# Inference and Risk Measurement with the Pari-Mutuel Model

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**Abstract**

We explore generalizations of the pari-mutuel model (PMM), a formalization of an intuitive way of assessing an upper probability from a precise one. We discuss a naive extension of the PMM considered in insurance, compare the PMM with a related model, the Total Variation Model, and generalize the natural extension of the PMM introduced by P. Walley and other pertained formulae. The results are subsequently given a risk measurement interpretation: in particular it is shown that a known risk measure, Tail Value at Risk (TVaR), is derived from the PMM, and a coherent risk measure more general than TVaR from its imprecise version. We analyze further the conditions for coherence of a related risk measure, Conditional Tail Expectation. Conditioning with the PMM is investigated too, computing its natural extension, characterising its dilation and studying the weaker concept of imprecision increase.

*Key words:* Pari-mutuel model, risk measures, natural extension, dilation, 2-monotonicity, imprecision increase.

## 1. Introduction

The *pari-mutuel model* (PMM) formalizes a very intuitive and therefore widely used method of assigning an upper probability starting from a precise probability. To introduce it, consider, following [3], a probability $P$ for event

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$A$ as a *fair* price for a bet which returns 1 unit to the bettor if $A$ is true, 0 units if $A$ is false, i.e. returns the *indicator* $I_A$ of $A$. The bettor's gain is $G = I_A - P(A)$, while that of his opponent, House, is $-G = G_H = P(A) - I_A$.

In most real-world betting schemes House is unwilling to accept such a game, which is fair to both counterparts (the expectations $E(G_H)$, $E(G)$ are 0), but asks for a positive gain expectation. It is so when House is a bookmaker, an insurance company, the organizer of a lottery, and so on. A way to achieve this goal is to raise the bettor's price without altering his reward, and a *naive method* multiplies $P$ by a constant greater than 1, say $1+\delta$, where $\delta > 0$ is a loading constant. The bettor pays $\overline{P}(A) = (1+\delta)P(A)$, while the gain for House is now $\overline{G}_H = (1+\delta)P(A) - I_A$. Alternatively, House might ask the same price to pay a reduced reward $(1-\tau)I_A$, where $0 < \tau < 1$ is interpreted as a commission, or also a taxation. This originates a gain $\overline{G}^*_H = P(A) - (1-\tau)I_A = (1-\tau)(\frac{P(A)}{1-\tau} - I_A) = (1-\tau)\overline{G}_H$ iff $\frac{1}{1-\tau} = 1+\delta$, i.e. iff $\tau = \frac{\delta}{1+\delta}$. Thus, up to a scaling factor, the two methods are equivalent if $\tau = \frac{\delta}{1+\delta}$; the latter is formally more adherent to common betting systems, called in fact *pari-mutuel systems*.[1]

In the theory of imprecise probabilities, $\overline{P}$ is an upper probability, but a slight adjustment to $\overline{P}$ is necessary to achieve the customarily adopted consistency notion of coherence, recalled in Section 2. In fact, Walley [16] terms pari-mutuel model the upper probability

$$\overline{P}(A) = \min\{(1+\delta)P(A), 1\}. \tag{1}$$

Intuitively, the correction is needed: when $P(A) > \frac{1}{1+\delta}$, it is $\overline{G}_H > 0$ in the naive method, i.e. a bettor suffers from a sure loss no matter whether $A$ is true or false.

This paper investigates further the pari-mutuel model, extending the analyses in [13, 16]. Preliminary issues are recalled in Section 2, very concisely in general, more extensively as for 2-monotone and 2-alternating previsions, since the upper probability $\overline{P}$ in (1) is 2-alternating. In Section 3 we discuss extensions of the PMM and compare it with a similar model. First, we consider alternative expressions for the natural extension $\overline{E}(X)$ of $\overline{P}$, defined on a field $\mathcal{A}$, to any $\mathcal{A}$-measurable gamble $X$. These expressions were stated in [16], but we make a more detailed analysis of the conditions ensur-

[1] The inventor of the pari-mutuel system was the French perfume maker Joseph Oller in 1865.

ing that $\overline{E}(X)$ is equal to a certain conditional prevision $(P(X|X > x_\tau))$, which has a risk measurement interpretation. In Section 3.1 we restrict to non-negative gambles and compare the natural extension $\overline{E}$ with the naive extension $\overline{P}_N(X) = \min\{(1+\delta)P(X), \sup X\}$, showing that quite often $\overline{P}_N$ is not coherent. The motivation for this work is that $\overline{P}_N$ is a premium in insurance, although with different premises: the starting point is not the PMM but a set of non-negative gambles. In Section 3.2 we compare the PMM with the Total Variation Model (TVM) [8] and compute the natural extension of the TVM to $\mathcal{A}$-measurable gambles. In Section 3.3 we generalize Walley's approach, obtaining a formula for $\overline{E}(X)$ when the PMM is given on a lattice of events and $X$ is not necessarily measurable: this makes it possible to evaluate $\overline{E}(X)$ for all $X$ defined on a suitable reference partition while assessing a probability on a relatively small set of events (this and other concepts are illustrated in Example 2).

These results have an interesting and, to the best of our knowledge, so far not considered interpretation in the realm of risk measurement. This is the main topic of Section 4, where the natural extension of the PMM defined on a field is shown to correspond to a coherent risk measure, called Tail Value-at-Risk or TVaR (in [5]; other authors may use a different terminology). When the PMM is defined on a lattice, we obtain a generalization of TVaR (not discussed in the risk literature), which replaces precise with imprecise uncertainty measures; we name it ITVaR. Thus the PMM supplies a motivation for introducing 'imprecise' risk measures: one of them, ITVaR, is the natural extension of a PMM assigned on a lattice. Conditioning the PMM defined on a field is discussed in Section 5. We specialize general formulae for the natural extension of 2-alternating and 2-monotone probabilities to the case of the PMM and explore the effect on them of *dilation* and of a weaker phenomenon, *imprecision increase*. We characterise dilation and give sufficient conditions for dilation or imprecision increase in Section 5.1. Further, operational conditions for the most relevant cases (when the commission $\tau$ is 'low' or event $A$ is 'rare') are given in Section 5.2. Ideas about varying the extent of dilation or imprecision increase are outlined in Section 5.3. Section 6 concludes the paper. The proofs of the main results are given in the Appendix.

## 2. Preliminaries

Upper ($\overline{P}$) and lower ($\underline{P}$) probabilities are customarily related by the *conjugacy* relation $\overline{P}(A) = 1 - \underline{P}(A^c)$, which lets one refer to either $\overline{P}$ or $\underline{P}$ only. Applying it to (1), the lower probability in the PMM is [16]

$$\underline{P}(A) = \max\{(1+\delta)P(A) - \delta, 0\}. \tag{2}$$

As noted in the Introduction, the parameter $\tau \in ]0; 1[$ can, and later will, alternatively describe $\overline{P}$, $\underline{P}$ in the PMM. We shall often exploit, without always recalling it, the relationship between $\tau$ and $\delta$:

$$\tau = \frac{\delta}{1+\delta} \quad ; \quad \delta = \frac{\tau}{1-\tau}. \tag{3}$$

An upper probability $\overline{P}$ defined by (1) for any $A$ in an *arbitrary* set of events $\mathcal{D}$ (or $\underline{P}$ defined by (2)) is *coherent* on $\mathcal{D}$, and probably the simplest way to see it is to apply our subsequent Proposition 2. In general, an *upper prevision* $\overline{P}$ is a mapping from a set $\mathcal{D}$ of *gambles* (bounded random variables), defined on a partition or possibility space $I\!P_u$, into the real line, and an *upper probability* is its special case that the domain $\mathcal{D}$ is made of (indicators of) events only.

The upper prevision $\overline{P}$ is *coherent* on $\mathcal{D}$ iff, $\forall n \in \mathbb{N}$, $\forall s_0, s_1, \ldots, s_n \geq 0$, $\forall X_0, X_1, \ldots, X_n \in \mathcal{D}$, defining $\overline{G} = \sum_{i=1}^{n} s_i(\overline{P}(X_i) - X_i) - s_0(\overline{P}(X_0) - X_0)$, it holds that $\sup \overline{G} \geq 0$.

There are several necessary conditions for coherence, in particular: *internality*, $\inf X \leq \overline{P}(X) \leq \sup X$, and *subadditivity*, $\overline{P}(X+Y) \leq \overline{P}(X) + \overline{P}(Y)$.

We refer to [16] for a thorough presentation of the theory of coherent upper and lower previsions. One of its most important notions is that of *natural extension* [16, Section 3].

In our framework, the natural extension $\overline{E}$ on $\mathcal{D}'$ of a coherent upper prevision (or probability) $\overline{P}$ defined on $\mathcal{D} \subset \mathcal{D}'$ is the *least-committal* coherent extension of $\overline{P}$ on $\mathcal{D}'$, i.e. $\overline{E}(X) = \overline{P}(X)$, $\forall X \in \mathcal{D}$, and for any coherent $\overline{P}^*$ such that $\overline{P}^* = \overline{P}$ on $\mathcal{D}$, $\overline{E}(X) \geq \overline{P}^*(X)$, $\forall X \in \mathcal{D}'$, i.e. $\overline{E}$ *dominates* $\overline{P}^*$. It can be shown that $\overline{E}$ always exists. Symmetrically, the natural extension $\underline{E}$ on $\mathcal{D}'_L$ of a coherent lower prevision $\underline{P}$ on $\mathcal{D}_L$ is such that $\underline{E} = \underline{P}$ (on $\mathcal{D}_L$), and every coherent extension $\underline{P}^*$ of $\underline{P}$ dominates $\underline{E}$ on $\mathcal{D}'_L$.

