mathcal{N}'$ over the same variables whose numerical parameters that specify the credal sets of nodes N' are all rational numbers greater than $2^{-\mathrm{poly}(b)}$ (numerical parameters related to nodes not in N' are kept unchanged), and such that there is a polynomial-time computable surjection (p, p') that associates any extreme p of the strong extension of $\mathcal{N}$ with an extreme p' of the strong extension of $\mathcal{N}'$ satisfying*

$$\max_{\mathbf{x}_A} |\mathrm{P}_{p'}(\mathbf{X}_A = \mathbf{x}_A) - \mathrm{P}_p(\mathbf{X}_A = \mathbf{x}_A)| \leq \varepsilon\,,$$

for any subset $\mathbf{X}_A \subseteq \mathbf{X}$ of the variables.

Proof. Take $\mathcal{N}'$ to be equal to $\mathcal{N}$ except that each computable number r used in the specification of $\mathcal{N}$ for nodes N' is replaced by a rational r' such that $|r' - r| < 2^{-(n+1)(v+1)-1}\varepsilon$, where n is the number of variables, and v is the maximum cardinality of the domain of any variable in $\mathcal{N}$. Because $\varepsilon \geq 2^{-\mathrm{poly}(b)}$, we can run the Turing machine M_r used to represent r on input $\mathrm{poly}(b) + (n+1)(v+1) + 1$ to obtain r' in time $O(\mathrm{poly}(\mathrm{poly}(b) + (n+1)(v+1) + 1)) = O(\mathrm{poly}(b))$. After obtaining r', add to it $2^{-(n+1)(v+1)-1}\varepsilon$ to ensure that $r < r' < r + 2^{-(n+1)(v+1)}\varepsilon$, that is, the approximation is from above. However, exactly one of the probability values in each distribution used to represent an extreme of a local credal set in $\mathcal{N}'$ is not approximated in that way but is computed as one minus the sum of the other numbers to ensure that its distribution adds up exactly to one; we can choose the greatest value for that (by trying each of the (at most) v states, which probability value is at least $1/v$), and its error with respect to the corresponding original computable number will be at most $(v-1) \cdot 2^{-(n+1)(v+1)}\varepsilon < 2^{-n(v+1)}\varepsilon$. This construction ensures that every created rational number is greater than $2^{-(n+1)(v+1)-1}\varepsilon > 2^{-\mathrm{poly}(b)}$ and have an "error" of at most $2^{-n(v+1)}\varepsilon$ to the original corresponding number.

Let $q_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)})$ and $q'_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)})$ denote, respectively, the parameters of $\mathcal{N}$ and $\mathcal{N}'$ (i.e. they are corresponding extreme distributions of the local credal sets $Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})$ in the two networks) such that $q'_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)})$ is the approximated version computed from $q_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)})$ as explained. Consider an assignment $\mathbf{x}$ to all variables in $\mathcal{N}$ (or in $\mathcal{N}'$). Let also p be an extreme of the strong extension of $\mathcal{N}$. Then p factorizes as $p(\mathbf{x}) = \prod_{i \in N} q_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)})$, for some combination of extreme distributions $q_i(\cdot|\mathbf{x}_{\mathrm{Pa}(i)})$ from $Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})$, $i \in N$. Finally, let p' be an extreme distribution in the strong extension of $\mathcal{N}'$ that satisfies $p'(\mathbf{x}) = \prod_{i \in N} q'_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)})$. By design, $|q'_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)}) - q_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)})| \leq 2^{-n(v+1)}\varepsilon$. It follows from the binomial expansion of the factorization of $p'(\mathbf{x})$ on any $\mathbf{x}$ that

$$\begin{aligned}
p'(\mathbf{x}) = \prod_{i \in N} q'_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)}) &\leq \prod_{i \in N} \left(2^{-n(v+1)}\varepsilon + q_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)})\right) \\
&= \sum_{A \subseteq N} (2^{-n-vn}\varepsilon)^{n-|A|} \prod_{i \in A} q_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)}) \\
&\leq 2^n 2^{-n-vn}\varepsilon + \prod_{i \in N} q_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)}) \\
&= p(\mathbf{x}) + 2^{-nv}\varepsilon\,.
\end{aligned}$$

The second inequality follows from the fact that there is one term for $p(\mathbf{x})$ in the expansion and $2^n - 1$ terms that can be written as a product of $2^{-n(v+1)}\varepsilon$ by nonnegative numbers less than or equal to one. With a similar reasoning, we can show that

$$p'(\mathbf{x}) \geq \prod_{i \in N} \left(q_i(x_i|\mathbf{x}_{\mathrm{Pa}(i)}) - 2^{-n(v+1)}\varepsilon\right) \geq p(\mathbf{x}) - 2^{-nv}\varepsilon\,.$$

Thus, $\max_{\mathbf{x}} |p'(\mathbf{x}) - p(\mathbf{x})| \leq 2^{-nv}\varepsilon$. Now consider a subset of the variables $\mathbf{X}_A$ and an assignment $\mathbf{x}_A \sim \mathbf{X}_A$. Since

$$\mathrm{P}_{p'}(\mathbf{X}_A = \mathbf{x}_A) = \sum_{\mathbf{x}' : \mathbf{x}'_A = \mathbf{x}_A} p'(\mathbf{x}')\,,$$

each term $p'(\mathbf{x}')$ in that sum satisfies $p'(\mathbf{x}') \leq p(\mathbf{x}') + 2^{-nv}\varepsilon$, and because there are less than $v^n \leq 2^{vn}$ terms being summed, it follows that

$$\mathrm{P}_{p'}(\mathbf{X}_A = \mathbf{x}_A) \leq \sum_{\mathbf{x}' : \mathbf{x}'_A = \mathbf{x}_A} \left(p(\mathbf{x}) + 2^{-vn}\varepsilon\right) \leq \mathrm{P}_p(\mathbf{X}_A = \mathbf{x}_A) + \varepsilon\,.$$

An analogous argument can be used to show that $\mathrm{P}_{p'}(\mathbf{X}_A = \mathbf{x}_A) \geq \mathrm{P}_p(\mathbf{X}_A = \mathbf{x}_A) - \varepsilon$. Note that the obtained mapping (p, p') is a surjection by construction. □

The above lemma has the following direct consequence on the computation of the STRONG-INF with polynomial-time computable numbers. The only restriction for its application is that the computable numbers must be either zero or greater than some ρ that is not "exponentially close to zero".

Corollary 1. *Consider a vertex-based credal network $\mathcal{N}$ whose numerical parameters are specified with polynomial-time computable numbers encoded by their respective machines (or directly given as rational numbers), such that no such number lies properly in $]0, \rho[$, for some $0 \leq \rho \leq 1$. Let b be the size of the encoding of the network. Given a subset of the nodes $N' \subseteq N$ of $\mathcal{N}$ and a rational number ε with $2^{-\text{poly}(b)} \leq \varepsilon < \rho$, we can construct in time* $\text{poly}(b)$ *a vertex-based credal network $\mathcal{N}'$ over the same variables whose numerical parameters defining credal sets related to nodes N' are all strictly positive rational numbers greater than* $2^{-\text{poly}(b)}$ *(numbers defining credal sets of nodes not in N' are kept unchanged), and such that* [6]

$$|\mathsf{STRONG\text{-}INF}(\mathcal{N}', t, x_t, O, \mathbf{x}_O) - \mathsf{STRONG\text{-}INF}(\mathcal{N}, t, x_t, O, \mathbf{x}_O)| \leq \varepsilon\,,$$

for any query $t, x_t, O, \mathbf{x}_O$ such that either $O = \emptyset$ or $\min_p \mathrm{P}_p(\mathbf{x}_O) > 0$ in $\mathcal{N}$.

Proof. According to Lemma 1, there is a polynomial-time computable network $\mathcal{N}'$ whose numerical parameters that specify the credal sets related to nodes N' are positive rational numbers and a polynomial-time computable surjection (p, p') such that p and p' are, respectively, extreme distributions of the strong extension of $\mathcal{N}$ and $\mathcal{N}'$, and satisfy $|\mathrm{P}_{p'}(\mathbf{x}_A) - \mathrm{P}_p(\mathbf{x}_A)| \leq \varepsilon^{n+1}/3$ for all $\mathbf{X}_A \subseteq \mathbf{X}$ and $\mathbf{x}_A \sim \mathbf{X}_A$. It follows that

$$\mathrm{P}_{p'}(x_t|\mathbf{x}_O) = \frac{\mathrm{P}_{p'}(x_t, \mathbf{x}_O)}{\mathrm{P}_{p'}(\mathbf{x}_O)} \geq \frac{\mathrm{P}_p(x_t, \mathbf{x}_O) - \varepsilon^{n+1}/3}{\mathrm{P}_p(\mathbf{x}_O) + \varepsilon^{n+1}/3}\,,$$

where p' is the image of p according to the surjection. If $\mathrm{P}_p(x_t, \mathbf{x}_O) = 0$, this equation is useless and vanishes. Otherwise, by Lemma 7 of Ref. [18], we have that

$$\frac{\mathrm{P}_p(x_t, \mathbf{x}_O) - \varepsilon^{n+1}/3}{\mathrm{P}_p(\mathbf{x}_O) + \varepsilon^{n+1}/3} \geq \mathrm{P}_p(x_t|\mathbf{x}_O) - \frac{2\varepsilon^{n+1}/3}{\rho^n} \geq \mathrm{P}_p(x_t|\mathbf{x}_O) - \varepsilon\,.$$

The other side of the inequality is obtained analoguously (using once more Lemma 7 of Ref. [18], except for the case of $\mathrm{P}_p(x_t, \mathbf{x}_O) = 0$, when the following reasoning is valid without the need of that lemma):

$$\begin{aligned}\mathrm{P}_{p'}(x_t|\mathbf{x}_O) &\leq \frac{\mathrm{P}_p(x_t, \mathbf{x}_O) + \varepsilon^{n+1}/3}{\mathrm{P}_p(\mathbf{x}_O) - \varepsilon^{n+1}/3} \leq \mathrm{P}_p(x_t|\mathbf{x}_O) + \frac{2\varepsilon^{n+1}/3}{\rho^n - \varepsilon^{n+1}/3} \\ &\leq \mathrm{P}_p(x_t|\mathbf{x}_O) + \frac{2\varepsilon}{3-\varepsilon} \leq \mathrm{P}_p(x_t|\mathbf{x}_O) + \varepsilon.\end{aligned}$$

[6] We abuse notation of STRONG-INF, as it is defined for opt $\in \{\min, \max\}$. We intend to mean that the equation is valid for both options of opt.

Hence, $|\mathrm{P}_{p'}(x_t|\mathbf{x}_O) - \mathrm{P}_p(x_t|\mathbf{x}_O)| \leq \varepsilon$. Let p be an extreme distribution in the strong extension of $\mathcal{N}$ such that $\mathrm{P}_p(x_t|\mathbf{x}_O) = \min_{q \in K_S} \mathrm{P}_q(x_t|\mathbf{x}_O)$, where K_S denotes the strong extension of $\mathcal{N}$. We have that

$$\min_{q \in K_S} \mathrm{P}_q(x_t|\mathbf{x}_O) = \mathrm{P}_p(x_t|\mathbf{x}_O) \geq \mathrm{P}_{p'}(x_t|\mathbf{x}_O) - \varepsilon \geq \min_{q' \in K'_S(X_t|\mathbf{x}_o)} \mathrm{P}_{q'}(x_t|\mathbf{x}_O) - \varepsilon\,,$$

where p' in the first inequality is the image of p according to the surjection, and K'_S in the last inequality is the strong extension of $\mathcal{N}'$. It follows from the above that

$$\min_{q' \in K'_S} \mathrm{P}_{q'}(x_t|\mathbf{x}_O) - \min_{q \in K_S} \mathrm{P}_q(x_t|\mathbf{x}_O) \leq \varepsilon\,.$$

The other side comes by contradiction. Suppose that

$$\min_{q \in K_S} \mathrm{P}_q(x_t|\mathbf{x}_O) - \min_{q' \in K'_S} \mathrm{P}_{q'}(x_t|\mathbf{x}_O) > \varepsilon \iff \min_{q \in K_S} \mathrm{P}_q(x_t|\mathbf{x}_O) - \varepsilon > \min_{q' \in K'_S} \mathrm{P}_{q'}(x_t|\mathbf{x}_O)\,.$$

Hence there would exist an extreme $q' \in K'_S$ such that $|\mathrm{P}_{q'}(x_t|\mathbf{x}_O) - \mathrm{P}_q(x_t|\mathbf{x}_O)| > \varepsilon$ for any $q \in K_S$, which is impossible because the mapping (q, q') is a surjection. An analogous proof works for showing that the upper bounds according to the two networks do not differ by more than ε. $\square$

We present the complexity results from the most complex models to the most simple ones. We start by stating the complexity of EPISTEMIC-INF in networks of unconstrained topology.

3.3 Precise-Vacuous Networks

To show that EPISTEMIC-INF is $\mathrm{NP}^{\mathrm{PP}}$-hard in arbitrary networks, we need the following result.

Proposition 2. *Consider a credal network whose root nodes are vacuous and non-root nodes are precise. Let t be a non-root node, $\tilde{x}_t$ an arbitrary value of X_t and $O = \emptyset$. Then STRONG-INF equals EPISTEMIC-INF.*

Proof. Let $\mathbf{X}_R$ be the vacuous variables associated to root nodes (hence to vacuous local credal sets), and $\mathbf{X}_I$ denote the remaining variables (which are precise). For every precise node i in I, let $q_i^{\mathbf{x}_{\mathrm{Pa}(i)}}(x_i)$ be the single distribution in the associated credal set $Q(X_i|\mathbf{x}_{\mathrm{Pa}(i)})$. Consider an arbitrary distribution p in the epistemic extension K_E, and let $<$ be a topological ordering of the nodes. For an assignment $\mathbf{x}$ to $\mathbf{X}$, we write $\mathbf{x}_{<i}$ to denote the coordinates $j < i$ of $\mathbf{x}$ according to the topological ordering. For every node i the set