If condition '$\forall s_0, s_1, \ldots, s_n \geq 0$' is replaced by '$\forall s_0, s_1, \ldots, s_n \in \mathbb{R}$' in the definition of coherent upper prevision, we obtain de Finetti's notion of *dF-coherent* (precise) prevision [3]. A dF-coherent prevision $P$ is coherent both as an upper and as a lower prevision. Moreover, a dF-coherent prevision corresponds to the expectation with respect to a *finitely* additive probability. The precise previsions or probabilities in the sequel are meant to be dF-coherent.

Although the domain of an upper prevision may be arbitrary, it will have a special structure in most of the paper, to exploit results on 2-alternating previsions.

More specifically, a set of events $\mathcal{A}$ is a *field* when $\emptyset \in \mathcal{A}$ and $A \vee B, A^c \in \mathcal{A}, \forall A, B \in \mathcal{A}$. If $\mathcal{A}$ is a field, a gamble $X$ is $\mathcal{A}$-*measurable* when the events $X > x$ and $X < x$ are in $\mathcal{A}$, $\forall x \in \mathbb{R}$.

A set of gambles $S$ is a *lattice* if $X$, $Y \in S$ implies $\max(X, Y) \in S$ and $\min(X, Y) \in S$, where $\max(X, Y)(\omega) = \max(X(\omega), Y(\omega))$ and $\min(X, Y)(\omega) = \min(X(\omega), Y(\omega))$, $\forall \omega \in I\!P_u$.

An upper prevision $\overline{P}$ defined on a lattice $S$ is called 2-*alternating* iff $\overline{P}(\max(X, Y)) \leq \overline{P}(X) + \overline{P}(Y) - \overline{P}(\min(X, Y))$, $\forall X, Y \in S$. A lower prevision $\underline{P}$ on $S$ is 2-*monotone* iff $\underline{P}(\max(X, Y)) \geq \underline{P}(X) + \underline{P}(Y) - \underline{P}(\min(X, Y))$, $\forall X, Y \in S$.

Results stated for 2-monotone previsions are easily reworded for 2-alternating ones (and vice versa), since the conjugate $\overline{P}(X) = -\underline{P}(-X)$ of a 2-monotone lower prevision is 2-alternating (and vice versa).

When $S$ is a set of (indicators of) events and $\overline{P}$ is therefore an upper probability, $S$ is a lattice iff $A$, $B \in S$ implies $A \vee B \in S$, $A \wedge B \in S$, and $\overline{P}$ is 2-alternating iff $\overline{P}(A \vee B) \leq \overline{P}(A) + \overline{P}(B) - \overline{P}(A \wedge B)$, $\forall A, B \in S$. Let $S^+$ be a lattice of events containing the impossible event $\emptyset$ and the sure event $\Omega$. With a mild additional condition, 2-alternating upper probabilities are coherent on $S^+$ [2]:

**Proposition 1.** *Let $\overline{P}$ be a 2-alternating upper probability on $S^+$. Then $\overline{P}$ is coherent iff $\overline{P}(\emptyset) = 0$ and $\overline{P}(\Omega) = 1$.*

One way to obtain coherent 2-alternating upper probabilities defines $\overline{P}$ as a special *distorted probability*, by the following result, adapted from [4], Example 2.1.

**Proposition 2.** *Let $P$ be a dF-coherent probability on $S^+$ and $\phi$ a (weakly) increasing concave function defined on $[0; 1]$ with $\phi(0) = 0$, $\phi(1) = 1$. Then the* distorted probability $\overline{P}(\cdot) = \phi(P(\cdot))$ *is a 2-alternating and coherent upper probability.*

Proposition 2 ensures that $\overline{P}$ in (1) is 2-alternating and coherent (put $\phi(x) = \min((1+\delta)x, 1)$), hence its conjugate $\underline{P}$ is 2-monotone and coherent.

To deal with the natural extension of the PMM in Section 3, the following Proposition 3 will be exploited.

*Notation* The natural extension of interest is that of $\overline{P}$ from $S^+$ to the set $\mathcal{L} = \mathcal{L}(I\!P_u)$ of all gambles defined on a 'universal' partition $I\!P_u$ (termed $\Omega$ in [16]). That is, $I\!P_u$ is a set of pairwise disjoint events, whose sum is the sure event $\Omega$, and such that its powerset $2^{I\!P_u}$ contains all the events of interest. In particular $S^+ \subseteq 2^{I\!P_u}$. Given $\overline{P} : S^+ \to \mathbb{R}$, its *outer (set) function* $\overline{P}^*$ is defined on $2^{I\!P_u}$ by $\overline{P}^*(B) = \inf\{\overline{P}(A) : A \in S^+, B \Rightarrow A\}$, $\forall B \in 2^{I\!P_u}$.

Two gambles $X$, $Y$ are *comonotonic* iff $X(\omega_1) < X(\omega_2)$ implies $Y(\omega_1) \leq Y(\omega_2)$, $\forall \omega_1, \omega_2 \in I\!P_u$. An upper prevision $\overline{P}$ is *comonotonic additive* iff $\overline{P}(X+Y) = \overline{P}(X) + \overline{P}(Y)$ for all comonotonic $X$, $Y$ in its domain.

**Proposition 3.** *[2] Let $\overline{P} : S^+ \to \mathbb{R}$ be a coherent 2-alternating upper probability. Its natural extension $\overline{E}$ on $\mathcal{L}$ is given by*

$$\overline{E}(X) = \inf X + \int_{\inf X}^{\sup X} \overline{P}^*(X > x)dx \tag{4}$$

*and is 2-alternating too. Further,*

*(a) The restriction of $\overline{E}$ on $2^{I\!P_u}$ coincides with the outer function $\overline{P}^*$.*

*(b) If $S^+ = 2^{I\!P_u}$, $\overline{E}$ is the* only *2-alternating, or equivalently the only comonotonic additive, coherent extension of $\overline{P}$ on $\mathcal{L}$.*

In Section 5 we shall be concerned with natural extensions on conditional events, like $\overline{E}(A|B)$ or $\underline{E}(A|B)$, while precise conditional previsions, like $P(X|X > x_\tau)$, appear in Section 3. In a conditional environment, the symbol $\mathcal{D}$ already introduced denotes more generally an arbitrary set of conditional gambles, i.e. its generic element is $X|B$, where $X$ is a gamble and $B$ a non-impossible event (when in particular $B{=}\Omega$, $X|B = X|\Omega = X$).

Although the paper presentation does not focus on coherence concepts in a conditional environment, our approach employs formally *Williams coherence* or *W-coherence*, in the version presented in [11], Definition 4, reported here:

*Definition* 1. $\underline{P} : \mathcal{D} \to \mathbb{R}$ is a *W-coherent conditional lower prevision* on $\mathcal{D}$ iff, for all $n \in \mathbb{N}$, $\forall X_0|B_0, \ldots, X_n|B_n \in \mathcal{D}$, $\forall$ $s_0, s_1, \ldots, s_n$ real and non-negative, defining $B = \bigvee_{i=0}^n B_i$ and $\underline{G} = \sum_{i=1}^n s_i I_{B_i}(X_i - \underline{P}(X_i|B_i)) - s_0 I_{B_0}(X_0 - \underline{P}(X_0|B_0))$, it holds that $\sup(\underline{G}|B) \geq 0$.

The term $g_i = s_i I_{B_i}(X_i - \underline{P}(X_i|B_i))$ $(i = 0, 1, \ldots, n)$ is interpreted as the gain from a *conditional* bet on $X_i$ with *stake* $s_i$. In fact, $g_i$ is obtained from the condition that the gambler's bet $s_i\underline{P}(X_i|B_i)$ is called off iff $B_i$ is false (in such a case $g_i = 0$), otherwise the gambler receives (if $i = 1, \ldots, n$) or pays (if $i = 0$) $s_i I_{B_i} X_i = s_i X_i$, and consequently $g_i = s_i(X_i - \underline{P}(X_i|B_i))$. Recall further that $\sup(\underline{G}|B) = \sup\{\underline{G}(\omega) : \omega \Rightarrow B\}$.

W-coherence reduces to the customary notion of coherence in [16] in the unconditional case, or to Walley's coherence in [16], Section 7.1.4 (b) when – and this is the case in the present paper – finitely many conditional events are involved (and some structure is imposed on $\mathcal{D}$ [11]).

Thus, the results in the paper may be equivalently interpreted in terms of the coherence concepts in [16]. A motivation for using W-coherence is that, since it requires no structure constraints on $\mathcal{D}$ and allows for rather general envelope and natural extension theorems also in a conditional framework, our results could be more simply extended to general conditional frameworks, where W-coherence is not necessarily equivalent to Walley's approach (cf. [16] for further comparisons of coherence concepts).

Several necessary conditions hold for W-coherence, whenever they are well-defined. Recall *internality*, $\inf(X|B) \leq \overline{P}(X|B) \leq \sup(X|B)$, and the *Generalized Bayes Rule* (GBR) $\overline{P}(I_A(X - \overline{P}(X|A))) = 0$, which in the case of precise previsions specialises to

$$P(XI_A) = P(X|A)P(A). \tag{5}$$

## 3. Extending the pari-mutuel model

The natural extension $\overline{E}$ of $\overline{P}(A) = \min\{(1 + \delta)P(A), 1\}$ from a field $\mathcal{A}$ to any $\mathcal{A}$-measurable gamble $X$ was shown in [16] to be

$$\overline{E}(X) = x_\tau + (1 + \delta)P((X - x_\tau)^+), \tag{6}$$

where $(X - x_\tau)^+ = \max\{X - x_\tau, 0\}$ and the (upper) *quantile* $x_\tau$ is defined as

$$x_\tau = \sup\{x \in \mathbb{R} : P(X \leq x) \leq \tau\} = \sup\{x \in \mathbb{R} : P(X > x) \geq 1 - \tau\}. \tag{7}$$

As appears from (7), $x_\tau$ is a quantile function, a well known concept in literature for defining an inverse for the *distribution function* $F_X(x) \stackrel{\text{def}}{=} P(X \leq x)$.

There are however slightly different ways of defining quantiles (see [4], Section 1.5.8); ours follows [16], Section 3.2.5. The upper quantile $x_\tau$ can be viewed as a threshold such that, for every $x > x_\tau$, $X$ exceeds $x$ with probability less than $1 - \tau$ (see also Example 1).

An alternative expression for $\overline{E}(X)$ is:[2]

$$\overline{E}(X) = (1-\varepsilon)P(X|X > x_\tau) + \varepsilon x_\tau,\ \varepsilon \stackrel{\text{def}}{=} 1 - (1+\delta)P(X > x_\tau). \tag{8}$$

It is also stated in [16] that $\overline{E}(X) = P(X|X > x_\tau)$ if $X$ has a continuous distribution function $F_X(x)$.

We shall now explore more thoroughly the relationship between $\overline{E}(X)$ and $P(X|X > x_\tau)$. The results will be exploited also in Section 4, where they will be reinterpreted in a risk measurement perspective.

To begin with, we gather some known or anyway elementary, but useful facts in the following proposition.

**Proposition 4.** *Let $X$ be $\mathcal{A}$-measurable and for $\tau \in ]0;1[$ define: $x_\tau$ by (7), $F_X(x_\tau^+) = \lim_{x\to x_\tau^+} F_X(x)$, $F_X(x_\tau^-) = \lim_{x\to x_\tau^-} F_X(x)$.*

*a) $\tau \in [F_X(x_\tau^-); F_X(x_\tau^+)]$; besides, all values of $\tau$ in $[F_X(x_\tau^-); F_X(x_\tau^+)[$ originate by (7) the same (upper) quantile $x_\tau$.*

*b) $\inf X \le x_\tau \le \sup X$.*

*c) $(X > x_\tau) = \emptyset$ iff $x_\tau = \sup X$; if $(X \le x_\tau) = \emptyset$ then $x_\tau = \inf X$.*

*d) It holds for $\varepsilon$ in (8) that $\varepsilon \lesseqqgtr 0$ iff $\tau \gtreqqless F_X(x_\tau)$.*[3]

**Corollary 1.** *If $(X > x_\tau) = \emptyset$, $\overline{E}(X) = \sup X$.*

*Proof.* Substitute (by Proposition 4, c)) $x_\tau = \sup X$ in (6), noting that $P((X - x_\tau)^+) = P(0) = 0$. □

*Remark* 1. When $P$ is $\sigma$-additive, $F_X(x_\tau^+) = F_X(x_\tau)$, i.e. $F_X$ is right-continuous. But an often neglected issue broadens the number of possible

[2]Equation (8) is stated without proof in [16], Note 3 to Section 3.2. A proof may be deduced from the later Proposition 8; see also the comments made there.

[3]We write $\lesseqqgtr$ or $\gtreqqless$ to summarize three conditions, here $\varepsilon < 0$ iff $\tau > F_X(x_\tau)$, $\varepsilon = 0$ iff $\tau = F_X(x_\tau)$, $\varepsilon > 0$ iff $\tau < F_X(x_\tau)$.

alternatives in comparing $\overline{E}(X)$ with $P(X|X > x_\tau)$ (and with another extension presented in the next Section 3.1): since $F_X$ is originated by a not necessarily $\sigma$-additive probability $P$, there may exist non-zero *adherent probabilities* at $x_\tau$ (cf. [3], Section 6.4.11; see also [9]). Precisely,

$$F_X(x_\tau^+) - F_X(x_\tau^-) = P_{x_\tau}^- + P_{x_\tau}^+ + P(X = x_\tau),$$

where $P_{x_\tau}^- = F_X(x_\tau) - F_X(x_\tau^-) - P(X = x_\tau)$ is the *left adherent* probability at $x_\tau$, $P_{x_\tau}^+ = F_X(x_\tau^+) - F_X(x_\tau)$ is the *right adherent* probability at $x_\tau$. Hence,

$$F_X(x_\tau) = F_X(x_\tau^-) + P_{x_\tau}^- + P(X = x_\tau). \tag{9}$$

While $P_{x_\tau}^+$ is zero iff $F_X$ is right-continuous at $x_\tau$ (always if $P$ is $\sigma$-additive), from (9), $F_X$ may be left-discontinuous in $x_\tau$ also when $P_{x_\tau}^- = 0$, if $P(X = x_\tau) > 0$ ($\sigma$-additivity of $P$ implies $P_{x_\tau}^- = 0$). □

**Proposition 5.** *a) If $P(X|X > x_\tau) = x_\tau$, then $\overline{E}(X) = P(X|X > x_\tau)$.*

*b) If $P(X|X > x_\tau) > x_\tau$, then $\overline{E}(X) \lesseqgtr P(X|X > x_\tau)$ iff $\tau \lesseqgtr F_X(x_\tau)$.*

*Proof.* Using (8), $\overline{E}(X) \lesseqgtr P(X|X > x_\tau)$ iff $\varepsilon(x_\tau - P(X|X > x_\tau)) \lesseqgtr 0$, from which a) follows immediately, b) using also Proposition 4, d). □

Proposition 5, a) considers a really extreme situation. Assuming from now that $P(X|X > x_\tau) > x_\tau$, Proposition 5, b) reduces the comparison between $\overline{E}(X)$ and $P(X|X > x_\tau)$ to comparing $\tau$ and $F_X(x_\tau)$ in the further subcases that can be identified. The most notable instances are:

i) $F_X$ is continuous at $x_\tau$. This implies $\tau = F_X(x_\tau)$, and $\overline{E}(X) = P(X|X > x_\tau)$.

ii) $F_X$ is right-continuous, but not continuous at $x_\tau$, and $\tau \neq F_X(x_\tau)$. This implies $F_X(x_\tau) = F_X(x_\tau^+) > \tau$, and $P(X|X > x_\tau) > \overline{E}(X)$.

Case ii) is the most obvious instance that ensures $P(X|X > x_\tau) > \overline{E}(X)$, but not the only one. By Proposition 4, a), it can be $\tau < F_X(x_\tau)$ also when $F_X$ is not right-continuous (while being left-discontinuous). Similarly, there are other cases when $P(X|X > x_\tau) = \overline{E}(X)$ because $\tau = F_X(x_\tau)$, apart from case i), which remains the most important one. And it is also possible that

iii) $P(X|X > x_\tau) < \overline{E}(X)$.

Obviously, case iii) cannot occur when $P$ is $\sigma$-additive, since it is equivalent to $\tau > F_X(x_\tau)$, hence to $\tau \in \, ]F_X(x_\tau); F_X(x_\tau^+)] = I^>$ and $I^> \neq \varnothing$ iff $P_{x_\tau}^+ > 0$.

When $P(X|X > x_\tau) > \overline{E}(X)$, then $P(X|X > x_\tau)$ is clearly not a coherent extension to $X$ of $\overline{P}$ in the PMM, while it is so when it coincides with $\overline{E}(X)$.

*3.1. Comparison with a naive extension*

In actuarial applications the upper probability $\overline{P}(A)$ in (1) is the price, determined by increasing $P$ by a loading $\delta > 0$, of an insurance policy which pays 1 unit if and only if event $A$ occurs. In analogy with (1), one could set the price of an insurance policy which refunds $x$ units iff the loss $X = x$ occurs, to $(1 + \delta)P(X)$, up to a maximum of $\sup X$. Here $P(X)$ is usually the expectation of $X$ computed from $P$. This procedure defines the *naive extension*:

$$\overline{P}_N(X) = \min\{(1 + \delta)P(X), \sup X\}.$$

It is important to discuss this extension both because it represents the intuitively simplest way to apply the PMM to gambles, and because it has been actually considered in risk theory. Precisely, only the multiplicative term $\overline{P}^*_N(X) = (1 + \delta)P(X)$ is employed, and referred to as *expected value principle*, in risk theory literature [7, p. 67], but we shall rather investigate $\overline{P}_N$, since the upper bound $\sup X$ is easily seen to be necessary for coherence. To fix the framework, suppose (throughout this section only) that $P$ is defined on the field $2^{I\!P_u}$, and that we are interested in extending it to some set $\mathcal{D}$ strictly contained in the cone $\mathcal{L}^+(I\!P_u)$ of the non-negative gambles in $\mathcal{L}(I\!P_u)$. Here we assume that the gambles in $\mathcal{D}$ are non-negative, bearing in mind the insurance framework $\overline{P}_N$ is applied to: each $X \in \mathcal{D}$ is a refund to the insured, hence $\inf X \geq 0$, $\forall X \in \mathcal{D}$.