$\{j \in N : j < i\}$ is a subset of $\mathrm{Nd}(i)$, and it follows from the definition of epistemic extension that $\mathrm{P}_p(x_i|\mathbf{x}_{<i}) = q_i^{\mathbf{x}_{\mathrm{Pa}(i)}}(x_i)$ for every precise node i and assignments x_i and $\mathbf{x}_{<i}$. By the Chain Rule we have that

$$\forall \mathbf{x} \sim \mathbf{X} : \quad \mathrm{P}_p(\mathbf{x}) = \mathrm{P}_p(\mathbf{x}_R) \prod_{i \in I} \mathrm{P}_p(x_i|\mathbf{x}_{<i}) = q(\mathbf{x}_R) \prod_{i \in I} q_i^{\mathbf{x}_{\mathrm{Pa}(i)}}(x_i)\,,$$

where q is any distribution of $\mathbf{X}_R$ (since these nodes are vacuous, any distribution satisfies the constraints in K_E for them). As stated, let $\tilde{x}_t$ be the value of interest of X_t. The result of the EPISTEMIC-INF is thus given by

$$\begin{aligned}
\min_{p \in \mathrm{K}_E} \mathrm{P}_p(X_t = \tilde{x}_t) &= \min_{q \in V(\mathbf{X}_R)} \sum_{\mathbf{x}} \delta_{\tilde{x}_t}(x_t) \cdot q(\mathbf{x}_R) \cdot \prod_{i \in I} q_i^{\mathbf{x}_{\mathrm{Pa}(i)}}(x_i) \\
&= \min_{q \in V(\mathbf{X}_R)} \sum_{\mathbf{x}_R} q(\mathbf{x}_R) \sum_{\mathbf{x}_I} \prod_{i \in I} q_i^{\mathbf{x}_{\mathrm{Pa}(i)}}(x_i) \delta_{\tilde{x}_t}(x_t) \\
&= \min_{q \in V(\mathbf{X}_R)} \sum_{\mathbf{x}_R} q(\mathbf{x}_R) \cdot g(\mathbf{x}_R)\,,
\end{aligned}$$

where $g(\mathbf{x}_R) \stackrel{\mathrm{def}}{=} \sum_{\mathbf{x}_I} \prod_{i \in I} \delta_{\tilde{x}_t}(x_t) q_i^{\mathbf{x}_{\mathrm{Pa}(i)}}(x_i)$. According to the last equality, the lower marginal probability of $X_t = \tilde{x}_t$ is a convex combination of $g(\mathbf{x}_R)$. Hence,

$$\min_{p \in \mathrm{K}_E} \mathrm{P}_p(X_t = \tilde{x}_t) \geq \min_{\mathbf{x}_R} \sum_{\mathbf{x}_{I \setminus \{q\}}} \prod_{i \in I} q_i^{\mathbf{x}_{\mathrm{Pa}(i)}}(x_i)\,.$$

The rightmost minimization is exactly the value of the lower marginal probability returned by STRONG-INF, and since the strong extension is contained in the epistemic extension, the inequality above is tight. An analogous result can be obtained for the upper probability by substituting minimizations with maximization and inverting the inequality above. □

The class of networks considered in the result above might seem restrictive at first sight. However, [7] showed that the STRONG-INF in any credal network of bounded treewidth whose local credal sets are represented in vertex-based form can be reduced in polynomial time to the same problem in a credal network whose non-root nodes are all precise and whose imprecise variables are all vacuous. The hardness of the EPISTEMIC-INF in such credal networks follows immediately from the result above, since the same is true for STRONG-INF, and one can efficiently reduce one problem to another, as the following corollary shows.

Corollary 2. *STRONG-INF and EPISTEMIC-INF are NP^{PP}-hard even if all variables are binary and all numerical parameters are strictly positive.*

Proof. [14] used a reduction from E-MAJSAT to STRONG-INF without evidence in a binary credal network whose root nodes are vacuous and non-root nodes are precise to show that such inference is NP^{PP}-hard. Since according to Proposition 2 the result of EPISTEMIC-INF is the same, EPISTEMIC-INF is also NP^{PP}-hard. In order to show that the result is valid also if all numerical parameters are strictly positive, we will only scketch the proof so as to avoid repeating all the formulation for the E-MAJSAT problem. Using Proposition 1 with epsilon $\varepsilon = 2^{-\text{poly}(b)}$ smaller than the precision of any number involved in any calculation, we build a new network where all numerical parameters are strictly positive and the variation in the result of STRONG-INF is negligible such that it can still decide E-MAJSAT (further details on how small ε has to be are omitted for simplicity, but the gap between instances is large enough that $2^{-O((n+m)^2)}$ will suffice, where n, m are the number of variables and clauses in the specification of E-MAJSAT, see the proof of Theorem 2 in [41]). Because EPISTEMIC-INF contains STRONG-INF, its result after applying Proposition 1 will be between the results of STRONG-INF in the new network and in the old network (the latter equals that of EPISTEMIC-INF). Hence, EPISTEMIC-INF in the new network with strictly positive numerical parameters also solves E-MAJSAT. □

Note that the result holds irrespective of how the local credal sets are represented, since vacuous and precise nodes can be mapped from constraint-based to vertex-based form and vice-versa in polynomial time.

3.4 Singly Connected Networks

We now turn our attention to singly connected networks. A first result is a direct consequence of Proposition 2 is the NP-hardness of the EPISTEMIC-INF in singly connected credal networks, since STRONG-INF is NP-hard in singly connected networks, even if we admit imprecision only on root nodes:

Corollary 3. *EPISTEMIC-INF is NP-hard in singly connected credal networks.*

Proof. [17] showed that STRONG-INF is NP-hard in singly connected networks. Since Proposition 2 shows that EPISTEMIC-INF can be reduced to STRONG-INF on the same input, the result follows. □

The proof of NP-hardness of STRONG-INF provided by [17] requires the variable domain cardinalities to be unbounded. We present here the stronger result of NP-hardness of credal inference in networks where imprecise variables are binary and precise ones are at most ternary. We can now show

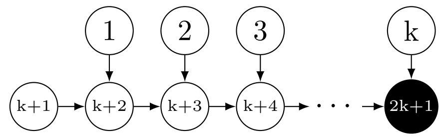


Figure 3: Credal network structure used to prove Theorem 1. The shaded node is the target.

Table 2: Local probability distributions used to prove Theorem 1

$q(x_i \mid x_{i-1}, x_{i-k-1})$	$x_i = 1$	$x_i = 2$	$x_i = 3$
$x_{i-1} = 1, x_{i-k-1} = 1$	2^{-v_i}	0	$1 - 2^{-v_i}$
$x_{i-1} = 2, x_{i-k-1} = 1$	0	1	0
$x_{i-1} = 3, x_{i-k-1} = 1$	0	0	1
$x_{i-1} = 1, x_{i-k-1} = 0$	1	0	0
$x_{i-1} = 2, x_{i-k-1} = 0$	0	2^{-v_i}	$1 - 2^{-v_i}$
$x_{i-1} = 3, x_{i-k-1} = 0$	0	0	1

NP-hardness of credal inference in singly connected networks with bounded variable cardinality.