The inclusion $\mathcal{D} \subsetneqq \mathcal{L}^+(I\!P_u)$ is strict because $\overline{P}_N$ cannot in general be coherent on a set $\mathcal{D}$ containing $X$, $X + k$, when $k \in \mathbb{R}^+$ is large enough. For instance, if $\overline{P}_N(X) = (1 + \delta)P(X) < \sup X$, then for $k \geq \frac{\sup X - (1+\delta)P(X)}{\delta}$, it holds that $\overline{P}_N(X + k) = \sup X + k > \overline{P}_N(X) + k$ , violating property (c) in [16], Section 2.6.1, which is a necessary condition for coherence.

The above argument points clearly out that $\overline{P}_N$ is not coherent when its domain $\mathcal{D}$ is sufficiently rich. Yet, $\overline{P}_N$ may be coherent on some very special set (for instance, $\mathcal{D} = \{X\}$) if, as usually done in insurance, the starting point for getting $\overline{P}_N$ is a set of gambles, and not the PMM. At a closer look, this is true when we only assess a prevision to compute $\overline{P}_N$, without preliminarily assigning a probability $P$ on $2^{I\!P_u}$, such that $P(X)$ is the expectation of $X$ under $P$ (for instance when $\mathcal{D} = \{X\}$ we may assess $P(X) \in [\inf X; \sup X]$ without eliciting any probability for events like $X = x$, or $X \leq x$). In the more customary case that $P(X)$ is an expectation, since the PMM is also determined by the probability $P$ we may always think of $\overline{P}_N$ as an extension of the PMM. Here $\overline{P}_N$ may be incoherent with the PMM, even when $\overline{P}_N$ is defined on very simple sets like $\mathcal{D} = \{X\}$, as the next example shows:

*Example* 1. Take $I\!P_u = \{\omega_0, \omega_1, \omega_2, \omega_3\}$, and let $X(\omega_i) = i$ $(i = 0, \ldots, 3)$, $P(X = 0) = 0$, $P(X = 1) = 0.1$, $P(X = 2) = 0.5$, $P(X = 3) = 0.4$ and $\delta = 1/10$. Then $P(X) = 2.3$ and hence $\overline{P}_N(X) = 2.53$. Let us now compute the natural extension in $X$. We have that $\tau = \frac{\delta}{1+\delta} = 1/11$, hence $x_\tau = 1$, as can be checked using $F_X$. Applying (6), $\overline{E}(X) = 1 + \frac{11}{10}P(\max\{X - 1, 0\}) = 1 + \frac{11}{10}1.3 = 2.43$. □

In Example 1, $\overline{P}_N(X) > \overline{E}(X)$. This is interesting because the natural extension is shown to lead to a price smaller than would be expected from the intuition at the basis of the PMM and also because $\overline{P}_N$ is incoherent with the PMM, being larger than $\overline{E}$.

The dominance relationship between $\overline{P}_N$ and $\overline{E}$ is analyzed in detail in [13], Proposition 6. It ensues from there that $\overline{P}_N$ is only occasionally and in quite special situations equal to $\overline{E}$. For instance, if $F_X$ is continuous at $x_\tau$ $\overline{P}_N$ is incoherent, unless the limiting evaluation $P(X|X \leq x_\tau) = 0$ $(\leq \inf X)$ applies.

As a final remark, we note that $\overline{P}_N$ is generally incoherent even when the assumption $X \geq 0$, $\forall X \in \mathcal{D}$, is removed. For instance, when $X < 0$, it is $\overline{P}_N(X) = (1 + \delta)P(X) < \inf X$ if $P(X) < \frac{\inf X}{1+\delta}$.

### *3.2. Comparison with the Total Variation Model*

While the naive extension $\overline{P}_N$ in Section 3.1 is directly inspired by the basic idea underlying the PMM, the *Total Variation Model* (TVM) may be introduced independently of the PMM [8], but functionally, as we shall see, is closely related to it. This motivates the investigation in this section.

Following [8], suppose a dF-coherent probability $P$ is given on a field $\mathcal{A}$, fix $\tau$ in the *open* interval $]0, 1[$, and let $\mathcal{M} = \{Q : \rho(P, Q) \stackrel{\text{def}}{=} \sup_{A \in \mathcal{A}} |P(A) - Q(A)| \leq \tau$, $Q$ dF-coherent probability on $\mathcal{A}\}$. The term $\rho(P, Q)$ is a total variation distance, and this names the model. So the TVM formalises an imprecise knowledge of a precise probability $P$: the 'true' probability may be any of those in $\mathcal{M}$.

If the following assumption is made

(a) $P(A) > 0$, $\forall A \in \mathcal{A} - \{\emptyset\}$,

the lower envelope of $\mathcal{M}$, i.e. $\underline{P}_{TVM}(A) = \inf_{Q \in \mathcal{M}}\{Q(A)\}$, $\forall A \in \mathcal{A}$, is

$$\underline{P}_{TVM}(A) = \max\{P(A) - \tau, 0\}, \forall A \in \mathcal{A} - \{\Omega\}, \quad (10)$$

while $\underline{P}_{TVM}(\Omega) = 1$.

It is then easy to derive $\overline{P}_{TVM}(A) = 1 - \underline{P}_{TVM}(A^c)$, getting

$$\overline{P}_{TVM}(A) = \min\{P(A) + \tau, 1\}, \forall A \in \mathcal{A} - \{\emptyset\}, \tag{11}$$

while $\overline{P}_{TVM}(\emptyset) = 0$.

By Proposition 2, $\overline{P}_{TVM}$ is a coherent, 2-alternating upper probability.

We emphasize that (10), (11) are obtained under assumption (a), which is rather restrictive: in the common case that $\mathcal{A} = 2^{I\!P_u}$, $I\!P_u$ must be countable. This is a difference with the PMM, whose probabilities (1), (2) need not comply with (a).

When assumption (a) applies, we may use (2), (3) to get

$$\underline{P}(A) = (1+\delta)\max\{P(A) - \tau, 0\} = \frac{1}{1-\tau}\underline{P}_{TVM}(A), \tag{12}$$

which holds iff $A \neq \Omega$. Analogously, $\overline{P}(A) = (1+\delta)\min\{P(A), \frac{1}{1+\delta}\} = \frac{1}{1-\tau}\min\{P(A), 1-\tau\} = \frac{1}{1-\tau}(\min\{P(A)+\tau, 1\} - \tau)$. Hence

$$\overline{P}(A) = \frac{1}{1-\tau}(\overline{P}_{TVM}(A) - \tau), A \neq \emptyset. \tag{13}$$

Thus equations (12), (13) explicit the relationship between PMM and TVM, when both are obtained from the same probability $P$.

The natural extension $\overline{E}_{TVM}$ of the TVM on any $\mathcal{A}$-measurable gamble is obtained using previous results: from (13) $\overline{P}_{TVM}(A) = (1-\tau)\overline{P}(A) + \tau$, and using (4)

$$\overline{E}_{TVM}(X) = \inf X + \int_{\inf X}^{\sup X} [(1-\tau)\overline{P}(X > x) + \tau]dx =$$
$$\inf X + (1-\tau)\int_{\inf X}^{\sup X} \overline{P}(X > x)dx + \tau(\sup X - \inf X).$$

Since (cf. [16], Section 3.2.3)

$$\int_{\inf X}^{\sup X} \overline{P}(X > x)dx = x_\tau + \frac{1}{1-\tau}P((X - x_\tau)^+) - \inf X,$$

we obtain

$$\overline{E}_{TVM}(X) = x_\tau + P((X - x_\tau)^+) + \tau(\sup X - x_\tau). \tag{14}$$

Although the relationship between PMM and TVM is functionally simple, not all the results concerning one model have a comparably simple counterpart within the other one. This is especially true when conditioning, cf. Section 5 for PMM and [8] for TVM.

*3.3. A generalization*

We shall derive here $\overline{E}$ in the more general framework of Proposition 3, that $\overline{P}$ is defined by the PMM on $S^+$ and $\overline{E}$ on $\mathcal{L}(I\!\!P_u)$. We first obtain an expression for $\overline{E}(B)$, for any event $B$ in $2^{I\!\!P_u}$.

**Proposition 6.** *In the PMM, the natural extension of $\overline{P} : S^+ \to \mathbb{R}$ on $2^{I\!\!P_u}$ is*

$$\overline{E}(B) = \min\{(1+\delta)\tilde{P}^*(B), 1\}, \tag{15}$$

*where the upper probability $\tilde{P}^*(B) = \inf\{P(A) : A \in S^+, B \Rightarrow A\}$ is the outer function of $P$.*

We emphasize that $\tilde{P}^*$ in (15) is generally not a precise, but an upper probability. In fact, by Proposition 3 (a), it coincides with the natural extension $\overline{E}_P$ on $2^{I\!\!P_u}$ of the probability $P$, when $P$ is interpreted as a special upper probability. As such, and since $P$ is obviously $n$-alternating, $\tilde{P}^*$ is $n$-alternating too (see [2]).

**Proposition 7.** *In the PMM, the natural extension of $\overline{P} : S^+ \to \mathbb{R}$ on $\mathcal{L}(I\!\!P_u)$ is, $\forall\ X \in \mathcal{L}(I\!\!P_u)$:*

$$\overline{E}(X) = x_\tau^u + (1+\delta)\overline{E}_P((X - x_\tau^u)^+) \tag{16}$$

*where $\overline{E}_P$ is the natural extension of $P$ (also of $\tilde{P}^*$) on $\mathcal{L}$, and $x_\tau^u$ is the (upper) quantile relative to $\tilde{P}^*$*

$$x_\tau^u = \sup\{x \in \mathbb{R} : \tilde{P}^*(X > x) \geq 1 - \tau\}. \tag{17}$$

Clearly, (16) generalizes (6). We might summarize the difference between the natural extension in (16) and that in (6) as follows: computing the natural extension of $\overline{P}$ on gambles which are not necessarily measurable with respect to the domain of $\overline{P}$ introduces imprecision by transforming the precise prevision $P((X - x_\tau)^+)$ in (6) into the upper prevision $\overline{E}_P((X - x_\tau^u)^+)$ in (16). Also the quantile $x_\tau$ refers to probability $P$ in (7), while $x_\tau^u$ employs the upper probability $\tilde{P}^*$ in (17).

But there is another attractive interpretation: $\overline{E}(B)$ in (15) can be viewed as a kind of *imprecise PMM*, defined via natural extension on $2^{\mathbb{P}_u}$ starting from a (precise) PMM on a narrower set $S^+$: then (16) describes the natural extension of this imprecise model.

Some properties of the natural extension of the PMM generalize to the natural extension of the imprecise PMM. The following proposition relaxes (8):

**Proposition 8.** *If* $(X > x_\tau^u) \neq \emptyset$, *it holds for the natural extension* $\overline{E}$ *on* $\mathcal{L}(\mathbb{P}_u)$ *of* $\overline{P} : S^+ \to \mathbb{R}$ *that*

$$\overline{E}(X) \leq \varepsilon^u x_\tau^u + (1 - \varepsilon^u)\overline{E}_P(X|X > x_\tau^u) \tag{18}$$

*where* $\varepsilon^u \stackrel{\text{def}}{=} 1 - (1 + \delta)\overline{E}_P(X > x_\tau^u)$.

Although the inequality in (18) can be strict (we omit proving this), when $P$ and hence $\overline{P}$ are defined on $2^{\mathbb{P}_u}$ then $\overline{E}_P$ is equal to $P$ (or to its extension using (5)), and $x_\tau^u$, $\varepsilon^u$ to $x_\tau$, $\varepsilon$ respectively. Since $P$ is additive (and considered the proof of Proposition 8 given in Appendix A) we get now $\overline{E}_P((X - x_\tau)^+) = P((X - x_\tau)^+) = \lambda$. Thus (18) reduces to (8).

The statement corresponding to Proposition 4 d) is $\varepsilon^u \gtreqless 0$ iff $\overline{E}_P(X > x_\tau^u) \lesseqgtr \frac{1}{1+\delta}$, or also $\varepsilon^u \gtreqless 0$ iff $\underline{E}_P(X \leq x_\tau^u) \gtreqless \tau$.

We know that $\varepsilon = 0$ when $F_X$ is continuous at $x_\tau$. An analogous relationship associates $\varepsilon^u$ and $\underline{F}_X$. In fact, when $\underline{F}_X(x) = \underline{E}_P(X \leq x) = 1 - \overline{E}_P(X > x)$ is continuous at $x_\tau^u$, then $\overline{E}_P(X > x_\tau^u) = 1 - \tau$. Hence $\varepsilon^u = 1 - (1 + \delta)(1 - \tau) = 0$.

## 4. Risk measurement interpretations

If $Y$ is a gamble, it is known [10] that $\overline{P}(-Y)$ may be interpreted as a *risk measure* for $Y$, i.e. a number measuring how risky $Y$ is, or also the amount of money to be reserved to cover potential losses from $Y$. Several risk measures were introduced in the literature, and there is often no unanimity on the terminology. To ensure comparisons with [5], we shall refer the risk measure to $X = -Y$; this corresponds, when $Y \leq 0$, to thinking in terms of losses and is frequently done in insurance, where $X$ represents the amount to be paid for insurance claims (however, $X$ is not necessarily non-negative in what follows).[4] Thus the upper previsions $\overline{E}(X)$ in (6), (8) and (16) may

[4] While ensuring compatibility with the prevailing literature and the formulae in [16], the convention of referring to losses modifies the range of the typical values for $\tau$. In

be seen as risk measures for $X$, and there is a strong correspondence with measures studied in the literature.

Consider equation (6): $x_\tau$ is the *Value-at-Risk* of $X$ at level $\tau$, $VaR_\tau(X)$, while $P((X-x_\tau)^+)$ is the *expected shortfall* $ES_\tau(X)$ (whenever $P$ is replaced by or thought of as an expectation) [5]. In fact, $(X - x_\tau)^+$ measures the shortfall, i.e. the residual loss in absolute value of an agent who reserves an amount of money equal to $VaR_\tau(X) = x_\tau$ to cover losses from $X$. Also $P(X|X > x_\tau)$ corresponds to a well-known risk measure (when $P$ is an expectation), termed *Conditional Tail Expectation* ($CTE_\tau$) in [5].

Equation (6) corresponds to (2.7) in [5], which defines another measure of risk, $TailVaR_\tau(X)$ or $TVaR_\tau(X)$. This equation is identical to (6), after replacing $\overline{E}$, $x_\tau$, $P((X - x_\tau)^+)$ with, respectively, $TVaR_\tau(X)$, $VaR_\tau(X)$, $ES_\tau(X)$:

$$TVaR_\tau(X) = VaR_\tau(X) + (1+\delta)ES_\tau(X).$$

Analogously, equation (8) corresponds to

$$TVaR_\tau(X) = (1-\varepsilon)CTE_\tau(X) + \varepsilon VaR_\tau(X). \tag{19}$$

The novel fact in our approach (apart from using previsions instead of expectations) is that $TVaR_\tau$ is derived as the natural extension of the PMM, while the starting point in the literature for defining this or other measures is usually a set of random variables, often a linear space equipped with a $\sigma$-additive probability measure, which is used to compute the expectations. In our notation, the usual approach would make the initial uncertainty assessments on the gambles defined on $\mathbb{P}_u$ rather than on the events in $2^{\mathbb{P}_u}$. Recalling also Proposition 3, we deduce the following properties for $TVaR_\tau$:

**Proposition 9.** *$TVaR_\tau(X)$ is the natural extension on $\mathcal{L}(\mathbb{P}_u)$ of the PMM defined on $2^{\mathbb{P}_u}$. Hence, it is the least-committal risk measure extending the PMM which is coherent. Actually, it is its* only *coherent extension which is 2-alternating, or equivalently comonotonic additive.*

$CTE_\tau$ complements $VaR_\tau$, in the sense that $VaR_\tau$, unlike $CTE_\tau$, is nearly uninformative about what are the losses, should the threshold $x_\tau$ be

this section $\tau$ should be fairly close to 1, representing the probability that the loss is not too high, while in the rest of the paper should rather be close to 0, being a taxation or commission.

exceeded. Unfortunately, neither $VaR_\tau$ nor $CTE_\tau$ is generally coherent, even though their linear combination in (19) originates a coherent risk measure. Conditions for coherence of $CTE_\tau$ are discussed in Section 3, and are commoner in practice than those ensuring coherence of $VaR_\tau$.[5] In the classical risk measurement approach using a $\sigma$-additive probability, the comparison between $CTE_\tau$ and $TVaR_\tau$ is limited to cases i), ii) in Section 3 which, as we pointed out there, are not exhaustive in general. Comonotonic additivity of $TVaR_\tau$ is also an interesting and sometimes required property in risk measurement [5]. The reason is that the risk of the sum $\rho(X+Y)$ is generally less than the sum of the single risks for $X$ and $Y$, because of a diversification effect; however, this effect is much weaker, making the equality $\rho(X+Y) = \rho(X) + \rho(Y)$ an often acceptable approximation, when $X$ and $Y$ tend to vary in the same direction, i.e. are comonotonic.

The generalization in Section 3.3 forms a basis for further results on the risk measurement side. This time, $\overline{E}(X)$ in (16) is the natural extension of the PMM defined on $S^+(\subset 2^{\mathbb{P}_u})$, and may again be interpreted as a risk measure, let us name it *Imprecise TailVar* or $ITVaR_\tau$. Using Proposition 3, $ITVaR_\tau$ is coherent and also 2-alternating. However, $ITVaR_\tau$ has no analogue in the risk measurement literature. The reason lies in the standard habit of defining risk measures from an underlying precise probability, an established custom which rules out potentially interesting risk measures that are functions of imprecise measures. And looking at (16), we notice that $ITVaR_\tau$ is a linear combination of other two measures which are imprecise versions of $VaR_\tau$ and $ES_\tau$: $x_\tau^u$ is defined in (17) as a function of the upper probability $\tilde{P}^*$, the shortfall $(X - x_\tau^u)^+$ is evaluated by the upper prevision $\overline{E}_P$.