Theorem 1. *STRONG-INF and EPISTEMIC-INF are NP-hard even if the network is singly connected and has treewidth at most two, all imprecise variables are binary, and all precise variables are (at most) ternary. Moreover, all numerical parameters in the network are strictly positive.*

Proof. We defer the treatment of zero numerical parameters to the final part. We build a singly connected credal network with underlying graph as in Figure 3. The variables (associated to nodes) on the upper row are binary and vacuous, namely $X_1, \ldots, X_k$, while the remaining variables $X_{k+1}, \cdots, X_{2k+1}$ are ternary and precise. The local credal sets associated to precise nodes are singletons such that $Q(X_{k+1})$ contains a uniform distribution $q(x_{k+1}) = 1/3$, and, for $i = k+2, \ldots, 2k+1$, $Q(X_i \mid x_{i-1}, x_{i-k-1})$ contains the conditional distribution $q(X_i \mid x_{i-1}, x_{i-k-1})$ specified in Table 2. The rational numbers v_i in the table shall be defined later on. Consider an extreme distribution $p(\mathbf{x})$ of the strong extension of the network. It follows for all $\mathbf{x}$ that

$$p(\mathbf{x}) = q(x_{k+1}) \prod_{i=k+2}^{2k+1} q(x_i \mid x_{i-1}, x_{i-k-1}) \prod_{i \in A} \delta_1(x_i) \prod_{i \notin A} \delta_0(x_i)\,,$$

for some $A \subseteq [k]$. This is because the extreme distributions of local vacuous sets over binary variables are δ_0 and δ_1, and each choice of a local extreme for a root node can be associated to a choice of either including or excluding its corresponding node in/from A. Let $\neg A \stackrel{\text{def}}{=} [k] \setminus A$ denote the complement of a set A with respect to $[k]$. We have that

$$p(\mathbf{x}) = \begin{cases} \frac{1}{3}\prod_{i=k+2}^{2k+1} q(x_i|x_{i-1}, x_{i-k-1}), & \text{if } \mathbf{x}_A = 1 \text{ and } \mathbf{x}_{\neg A} = 0\,; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\mathrm{P}_p(X_{2k+1}=1) = \sum_{\mathbf{x}} p(\mathbf{x})\delta_1(x_{2k+1}) = \frac{2^{-\sum_{i\in A} v_i}}{3} \text{ and } \mathrm{P}_p(X_{2k+1}=2) = \frac{2^{\sum_{i\in A} v_i - 2}}{3}\,.$$

We show the NP-hardness of credal inference by reducing the NP-complete PARTITION problem [26] to the computation of $\max_{p\in K_S} \mathrm{P}_p(X_{2n+1} = 3)$. We define PARTITION as follows.

PARTITION
Input: List of positive integers $z_1, \ldots, z_k$.
Output: Is there a subset $A \subseteq [k]$ such that

$$\sum_{i\in A} z_i = \sum_{i\in\neg A} z_i \quad ?$$

Notice that the above equality is equivalent to

$$\sum_{i\in A} z_i/z = 1\,, \quad \text{where } z = \frac{1}{2}\sum_{i=1}^{k} z_i\,.$$

Define the exponents in Table 2 as $v_i \stackrel{\text{def}}{=} z_i/z$, and let $v_A \stackrel{\text{def}}{=} \sum_{i\in A} v_i$. It follows for any A that $v_A = 2 - \sum_{i\in\neg A} v_i$. If an instance of the PARTITION is a *yes-instance* (i.e., if the output of PARTITION is yes), then there is A for which $v_A = 1$, whereas if it is a *no-instance* (i.e., if the output is no), then for any A, it follows that $|v_A - 1| \geq 1/(2z)$ because the numbers in the input are integers and hence the sums of two different sets are either equal or differ by at least one. Consider the function

$$h(v_A) = \frac{2^{-(v_A-1)} + 2^{v_A-1}}{2}\,.$$

The graph of the function is depicted in Figure 4. Seen as a function

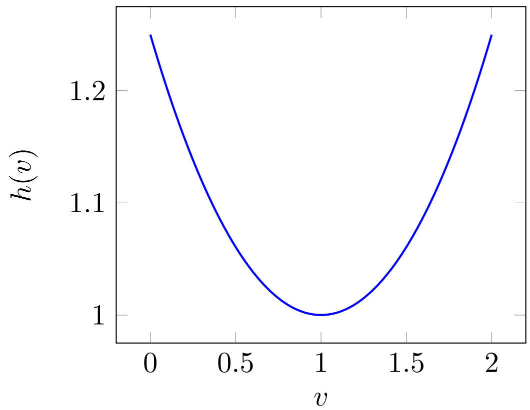


Figure 4: Function used in the reduction in the proof of Theorem 1.

of a continuous variable $v_A \in [0, 2]$, the function above is strictly convex, symmetric around one, and achieves the minimum value of one at $v_A = 1$. Thus, if PARTITION returns yes then $\min_A h(v_A) = 1$, while if it returns no we have that

$$\min_A h(v_A) \geq 2^{-1/(2z)-1} + 2^{1/(2z)-1} \geq 2^{(2z)^{-4}} > 1 + (2z)^{-4}/2 = 1 + 1/(32z^4)\,,$$

where the second inequality is due to Lemma 24 in Ref. [37], and the strict inequality follows from the first-order Taylor expansion of $2^{(2z)^{-4}}$. Let $\alpha \overset{\text{def}}{=} (1 + z^{-4}/64)/3$. By computing STRONG-INF with query $X_{2n+1}=3$ and no evidence, we can decide PARTITION, as

$$\begin{aligned} 1 - \max \mathrm{P}_p(X_{2k+1}=3) &= \min_p \left(\mathrm{P}_p(X_{2k+1}=1) + \mathrm{P}_p(X_{2k+1}=2)\right) \\ &= \min_A \frac{h(v_A)}{3} \leq \alpha\,, \end{aligned}$$

if and only if the result of PARTITION is yes. It remains to show that we can polynomially encode the numbers $2^{-z_i/z}$. This is done by applying Lemma 1 with a small enough ε computable in time polynomial in the size of the partition problem: $\varepsilon = 1/(3 \cdot 64z^4)$ suffices. Note that if we only apply Lemma 1 to the non-root nodes and leave the root nodes untouched as vacuous, then according to Lemma 2, the outcome of EPISTEMIC-INF is the same, proving also its NP-hardness. By applying Lemma 1 to all the nodes, we ensure all numerical parameters are strictly positive and still yield a result that can be used to decide PARTITION. Because EPISTEMIC-INF contains STRONG-INF, its result after applying Proposition 1 but keeping root nodes

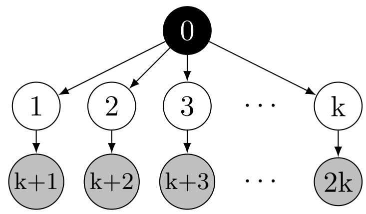


Figure 5: ADG of the credal tree used to prove Theorem 2.

vacuous (such that we know EPISTEMIC-INF equals STRONG-INF) will be between the results of STRONG-INF in the new network and in that network with vacuous root nodes. Hence, EPISTEMIC-INF in the new network with strictly positive numerical parameters also solves PARTITION. $\square$

3.5 Credal Trees

The previous complexity results showed that, from a theoretical standpoint, computing the EPISTEMIC-INF is just as difficult as solving STRONG-INF. When the underlying graph is a tree, [22] showed that EPISTEMIC-INF can be computed efficiently, and it was previously unknown whether a similar result could be obtained for STRONG-INF. The next result shows that in this case the equivalence on the tractability under the two different irrelevance concepts does not hold unless P equals NP.