We may conclude that the PMM provides also a formal justification for the existence of a new kind of risk measures, those defined in terms of imprecise uncertainty measures. This topic is still largely not investigated; an exception is the generalization of Dutch risk measures introduced in [1].

Finally, note that the natural extension $\overline{E}_{TVM}$ of the TVM in equation (14) provides us with another 2-alternating (and comonotonic additive) coherent risk measure, let us call it $TVRM_\tau$ (Total Variation risk measure). Its expression is

$$TVRM_\tau(X) = VaR_\tau(X) + ES_\tau(X) + \tau(\sup X - x_\tau).$$

[5] For $VaR_\tau$, see the discussion in [10].

We are not aware whether $TVRM_\tau$ has been previously employed in the relevant literature, seemingly not, but similarly to $TVaR_\tau$ it can be viewed as the least-committal coherent risk measure which extends the TVM defined on $2^{\mathbb{P}_u}$.

To conclude this section, we present a simple example illustrating several of the concepts discussed so far.

*Example* 2. Let $\mathbb{P}_u = \mathbb{R}^+$ denote the set of possible outcomes of a certain stock index $Y$ on a given day. For all $\omega \in \mathbb{P}_u$, define $X$ on $\mathbb{P}_u$ by

$$X(\omega) = \begin{cases} y_1 & \text{if } \omega \leq y_1 \\ y_2 & \text{if } \omega \geq y_2 \\ \omega & \text{otherwise,} \end{cases}$$

where $0 < y_1 < y_2$. We regard $X$ as the return of an *investment certificate* with equity protection: $X$ replicates $Y$ within some chosen bounds, a *floor* $y_1$ and a *cap* $y_2$.

We would like to sell $X$ at price $p$, with $y_1 < p < y_2$, thus making a profit $p - X$. We aim at calculating a risk measure for the profit, by calculating the upper prevision of the potential loss $X - p$. We consider a threshold $t$, with $p < t < y_2$, such that $X \geq t$ is deemed to lead to a critical loss.

We will calculate the upper prevision of $X - p$ as the natural extension of a PMM defined on a lattice, which means that the risk measure will be $ITVaR_\tau$. To this end, we first consider some events that are important for assessing the risk:

- $E_1$ is defined as the event '$X \leq p$'; it corresponds to having a gain (in the limit zero, if $X = p$);
- $E_1^c$, the complement of $E_1$, obviously means that there is a loss;
- $E_2 =$ '$X \geq t$' corresponds to experiencing a critical loss.

The minimum lattice that includes the three events above is $S^+ = \{E_1, E_1^c, E_2, E_1 \vee E_2, \emptyset, \Omega\}$ (note that $E_1 \wedge E_2 = \emptyset$). We assess a probability $P$ on this lattice, to compute $\overline{E}(X - p) = \overline{E}(X) - p$ using Proposition 7.

Now we have to fix $\tau \in ]0; 1[$. We choose $\tau$ so that $P(E_1) \leq \tau < 1 - P(E_2)$. The rationale is the following. Remember that the meaning of $\tau$ is that of a threshold (cf. Footnote 4): the probability of not experiencing a critical loss must be larger than $\tau$. Using the introduced events, this means that

$\tau < 1 - P(E_2)$. On the other hand, if it was the case that $P(E_1) > \tau$, recalling that $\tau$ is a number close to 1, that would correspond to requiring an unrealistically high probability of making a gain.

We may conveniently apply the following formula (cf. (30) in the proof of Proposition 7) to obtain $\overline{E}(X)$:

$$\overline{E}(X) = x_\tau^u + \frac{1}{1-\tau}\int_{x_\tau^u}^{\sup X} \tilde{P}^*(X > x)dx, \tag{20}$$

where $x_\tau^u$ and $\tilde{P}^*$ are defined in (17) and Proposition 6, respectively.

Let us calculate $x_\tau^u$.

- If $x < p$, then the event '$X > x$' implies only the element $\Omega$ of $S^+$, and hence $\tilde{P}^*(X > x) = 1$.
- If $p \leq x < t$, then the elements in $S^+$ implied by '$X > x$' are $E_1^c$ and $\Omega$, and hence $\tilde{P}^*(X > x) = P(E_1^c)$.
- If $t \leq x < y_2$, then the elements in $S^+$ implied by '$X > x$' are $E_1^c$, $E_2$, $E_1 \vee E_2$ and $\Omega$, and hence $\tilde{P}^*(X > x) = P(E_2)$.
- If $x \geq y_2$, then the event '$X > x$' is $\emptyset$, and hence $\tilde{P}^*(X > x) = 0$.

Recalling that $P(E_1) \leq \tau < 1 - P(E_2)$, we obtain that $x_\tau^u = t$. Now, using (20)

$$\overline{E}(X - p) = (t - p) + \frac{P(E_2)}{1-\tau}(y_2 - t).$$

This means that the amount of money to be reserved is made of a fixed term $(t-p)$ to cover non-critical losses, plus a term to cover the critical part of the loss, which for a given $\tau$ is proportional to the probability of experiencing a critical loss.

As the reader can easily verify, extending $P$ on the field (and lattice) $S^+ \cup \{E_2^c\}$ and modifying the PMM accordingly does not change the value of $\overline{E}(X-p)$, since $E_2^c$ is not implied by any event '$X > x$' when $x < y_2$. $\square$

It is worth noting that the less general formula (6) [16] can not be applied in Example 2 when $P$ is assessed only on $S^+$ or even on $S^+ \cup \{E_2^c\}$, as it requires $P$ to be defined on a much larger domain, (at least) the smallest field containing the events '$X > x$' and '$X < x$', $x \in [y_1; y_2]$. This highlights an operationally important feature of the generalization of the PMM introduced in Section 3.3: it supplies us with uncertainty evaluations for gambles which are not necessarily measurable with respect to the set of events the PMM is defined on. This set must be a lattice (like $S^+$). In particular it may also be a field (like $S^+ \cup \{E_2^c\}$), but could be considerably smaller than the minimal field meeting the measurability requirements underlying (6). This is a quite useful fact when eliciting too many beliefs may be hard, or somewhat arbitrary.

## 5. Conditioning the pari-mutuel model

Reconsider the basic PMM, with $\overline{P}(A)$, $\underline{P}(A)$ given by (1), (2), $A \in \mathcal{D}$, and $\mathcal{D}$ is now a *field* of events. We shall compute the natural extensions $\overline{E}(A|B)$, $\underline{E}(A|B)$ of $\overline{P}$ and $\underline{P}$ on $A|B$, with $B \in \mathcal{D}$, $B \neq \emptyset$. Since $\overline{P}$ and $\underline{P}$ are, respectively, 2-alternating and 2-monotone, from a well-known result ([15], Thm. 7.2; see also [12]), when $\underline{P}(B) > 0$:

$$\begin{aligned} \overline{E}(A|B) &= \frac{\overline{P}(A \wedge B)}{\overline{P}(A \wedge B) + \underline{P}(A^c \wedge B)}, \\ \underline{E}(A|B) &= \frac{\underline{P}(A \wedge B)}{\underline{P}(A \wedge B) + \overline{P}(A^c \wedge B)}. \end{aligned} \tag{21}$$

When $\underline{P}(B) = 0$, equations (21) do not apply, but it can be shown (directly, using Williams coherence, or alternatively from results in [16]) that

**Lemma 1.** *Given a coherent lower probability $\underline{P}$ on a set $\mathcal{D}$ of (unconditional) events, let $B \in \mathcal{D}$, $\underline{P}(B) = 0$. The natural extension $\underline{E}$ of $\underline{P}$ on $\mathcal{D} \cup \{A_1|B, \ldots, A_n|B\}$ is $\underline{E}(A_i|B) = 1$ if $B \Rightarrow A_i$, $\underline{E}(A_i|B) = 0$ otherwise, for $i = 1, \ldots, n$.*

Applying Lemma 1 for $n = 2$, $A_1 = A$, $A_2 = A^c$ and using conjugacy, it follows that, when $\underline{P}(B) = 0$ in the PMM, then $\underline{E}(A|B) = 0$, $\forall A$ such that $B \not\Rightarrow A$, and $\overline{E}(A|B) = 1$, $\forall A$ such that $A \wedge B \neq \emptyset$.

We assume in the sequel $\underline{P}(B) > 0$; note that by (2) $\underline{P}(B) > 0$ iff $P(B) > \frac{\delta}{\delta+1} = \tau$. Further, $\underline{P}(B) > 0$ ensures that the denominators in (21) are non-zero. Take $\overline{E}(A|B)$: using property 2.7.4 (d) in [16], $\overline{P}(A \wedge B) + \underline{P}(A^c \wedge B) \geq \underline{P}(B) > 0$. Similarly for $\underline{E}(A|B)$.

To derive $\overline{E}(A|B)$, from (21), two alternatives occur:

a) $\underline{P}(A^c \wedge B) = \max\{(1+\delta)P(A^c \wedge B) - \delta, 0\} = 0$. Hence $\overline{E}(A|B) = 1$.

$$
\begin{aligned}
\overline{P}(A) &= \begin{cases} \frac{P(A)}{1-\tau} & \text{if } \tau < P(A^c) \\ 1 & \text{if } \tau \geq P(A^c) \end{cases} \\
\underline{P}(A) &= \begin{cases} \frac{P(A)-\tau}{1-\tau} & \text{if } \tau < P(A) \\ 0 & \text{if } \tau \geq P(A) \end{cases} \\
\overline{E}(A|B) &= \begin{cases} \frac{P(A\wedge B)}{P(B)-\tau} & \text{if } \tau < P(A^c \wedge B) \\ 1 & \text{if } \tau \geq P(A^c \wedge B) \end{cases} \\
\underline{E}(A|B) &= \begin{cases} \frac{P(A\wedge B)-\tau}{P(B)-\tau} & \text{if } \tau < P(A \wedge B) \\ 0 & \text{if } \tau \geq P(A \wedge B) \end{cases}
\end{aligned}
$$

Table 1: Values of $\overline{P}(A)$, $\underline{P}(A)$, $\overline{E}(A|B)$, $\underline{E}(A|B)$.

b) $\max\{(1+\delta)P(A^c \wedge B) - \delta, 0\} > 0$. This happens iff $P(A^c \wedge B) > \frac{\delta}{1+\delta} = \tau$ and implies $\min\{(1+\delta)P(A \wedge B), 1\} < 1$ (otherwise $P(A \wedge B) \geq \frac{1}{1+\delta}$ and $P(B) > \frac{\delta}{\delta+1} + \frac{1}{1+\delta} = 1$). Hence we get $\overline{E}(A|B) = \frac{(1+\delta)P(A\wedge B)}{(1+\delta)(P(A\wedge B)+P(A^c\wedge B))-\delta} = \frac{P(A\wedge B)}{P(B)-\tau}$.

The derivation of $\underline{E}(A|B)$ is analogous:

a) If $\underline{P}(A \wedge B) = \max\{(1+\delta)P(A \wedge B) - \delta, 0\} = 0$, $\underline{E}(A|B) = 0$.

b) If $\max\{(1+\delta)P(A \wedge B) - \delta, 0\} > 0$, this implies $\tau < P(A \wedge B)$ and $\min\{(1+\delta)P(A^c \wedge B), 1\} < 1$; then $\underline{E}(A|B) = \frac{P(A\wedge B)-\tau}{P(B)-\tau}$.

Table 1 lists the values of $\overline{P}(A)$, $\underline{P}(A)$, $\overline{E}(A|B)$, $\underline{E}(A|B)$. They are written as functions of $\tau$, to simplify the inequalities in the 'if' clauses (referring to $\delta$, the clauses involve ratios of probabilities instead of probabilities). Note that the expressions for $\overline{E}(A|B)$, $\underline{E}(A|B)$ reduce to those for $\overline{P}(A)$, $\underline{P}(A)$ when $B = \Omega$.

### *5.1. Dilation and imprecision increase*

How does imprecision in the evaluations vary when conditioning in the PMM model? To supply some answers, we first recall two concepts.

*Definition* 2. Given a partition of non-impossible events $\mathbb{P}$, we say that *(weak) dilation* occurs (with respect to $A$ and $\mathbb{P}$) when

$$\underline{P}(A|B) \leq \underline{P}(A) \leq \overline{P}(A) \leq \overline{P}(A|B), \forall B \in \mathbb{P}, \tag{22}$$

while there is an *imprecision increase* when

$$\overline{P}(A) - \underline{P}(A) \leq \overline{P}(A|B) - \underline{P}(A|B), \forall B \in \mathbb{P}. \tag{23}$$

Dilation is a so far little investigated phenomenon (see [8, 14]), which implies that our a posteriori opinions on $A$ will be *vaguer* and hence also *more imprecise* (at least in a weak sense, if the first or last weak inequalities in (22) are equalities) than the a priori ones, *no matter* which $B \in \mathbb{P}$ is true. Even though dilation is $\mathbb{P}$-dependent (so that we may hope that a well-chosen partition $\mathbb{P}$ avoids dilation), it is a puzzling phenomenon. Clearly, dilation implies the weaker concept of imprecision increase, which captures one of the two basic features of dilation, the growth in the degree of imprecision.

To discuss the occurrence of dilation or imprecision increase in the PMM, we assume that the conditional imprecise probabilities on each $A|B$ are the natural extensions $\overline{E}(A|B)$, $\underline{E}(A|B)$ in Table 1. In this way we obtain the (W-coherent) natural extension of $\underline{P}$ (or its conjugate $\overline{P}$) on $\mathcal{D} \cup \{A|B : B \in \mathbb{P}\}$, cf. the Appendix in [11]. As for partition $\mathbb{P}$, our prior assumption $\underline{P}(B) > 0$, $\forall B \in \mathbb{P}$, being equivalent by (2) to $P(B) > \tau$, $\forall B \in \mathbb{P}$, implies that $\mathbb{P}$ is *finite*. Removing this restriction would lead to versions of the following results formally, but not operationally more general. This is because conditioning the PMM probabilities on the events of a partition $\mathbb{P}$ may produce non-vacuous and non-trivial results only for finitely many $B \in \mathbb{P}$. In fact, there is a finite number of $B$ such that $P(B) > \tau$ for a given $\tau$, while when $P(B) \leq \tau$ then $\underline{P}(B) = 0$ and the PMM natural extension on $A|B$ is either trivial or vacuous by Lemma 1.

We present now a number of results concerning dilation.

**Proposition 10.** *Let $\underline{P}$, $\overline{P}$ be defined by the PMM on a field of events $\mathcal{D}$ and let $\underline{E}(A|B)$, $\overline{E}(A|B)$ be the corresponding natural extensions on $A|B$, with $B \in \mathbb{P}$. Then,*

$$\underline{E}(A|B) \leq \underline{P}(A) \text{ iff } \quad P(A^c \wedge B^c) = 0 \text{ or } \tau \geq \tfrac{P(A \wedge B) - P(A)P(B)}{P(A^c \wedge B^c)}, \tag{24}$$

$$\overline{E}(A|B) \geq \overline{P}(A) \text{ iff } \quad P(A \wedge B^c) = 0 \text{ or } \tau \geq \tfrac{P(A)P(B) - P(A \wedge B)}{P(A \wedge B^c)}. \tag{25}$$

*Remark* 2. At most one of the two weak inequalities in (24), (25) has to be checked, since the signs of the numerators of the fractional terms are opposite and $\tau > 0$. □

**Proposition 11.** *In the PMM, define:* $M_L = 0$ *if* $P(A^c \wedge B^c) = 0 \ \forall B \in I\!\!P$, $M_L = \max\{\frac{P(A\wedge B)-P(A)P(B)}{P(A^c\wedge B^c)} : B \in I\!\!P, P(A^c \wedge B^c) \neq 0\}$ *otherwise;* $M_U = 0$, *if* $P(A \wedge B^c) = 0 \ \forall B \in I\!\!P$, $M_U = \max\{\frac{P(A)P(B)-P(A\wedge B)}{P(A\wedge B^c)} : B \in I\!\!P, P(A \wedge B^c) \neq 0\}$ *otherwise;* $M = \max\{M_L, M_U\}$. *Then dilation occurs if and only if* $\tau \geq M$.

**Proof.** (22) holds iff both (24) and (25) hold $\forall B \in I\!\!P$, i.e. iff $\tau \geq M$. □

Proposition 11 fully solves the problem of characterising dilation for the natural extension of the PMM. Yet there are some interesting sufficient conditions for dilation, given in the following corollary.

**Corollary 2.** *Dilation occurs in the PMM* for any $\tau \in ]0;1[$, *if either of the following conditions holds:*

*(a)* $P(A \wedge B) = P(A)P(B), \forall B \in I\!\!P$.

*(b)* $P$ *is uniform on* $I\!\!P = \{B_1, \dots, B_n\}$ *and* $P(A \wedge B_i) = k > 0$ $(i = 1, \dots, n)$.

**Proof.** Condition (a) ensures dilation, as it implies $M = 0$ in Proposition 11. As for (b), it implies $P(B_i) = \frac{1}{n}$, $P(A \wedge B_i) = \frac{P(A)}{n}$ and therefore condition (a). □

Note that the conditions in Corollary 2 are independent of $\tau$ and that like other models, including the TVM and the $\varepsilon$-contamination model [8], non-correlation under $P$ between $A$ and each $B \in I\!\!P$ causes dilation.

Concerning imprecision increase, the following sufficient condition holds:

**Proposition 12.** *Imprecision increases if* $(\tau < \min\{P(A \wedge B), P(A^c \wedge B)\}$ *or* $\tau \geq \max\{P(A \wedge B), P(A^c \wedge B)\})$, $\forall B \in I\!\!P$.

**Proof.** Check that (23) holds, using Table 1. □

In the special case of Proposition 12 that $\tau \geq \max\{P(A\wedge B), P(A^c\wedge B)\}$, $\forall B \in I\!\!P$, dilation is ensured too, but the inferences via natural extension are trivial. See Example 3 for a non-trivial case of equivalence between dilation and imprecision increase.

### *5.2. Imprecision variation in practice*

Since Proposition 11 characterises dilation in the PMM, we just have to check whether its condition $\tau \geq M$ applies or not to shape dilation in any practical problem. The matter is less immediate with imprecision increase, when it is not already implied by dilation: only the sufficient condition in Proposition 12 is available. As a general remark, this kind of investigation should distinguish more cases, according to the relative orderings in $[0; 1]$ of $P(A \wedge B)$, $P(A^c \wedge B)$, $\forall B \in I\!P$, and $\tau$. However, the importance of each case varies greatly in the applications. We supply some results for the most significant ones, while the remaining may be analyzed using Table 1 to check (23), as demonstrated in Example 4. Consider then the following situations

(a) the commission $\tau$ is 'low';

(b) event $A$ is 'rare'.