Theorem 2. *STRONG-INF in tree-shaped credal networks is NP-hard, even if only one variable is ternary and precise and all the rest are binary, and even if all numerical parameters are strictly positive.*

Proof. We show hardness by a reduction from PARTITION as defined previously. As before, we define $v_i \overset{\text{def}}{=} z_i/z$, and $v_A \overset{\text{def}}{=} \sum_{i \in A} v_i$, and note that $v_A = 2 - \sum_{i \in \neg A} v_i$. We also let $h(v_A)$ to be as before (thus h is strictly convex on $[0, 2]$, symmetric around one, and achieves the minimum value of one at $v_A = 1$). Given an instance of PARTITION (i.e., a list of integers), we build a credal tree $\mathcal{N}$ over variables $X_0, \ldots, X_{2k}$ with ADG as in Figure 5. The root variable X_0 takes values in $\{1, 2, 3\}$, and is precise and uniformly distributed (i.e., its local credal set contains only the distribution $p_0(x_0) = 1/3$). The remaining variables are all binary and take values in $\{0, 1\}$. For $i = 1, \ldots, k$, we specify the local conditional credal sets $Q(X_i|x_0)$

as singletons $\{p_i^{x_0}\}$ such that

$$p_i^{x_0}(1) = \begin{cases} 2^{-v_i}/(1+2^{-v_i}), & \text{if } x_0 = 1, \\ 1/(1+2^{-v_i}), & \text{if } x_0 = 2, \\ 1/2, & \text{if } x_0 = 3. \end{cases}$$

For $i = 1+k, \dots, 2k$ we specify the local credal sets $Q(X_i|x_{i-k}) = \{p \in V(X_i) : \epsilon \leq p(1) \leq 1\}$, where $\epsilon = 2^{-k-3}/(64z^4)$. Each of these local credal sets can be represented either in vertex-based form by two extreme distributions or by a couple of constraints.

Let

$$\mu \stackrel{\text{def}}{=} \max_{p \in K_S} \mathrm{P}_p(X_0 = 3 | \mathbf{X}_O = \mathbf{x}_O) = \max_{p \in K_S} \frac{\mathrm{P}_p(X_0 = 3, \mathbf{X}_O = \mathbf{x}_O)}{\mathrm{P}_p(\mathbf{X}_O = \mathbf{x}_O)} \,.$$

Hence, μ is the solution of

$$\max_{p \in K_S} \left[\sum_{x_0} (\delta_3(x_0) - \mu) \mathrm{P}_p(X_0 = x_0, \mathbf{X}_O = \mathbf{x}_O) \right] = 0 \,.$$

By definition, any extreme distribution p in the strong extension K_S satisfies for $\mathbf{x} \sim \mathbf{X}$ such that $x_{k+1} = x_{k+2} = \cdots = x_{2k} = 1$ the equality

$$p(\mathbf{x}) = p_0(x_0) \prod_{i \in [k]} p_i^{x_0}(x_i) \alpha_i^{x_i} \,,$$

where each $\alpha_i^{x_i}$ is a number in $[\epsilon, 1]$. Let $O = \{k+1, \dots, 2k\}$ and $\mathbf{x}_O = (1, \dots, 1)$. It follows that μ is the solution of

$$\max \sum_{x_0, \dots, x_k} (\delta_3(x_0) - \mu) p_0(x_0) \prod_{i \in [k]} p_i^{x_0}(x_i) \alpha_i^{x_i} = 0 \,,$$

where the maximization is performed on α_i^0, α_i^1, for $i = 1, \dots, k$. Consider $j \in [k]$ and let

$$\beta_j^{x_j} \stackrel{\text{def}}{=} \sum_{x_0, \dots, x_{j-1}} \sum_{x_{j+1}, \dots, x_k} (\delta_3(x_0) - \mu)\, p_0(x_0) p_j^{x_0}(x_j) \prod_{i \in [k], i \neq j} [p_i^{x_0}(x_i) \alpha_i^{x_i}] \,.$$

Then,

$$\max \sum_{x_0, \dots, x_k} (\delta_3(x_0) - \mu)\, p_0(x_0) \prod_{i \in [k]} [p_i^{x_0}(x_i) \alpha_i^{x_i}] = \max \left(\alpha_j^0 \beta_j^0 + \alpha_j^1 \beta_j^1 \right) \,.$$

Since α_j^0 and α_j^1 are both positive, the maximization on the right-hand side above equals zero only if both β_j^0 and β_j^1 are zero or they have different signs. In the former case, any value of α_j^0 and α_j^1 maximizes the expression, and we can assume that (α_j^0, α_j^1) equals $(\epsilon, 1)$ or $(1, \epsilon)$. In the latter case, $\beta_j^0 < \beta_j^1$ implies that (α_j^0, α_j^1) equals $(\epsilon, 1)$ in order to maximize the expression, and $(\alpha_j^0, \alpha_j^1) = (1, \epsilon)$ would do it otherwise. Since we selected j arbitrarily, the result holds for all j. Thus, the maximization is equivalent to selecting, for $i = 1, \ldots, k$, a value y_i in $\{0, 1\}$ such that $\alpha_i^0 = \epsilon^{1-y_i}$ and $\alpha_i^1 = \epsilon^{y_i}$. It follows that

$$\max \sum_{x_0,\ldots,x_k} (\delta_3(x_0) - \mu)\, p_0(x_0) \prod_{i\in[k]} [p_i^{x_0}(x_i)\alpha_i^{x_i}] =$$

$$\max_{\mathbf{y}\in\{0,1\}^k} \sum_{x_0,\ldots,x_k} (\delta_3(x_0) - \mu)\, p_0(x_0) \prod_{i\in[k]} [p_i^{x_0}(x_i)\epsilon^{(1-x_i)(1-y_i)}\epsilon^{x_i y_i}]\,.$$

By rearranging terms, we obtain

$$\max_{\mathbf{y}\in\{0,1\}^k} \sum_{x_0} (\delta_3(x_0) - \mu)\, p_0(x_0) \prod_{i\in[k]} \left[p_i^{x_0}(0)\epsilon^{1-y_i} + p_i^{x_0}(1)\epsilon^{y_i}\right] ,$$

which by design equals

$$\max_{\mathbf{y}\in\{0,1\}^k} -\left(\frac{\mu}{3}\gamma \prod_{i\in[k]} [\epsilon^{1-y_i}+2^{-v_i}\epsilon^{y_i}]+\frac{\mu}{3}\gamma \prod_{i\in[k]} [2^{-v_i}\epsilon^{1-y_i}+\epsilon^{y_i}]+\frac{\mu-1}{3} \prod_{i\in[k]} \frac{1+\epsilon}{2}\right),$$

where $\gamma = \prod_{i\in[k]} p_i^2(1)$ (recall that $p_i^2(1)$ is the probability value of $X_i = 1|X_0 = 2$). The binary vector $\mathbf{y}$ can be seen as the characteristic vector of a subset $A \subset [k]$. Define

$$b_A \stackrel{\text{def}}{=} \prod_{i\in A}(2^{-v_i} + \epsilon) \prod_{i\in\neg A} (1 + 2^{-v_i}\epsilon)$$

for every subset A of $[k]$. The optimization on $\mathbf{y}$ can be rewritten as the following optimization over subsets A: find μ such that

$$\left(-\frac{\mu-1}{3}\left(\frac{1+\epsilon}{2}\right)^k + \max_A -\frac{\mu}{3}\gamma\,(b_A + b_{\neg A})\right) = 0\,.$$

Solving the expression above for μ, we get to

$$\mu = \left(1 + \left(\frac{2}{1+\epsilon}\right)^k \gamma \min_A\,(b_A + b_{\neg A})\right)^{-1}.$$

Define the function $g(a)$ as

$$g(a) \stackrel{\text{def}}{=} 1 + \left(\frac{2}{1+\epsilon}\right)^k \gamma(1+a)$$

for any real number a, and let $a_A \stackrel{\text{def}}{=} b_A + b_{\neg A} - 1$ for any $A \subset [k]$. Now $\mu = -(\min_A g(a_A))^{-1}$. Note that $g(a_A) > 1 + (1+a_A)2^{-k}$, because $\gamma > 2^{-k}$ (this will be used later). It follows from the Binomial Theorem that the value of b_A is very close to the value of 2^{-v_A} from above.

$$\begin{aligned} 2^{-v_A} \le b_A &\le (2^{-v_A} + 2^k\epsilon)(1+\epsilon)^k \\ &\le (2^{-v_A} + 2^k\epsilon)(1+2k\epsilon) \\ &\le 2^{-v_A} + 2^{k+2}\epsilon\,, \end{aligned}$$

where we have used the inequality $(1+r/c)^c \le 1 + 2r$ valid for $r \in [0,1]$ and positive integer c [36, Lemma 37]. Thus we conclude that the value a_A is very close (again from above) to $h(v_A) - 1$.

$$h(v_A) - 1 \le a_A \le h(v_A) - 1 + 2^{k+3}\epsilon = h(v_A) - 1 + 1/(64z^4)\,.$$

Now if the partition problem is a yes-instance, then $h(v_A) = 1$ (recall the behavior of h from the proof of Theorem 3) and thus $a_A \le 1/(64z^4)$, while if it is a no-instance, then $h(v_A) > 1 + 1/(32z^4)$ and thus $a_A > 1/(32z^4)$. Hence, there is a gap of at least $1/(64z^4)$ in the value of a_A between yes- and no-instances, and we can decide the partition problem by verifying whether $\mu \le -g(3/(128z^4))^{-1}$. This proof shall be completed with the guarantee that we can approximate in polynomial time the irrational numbers used to specify the credal tree and $g(a)$ well enough so that $-g(3/(128z^4))^{-1}$ falls exactly in the middle of the gap between the values of μ for yes- and no-instances (because g is linear in a). First, note that

$$g\left(\frac{1}{32z^4}\right) - g\left(\frac{1}{64z^4}\right) = \frac{1}{64z^4}\left(\frac{2}{1+\epsilon}\right)^k \gamma\,,$$

which is greater than $2^{-k}/(64z^4)$ (since $\gamma > 2^{-k}$). The gap in the value of μ is at least

$$\begin{aligned} \frac{1}{g(1/(64z^4))} - \frac{1}{g(1/(32z^4))} &= \frac{g(1/(32z^4)) - g(1/(64z^4))}{g(1/(64z^4))g(1/(32z^4))} \\ &> \frac{g(1/(32z^4)) - g(1/(64z^4))}{g(1/(32z^4))^2} \\ &> \frac{2^{-k}/(64z^4)}{(1 + (1+\frac{1}{32z^4})2^{-k})^2} > \frac{2^{-k}}{4 \cdot 64z^4}\,. \end{aligned}$$

So we apply Corollary 1 with $\varepsilon = \frac{1}{2}\frac{2^{-k}}{4 \cdot 64 z^4}$ to obtain from $\mathcal{N}$ a network $\mathcal{N}'$ made only of positive rational numbers. Such ε guarantees that the separation between yes-instances and no-instances of PARTITION will continue to exist. □

The credal network used in the reduction that proves the previous result is in a sense the simplest structure on which solving STRONG-INF is hard, since the problem would be polynomial-time solvable if the root node were replaced with a binary variable. It is also interesting as it describes a naive Bayes structure with a single layer of latent variables, a useful topology for robust classification problems on non-linearly separable feature spaces.

3.6 Imprecise hidden Markov models

An imprecise hidden Markov model (HMM) is a credal tree whose nodes can be partitioned into *hidden* and *manifest* nodes such that the hidden nodes form a chain (i.e., a sequence of nodes with one node linking to the next and to no other in the sequence), and manifest nodes are leaves of the graph. HMMs are widely used to represent discrete dynamic systems whose output at any given time step can be stochastically determined by the current state of the system, which is assumed to be only partially observable.

Since an HMM is simply a credal tree, the algorithm of [22] can be used to efficiently compute EPISTEMIC-INF in HMMs, while 2U can be used to solve STRONG-INF if all variables all binary. For networks with variables taking on more than two values, no polynomial-time is known for STRONG-INF. In this section, we show that when there is no evidence variables farther (in the sense of number of nodes in the path) from the root node than the queried variable, the outcomes of the STRONG-INF and EPISTEMIC-INF coincide. On these cases, we can run the [22] algorithm to compute STRONG-IN in polynomial time. This is however not always true, that is, there are types of queries in which the results of STRONG-INF and EPISTEMIC-INF differ, as the following example shows.

Example 5. *Consider an HMM of length two whose topology is depicted in Figure 6. All variables are binary and take values in* $\{0, 1\}$*. Variables* X_1 *and* X_2 *are hidden, while variables* X_3 *and* X_4 *are manifest. The local credal sets are given by* $Q(X_1) = Q(X_2|0) = Q(X_4|0) = \{p \in V(X_4) : p(1) = 1/4\}$*,* $Q(X_2|1) = Q(X_4|1) = \{p \in V(X_4) : p(1) = 3/4\}$*, and* $Q(X_3|0) = \{p \in V(X_3) : 1/2 \leq p(1) \leq 3/4\}$ *and* $Q(X_3|1) = \{p \in V(X_3) : 1/4 \leq p(1) \leq 1/2\}$*. Thus, variable* X_3 *is imprecise, and the remaining variables are precise.*

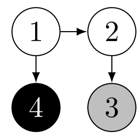


Figure 6: Credal HMM in Example 5.

Consider a query with target $X_4=0$ and evidence $X_3=0$. The lower bound of STRONG-INF *is the value of μ that solves the equation*

$$\min \sum_{x_2} p_3^{x_2}(0) g_\mu(x_2) = \sum_{x_2} \min p_3^{x_2}(0) g_\mu(x_2) = 0 \,,$$

where the minimizations are performed over $p_3^{x_2} \in M(X_3|x_2)$, $x_2 = 0, 1$, and

$$g_\mu(x_2) \stackrel{def}{=} \sum_{x_1,x_4} (\delta_0(x_4) - \mu)\, p_1(x_1) p_2^{x_1}(x_2) p_4^{x_1}(x_4) \,,$$

with $p_1 = p_2^0 = p_4^0 = (3/4, 1/4)$ and $p_2^1 = p_4^1 = (1/4, 3/4)$. The values of $p_3^{x_2}(0)$ depend only on the signs of $g_\mu(0)$ and $g_\mu(1)$, which ought to be different for the expression to vanish. Solving for μ for each of the four possibilities, and taking the minimum value of μ, we find that $\mu = 4/7 > 1/2$.

The lower bound of EPISTEMIC-INF *is the value of μ that solves*

$$\begin{aligned} \min \sum_{x_1,x_2,x_4} & p_1(x_1) p_2^{x_1}(x_2) p_4^{x_1}(x_4) p_{x_1,x_2,x_4}(0) h_\mu(x_4) = \\ & (1-\mu) \sum_{x_1,x_2} p_1(x_1) p_2^{x_1}(x_2) p_4^{x_1}(0) \min p_{x_1,x_2,0}(0) \\ & \quad - \mu \sum_{x_1,x_2} p_1(x_1) p_2^{x_1}(x_2) p_4^{x_1}(1) \max p_{x_1,x_2,1}(0) = 0 \,, \end{aligned}$$

where $h_\mu(x_4) = \delta_0(x_4) - \mu$, p_1, $p_2^{x_1}$ and $p_4^{x_1}$ are defined as before, and $p_{x_1,x_2,x_4} \in Q(X_3|x_2)$ for every x_1, x_2, x_4. Solving the equation above for μ we obtain $\mu = 13/28 < 1/2$. □

The above example shows that STRONG-INF and EPISTEMIC-INF might differ, even in the simple case of HMMs with binary variables. It is currently unknown whether this type of inference is hard for STRONG-INF. The following result shows that at least for a particular case, the computations of the STRONG-INF and EPISTEMIC-INF in HMMs coincide.

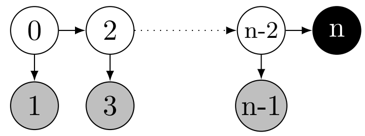


Figure 7: ADG of the HMM considered in Theorem 3.

Theorem 3. *Consider a separately specified HMM over variables $X_0, \dots, X_n$. The variables associated to odd numbers are manifest, and the remaining variables are hidden (see Figure 7). Consider also a target hidden node $X_n = \tilde{x}_n$, and evidence $\mathbf{X}_O = \tilde{\mathbf{x}}_O$ on a subset O of the manifest nodes. Then the outcomes of STRONG-INF and EPISTEMIC-INF are the same.*

Proof. Define $f_\mu(x_n) \stackrel{\text{def}}{=} \delta_{\tilde{x}_n}(x_n) - \mu$ for any given μ, and consider the distribution p in the epistemic extension K_E that minimizes

$$\sum_{\mathbf{x} \sim \mathbf{X} : \mathbf{x}_O = \tilde{\mathbf{x}}_O} f_\mu(x_n) p(\mathbf{x}) \,.$$

Let $<$ be any topological ordering of the nodes. By the Chain Rule, we have for all $\mathbf{x}$ that p factorizes as $p(\mathbf{x}) = \mathrm{P}_p(x_0) \prod_{i \in [n]} \mathrm{P}_p(x_i | \mathbf{x}_{<i})$, where $\mathbf{x}_{<i}$ denotes the coordinates x_j of $\mathbf{x}$ with $j < i$ according to the topological ordering (we also write $\mathbf{x}_{\geq i}$ and $\mathbf{x}_{>i}$ to denote analogous projections). Assume that for some nonnegative integer i less than or equal to n it holds that

$$\sum_{\mathbf{x} : \mathbf{x}_O = \tilde{\mathbf{x}}_O} f_\mu(x_n) p(\mathbf{x}) \geq \sum_{\mathbf{x} : \mathbf{x}_O = \tilde{\mathbf{x}}_O} f_\mu(x_n) \prod_{j < i} \mathrm{P}_p(x_j | \mathbf{x}_{<j}) \prod_{j \geq i} p_j^{\mathbf{x}_{\mathrm{Pa}(j)}}(x_j) \,,$$

where each $p_j^{\mathbf{x}_{\mathrm{Pa}(j)}}$ is recursively defined as the extreme distribution of the local credal set $Q(X_j | \mathbf{x}_{\mathrm{Pa}(j)})$ that minimizes either

$$\sum_{x_j} p_j^{\mathbf{x}_{\mathrm{Pa}(j)}}(x_j) \sum_{\mathbf{x}_{>j}} f_\mu(x_n) \prod_{k > j} p_k^{\mathbf{x}_{\mathrm{Pa}(k)}}(x_k) \,,$$

if j is not in O, or

$$p_j^{x_{\mathrm{Pa}(j)}}(\tilde{x}_j) \sum_{\mathbf{x}_{>j}} f_\mu(x_n) \prod_{k > j} p_k^{\mathbf{x}_{\mathrm{Pa}(k)}}(x_k) \,,$$

if j is in O, where $\tilde{x}_j$ is the value of X_j compatible with $\tilde{\mathbf{x}}_O$. We will show by induction in $i = n, \dots, 0$ that the assumption is true. If $i - 1$ is not in O

then

$$\sum_{\mathbf{x}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} f_\mu(x_n) \prod_{j<i} \mathrm{P}_p(x_j|\mathbf{x}_{<j}) \prod_{j\geq i} p_j^{\mathbf{x}_{\mathrm{Pa}(j)}}(x_j) =$$

$$\sum_{\mathbf{x}_{<i-1}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} \prod_{j<i-1} \mathrm{P}_p(x_j|\mathbf{x}_{<j}) \sum_{x_{i-1}} \mathrm{P}_p(x_{i-1}|\mathbf{x}_{<i-1}) \sum_{\mathbf{x}_{\geq i}} f_\mu(x_n) \prod_{k\geq i} p_k^{\mathbf{x}_{\mathrm{Pa}(k)}}(x_k) \geq$$

$$\sum_{\mathbf{x}_{<i-1}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} \prod_{j<i-1} \mathrm{P}_p(x_j|\mathbf{x}_{<j}) \min_{q\in Q(X_{i-1}|\mathbf{x}_{\mathrm{Pa}(i-1)})} \sum_{x_{i-1}} q(x_{i-1}) \sum_{\mathbf{x}_{\geq i}} f_\mu(x_n) \prod_{k>i} p_k^{\mathbf{x}_{\mathrm{Pa}(k)}}(x_k) =$$

$$\sum_{\mathbf{x}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} f_\mu(x_n) \prod_{j<i-1} \mathrm{P}_p(x_j|\mathbf{x}_{<j}) \prod_{j\geq i-1} p_j^{\mathbf{x}_{\mathrm{Pa}(j)}}(x_j)\,,$$

where the inequality follows from the definition of epistemic extension, which implies that $\sum_{x_i} h(x_i)\mathrm{P}_p(x_i|\mathbf{x}_{\mathrm{Nd}(i)}) \geq \min_{q\in Q(X_i|x_{\mathrm{Pa}(i)})} \sum_{x_i} h(x_i)q(x_i)$ for any function h on x_i (note that $\mathrm{Nd}(i) \supseteq \{j < i\}$, and that the minimization on the right is constant w.r.t. values $\mathbf{x}_{j<i:j\notin \mathrm{Pa}(i)}$). The case of a node i in O is analogous with the sum substituted by a single term. For $i = n$, it follows that

$$\begin{aligned}\sum_{\mathbf{x}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} f_\mu(x_n)p(\mathbf{x}) &= \sum_{\mathbf{x}_{j<n}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} \prod_{j\leq n} \mathrm{P}_p(x_j|\mathbf{x}_{<j}) \sum_{x_t} f_\mu(x_n)\mathrm{P}_p(x_t|\mathbf{x}_{<n}) \\ &\geq \sum_{\mathbf{x}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} f_\mu(x_n)p_n^{x_{n-2}}(x_n) \prod_{j<n} \mathrm{P}_p(x_j|\mathbf{x}_{<j})\,,\end{aligned}$$

so that the basis of the induction holds. For $i = 0$, we have that

$$\sum_{\mathbf{x}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} f_\mu(x_n)p(\mathbf{x}) \geq \sum_{\mathbf{x}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} f_\mu(x_n)p_0(x_0) \prod_{i\in[n]} p_j^{x_{\mathrm{Pa}(j)}}(x_j)\,,$$

which is the lower bound of STRONG-INF. Thus, since the epistemic extension contains the strong extension, the inequality above is tight. In particular, the equality holds if μ is the lower bound of EPISTEMIC-INF, and it follows that

$$\min_{p\in K_S} \sum_{\mathbf{x}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} f_\mu(x)p(\mathbf{x}) = \min_{p\in K_E} \sum_{\mathbf{x}:\mathbf{x}_O=\tilde{\mathbf{x}}_O} f_\mu(x)p(\mathbf{x}) = 0\,,$$

where K_S denotes the strong extension. An analogous proof shows that also the upper bounds coincide. □

The previous result shows that at least for the particular case where one seeks the probability of "last" variable, STRONG-INF can be computed in polynomial time. Although restrictive, this type of inference is highly relevant, as it corresponds to predicting the future state of a partially observable dynamic system whose future state depends in some level only on its current (unknown) state. There is also another type of inference in trees which is insensitive to the irrelevance concept adopted, which is the case of marginal inferences:

Corollary 4. *Consider a tree-shaped network $\mathcal{N}$ and a target $X_t = x_t$. Then*

$$\mathit{STRONG\text{-}INF}(\mathcal{N}, t, x_t, \emptyset, \cdot) = \mathit{EPISTEMIC\text{-}INF}(\mathcal{N}, t, x_t, \emptyset, \cdot)\,.$$

Proof. We say that a node is *barren* if it is not an ancestor of any target or evidence node. It is well-known that removing barren nodes from a Bayesian network does not affect the outcome of BN-INF [30]. Since inference under strong independence can be seen as (exponentiatly many) inferences in Bayesian networks, the result of STRONG-INF is also unaltered if remove barren nodes. Moreover, since $\mathcal{N}$ is a tree, removing barren nodes leaves with a chain of ancestors of t. According to Theorem 15 of Ref. [12], the epistemic extension of $\mathcal{N}$ projected on the ancestors of t, that is, the set of marginal distributions on $\mathbf{x}_A \sim \mathbf{X}_A$ induced from joint distributions in the epistemic extension, where A denotes the ancestors of t, is the epistemic extension of the network we get by removing nodes not in A. This implies that barren nodes can be discarded also in inferences under epistemic irrelevance, and the result follows. □

[55] developed a linear-time to compute marginal inferences under strong independence in trees as a by-product of their work on imprecise tree-augmented naive Bayes classifiers. The result above shows that the same algorithm can be used to compute marginal inferences in trees under epistemic irrelevance; conversely, [22] algorithm for epistemic trees can be used to compute marginal inferences in strong trees.

3.7 Imprecise Markov chains

The simplest ADG structure forming a connected graph is that of a chain, that is, of a network in which each variable has at most one parent and one child. Credal chains are more usually known as (imprecise) Markov chains. As a chain is also a tree, computing EPISTEMIC-INF can be done in polynomial time; this is also the case for STRONG-INF on chains of binary variables, as this is a subcase of binary polytrees. A chain can be seen as

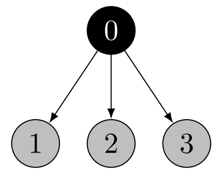


Figure 8: ADG of a Naive Bayes with 3 feature variables.

an HMM where the values of the manifest variables are deterministically determined by the values of the hidden variables. As such, the equivalence of both types of inference in certain types of HMMs extends to chains:

Corollary 5. *Consider a credal chain $X_0 \to \cdots \to X_n$, a target $X_n = x_n$ on the single leaf variable of a separately specified (imprecise) Markov chain, and some evidence $\mathbf{X}_O = \tilde{\mathbf{x}}_O$ on arbitrary non-leaf variables. Then the outcomes of STRONG-INF and EPISTEMIC-INF coincide.*

Proof. The same proof of Theorem 3 applies here, if we omit manifest nodes. □

3.8 Imprecise Naive Bayes

A widely used ADG structure is the Naive, where a node (usually called *class*) has all other nodes (called *features*) as its children, and no other arc is present. Figure 8 depicts a Naive structure with class variable X_0 and features X_1, X_2 and X_3. Such ADG constitute the structure behind the Naive Bayes and the Naive Credal classifiers [53]. As it is a tree, computing EPISTEMIC-INF can be done in polynomial time; this is also the case for STRONG-INF when the target node is the class [53]. We show next that this similar tractability is not coincidental: both inferences yield the same result, even if the target is not the class node. We achieve such a result by building an HMM where the first hidden variable is the class and all other hidden variables have the same state space as the class and are deterministically determined by its value, while the manifest variables are the features in the Naive structure. As such, the equivalence of inferences under both types of irrelevance extends to queries in any node of a Naive structure:

Corollary 6. *Consider a separately specified Naive Credal classifier $X_0 \to X_1, X_0 \to X_2, \cdots, X_0 \to X_n$, a target $X_t = x_t$ on a node t, and some evidence $\mathbf{X}_O = \tilde{\mathbf{x}}_O$ on arbitrary features (leaf variables). Then the outcomes of STRONG-INF and EPISTEMIC-INF coincide.*

Proof. Let $X'_0 = X_0$ and $X'_1, \cdots, X'_n$ be precise variables with the same state space as the class X'_0 and probability distributions $q(x'_i|x'_{i-1}) = 1$ if $x'_i = x'_{i-1}$ and zero otherwise, for all $i = 1, \ldots, n$. Define $Q(X_i|X'_i = x'_i)$ by using the credal set $Q(X_i|X_0 = x_0)$ of the original Naive Credal classifier, whenever $x'_i = x_0$, for $i = 1, \ldots, n$. Without loss of generality, assume that $t = n$ (if the query was in X_0, then use X'_n as query instead of X_n). This procedure has created an HMM with hidden nodes $X'_0, \ldots, X'_n$, manifest nodes $X_1, \ldots, X_{n-1}$, and a final query $t = n$ with $X_t = x_t$. This HMM clearly yields the same inferential result as does the Naive Credal classifier for STRONG-INF. By Theorem 3, the results of STRONG-INF and EPISTEMIC-INF coincide in this HMM, hence the result of EPISTEMIC-INF in this HMM is equal to the result of STRONG-INF in the original Naive structure. By construction, EPISTEMIC-INF in this HMM contains the result of EPISTEMIC-INF in the original Naive structure (that is, the latter is equal or lies inside the former). Because EPISTEMIC-INF always contains STRONG-INF, and in particular in the Naive structure, they must all coincide. □

4 Conclusion

Credal networks generalize Bayesian networks to allow for the representation of uncertain knowledge in the form of credal sets, closed and convex sets of probability distributions. The use of credal sets arguably facilitates the constructions of complex models, but presents a challenge to the computation of inferences with the model.

In this paper, we studied the theoretical complexity of inferences in credal networks, in what concerns the topology of the network, the semantics of the arcs (i.e., whether epistemic irrelevance or strong independence is assumed), and the cardinality of variable domains. In a nutshell, computing with credal networks is NP-hard except in the cases of tree-shaped models under epistemic irrelevance, and polytree-shaped models under strong independence. A notable exception is the computation of probability bounds on the value of the last variable in a imprecise hidden Markov models, in which case we have shown that inferences under epistemic irrelevance and strong independence coincide, which implies that the latter is polynomial-time computable. We leave as an open question the complexity of generic inferences in imprecise HMMs under strong independence.

Another possible avenue for future research is investigating the complexity of approximate inference. [17] showed that approximating inference under strong independence is NP-hard, even if we consider only singly connected networks of bounded treewidth. This is however not the case if variables are binary, as in this case we can run the 2U algorithm to obtain the exact value. [38] showed that for any network of bounded treewidth whose variables have bounded cardinality there exists a fully polynomial time approximation scheme for performing inference under strong independence, that is, an algorithm that given a rational $\epsilon > 0$ finds solutions which are within a factor $1+\epsilon$ of the true value in time polynomial in the input size and in $1/\epsilon$. Apart from its tractability on credal trees, nothing is known about the complexity of approximate inference under epistemic irrelevance, unless for the case of precise-vacuous networks, which we showed here to provide the same inferences under strong independence or epistemic irrelevance, so the NP-hardness of approximate inference under the former extends to the latter.

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