Case (a) is probably the most important: $\tau$ will often be rather low, recalling that it has the meaning of a commission or taxation (this happens for instance with Internet betting). If 'low' means that $\tau < M$, dilation does not occur by Proposition 11. However, and perhaps surprisingly, imprecision increases if $\tau$ is 'too low', meaning with this that $\tau < \min\{P(A \wedge B), P(A^c \wedge B)\}$, $\forall B \in I\!P$. In fact, the 'if' condition in Proposition 12 is true under this assumption, which also tends to rule out case (b).

We do not necessarily meet case (a) when $P(A)$ is smaller than the commission $\tau$ in favour of House or of an insurer (this is relatively frequent in non-life insurance). In such instances, the study of imprecision increase is typically split into a number of sub-cases, cf. Example 4 later on.

An important instance is that of rare or *extremal* events [6], a basic concept in several applications such as large insurance claims, stock market shocks, climate phenomena, and so on. It corresponds to case (b). We present a simple, although common situation of this kind in Example 3. Here, like Example 4, the partition of conditioning events $I\!P = \{B, B^c\}$ is binary.

*Example* 3. Suppose that $A$ is 'rare', while $A^c \wedge B$, $A^c \wedge B^c$ are not. Thus we may assume that $P(A) \leq \tau < \min\{P(A^c \wedge B), P(A^c \wedge B^c)\}$. Then (see Table 1) $\overline{P}(A) = P(A)/(1-\tau)$, $\overline{E}(A|B) = P(A \wedge B)/(P(B) - \tau)$, $\overline{E}(A|B^c) = P(A \wedge B^c)/(P(B^c) - \tau)$, $\underline{E}(A|B) = \underline{E}(A|B^c) = \underline{P}(A) = 0$. Substituting these values into (22) and (23), we see that the inequalities in (22) are the same as those in (23). That is, *in this case*, there is dilation iff there is imprecision increase. We may then apply Proposition 11 to determine both by computing $M$, or just $M_U$ in this example. In fact $M = M_U$ since condition $\tau \geq M_L$ ensures that $\max\{\underline{E}(A|B), \underline{E}(A|B^c)\} \leq \underline{P}(A)$ holds, which we already know to be trivially true in the form $0 = 0$. For instance, if $P(A) = 0.02$, $P(A \wedge B) = 0.005$, $P(A \wedge B^c) = 0.015$, $P(B) = 0.4$, we obtain $M_U = \frac{1}{5}$ (and $M_L = \frac{3}{395} < P(A) \leq \tau$), thus there is dilation (and imprecision increase) for $\tau$ such that $0.2 \leq \tau < 0.395 = \min\{P(A^c \wedge B), P(A^c \wedge B^c)\}$, none of them for $\tau \in [0.02; 0.2[$. □

*Example* 4. Assign $P$ as follows: $P(A \wedge B) = \frac{1}{10}$, $P(A \wedge B^c) = P(A^c \wedge B^c) = \frac{1}{5}$, $P(A^c \wedge B) = \frac{1}{2}$. Consequently $P(A) = \frac{3}{10}$, $P(B) = \frac{3}{5}$, $P(A|B) = \frac{1}{6}$, $P(A|B^c) = \frac{1}{2}$. To study dilation for the corresponding PMM using Proposition 11, we calculate $\Delta = P(A \wedge B^c) - P(A)P(B^c) = P(A)P(B) - P(A \wedge B) = \frac{2}{25} > 0$, $M_L = \frac{\Delta}{P(A^c \wedge B)} = \frac{4}{25}$, $M_U = \frac{\Delta}{P(A \wedge B^c)} = \frac{2}{5}$. Therefore, Proposition 11 guarantees that dilation occurs iff $\tau \geq \max\{M_L, M_U\} = \frac{2}{5}$.

As for imprecision increase, it is ensured by Proposition 12 when $\tau < \frac{1}{10}$ or $\tau \geq \frac{1}{2}$, but in the latter case we already know that dilation, and hence imprecision increase as well, occurs for $\tau \geq \frac{2}{5}$. For $\tau \in [\frac{1}{10}; \frac{2}{5}[$, we have to check whether the inequalities (23) hold, distinguishing more subcases according to the different expressions for $\overline{P}(A)$, $\underline{P}(A)$, $\overline{E}(A|B)$, $\underline{E}(A|B)$, $\overline{E}(A|B^c)$, $\underline{E}(A|B^c)$. Conditioning on $B^c$, we should check whether

$$\overline{E}(A|B^c) - \underline{E}(A|B^c) \geq \overline{P}(A) - \underline{P}(A). \tag{26}$$

Now, $\overline{E}(A|B^c) - \underline{E}(A|B^c) = 1$ and (26) therefore holds if $\tau \in [\frac{1}{5}; \frac{2}{5}[$, while (26) specialises into $\frac{\tau}{P(B^c)-\tau} \geq \frac{\tau}{1-\tau}$ when $\tau \in [\frac{1}{10}; \frac{1}{5}[$, and this inequality is true. Therefore (26) is verified for $\tau \in [\frac{1}{10}; \frac{2}{5}[$, and imprecision increase in this interval depends only on whether the inequality $\overline{E}(A|B) - \underline{E}(A|B) \geq \overline{P}(A) - \underline{P}(A)$ holds. Noting that $\overline{E}(A|B) - \underline{E}(A|B) = \frac{P(A \wedge B)}{P(B)-\tau} = \frac{1}{6-10\tau}$, $\forall \tau \in [\frac{1}{10}; \frac{2}{5}[$, we have to check whether:

$$\begin{array}{lll} \frac{1}{6-10\tau} & \geq \frac{P(A)}{1-\tau} = \frac{3}{10(1-\tau)} & \text{if } \tau \in [\frac{3}{10}; \frac{2}{5}[ \\ \frac{1}{6-10\tau} & \geq \frac{\tau}{1-\tau} & \text{if } \tau \in [\frac{1}{10}; \frac{3}{10}[. \end{array}$$

The former inequality has no solution in $[\frac{3}{10}; \frac{2}{5}[$, the latter is true for $\tau \in [\frac{1}{10}; \frac{1}{5}]$. Conclusions: dilation occurs iff $\tau \in [\frac{2}{5}; 1[$, imprecision increase but not dilation iff $\tau \in ]0; \frac{1}{5}]$, neither of them iff $\tau \in ]\frac{1}{5}; \frac{2}{5}[$. □

*5.3. Imprecision variation and partition refinement*

Limiting dilation or imprecision increase in the PMM is not straightforward. This may be achieved by an appropriate choice of $\tau$ in some, but not all cases (for instance, $\tau \in [\frac{1}{5}; \frac{2}{5}[$ might be too high a percentage in Example 4). More generally, an obvious alternative is to choose a coherent extension other than the natural extension. This often shrinks imprecision, by the dominance properties of the natural extension, but finding a computationally simple such extension may be not so easy in practice.

In [8] the following interpretation for dilation is outlined: one may think that $\underline{P}(A|B)$ ($\overline{P}(A|B)$) is the lower (upper) envelope of a set of precise probabilities, each representing the opinion of a single expert, and that $\underline{P}(A|B) \leq \underline{P}(A)$ and $\overline{P}(A|B) \geq \overline{P}(A)$ because there is disagreement among the experts about the effect of observing $B$ on the probability of $A$. This suggests that we might refine the starting partition $I\!\!P$, for instance splitting $B$ into $B_1 \vee B_2 \ldots \vee B_m$, $B_i \wedge B_j = \emptyset$ if $i \neq j$, such that there is a larger consensus on the effect of each $B_i$ on $A$. Intuitively, refining the partition of the conditioning events should hinder dilation, since the number of constraints in Proposition 10 increases. One might expect a decrease in imprecision too, but this cannot be taken for sure. To give an idea of the situation, consider the special case that $A \Rightarrow B_1 \Rightarrow B$. If a dF-coherent precise probability $P$ is assessed on the relevant conditional events, it is well known that $P(A|B_1)P(B_1|B) = P(A|B)$, hence $P(A|B_1) \geq P(A|B)$: uncertainty decreases (or does not increase, at least) when conditioning in a narrower environment. What happens in the same case, if $P$ is replaced by imprecise probabilities? The next proposition gives the answer:

**Proposition 13.** *Let $A \Rightarrow B_1 \Rightarrow B$, and let $\overline{P}$ ($\underline{P}$) be a W-coherent upper (lower) probability, such that the expressions below are well-defined. Then*

$$\overline{P}(A|B_1) \geq \overline{P}(A|B), \tag{27}$$

$$\underline{P}(A|B_1) \geq \underline{P}(A|B). \tag{28}$$

From Proposition 13, introducing a more refined partition has an upward *shift effect* on the conditional imprecise probabilities concerning $A$. While this tends to make dilation more difficult, but not necessarily impossible (note that the shift in (28) tends to reduce dilation, or even to prevent it if such that $\underline{P}(A|B_1) > \underline{P}(A)$), the overall effect on imprecision variation is unsure (the shift in (27) tends to increase it), and it is possible that $\overline{P}(A|B_1) - \underline{P}(A|B_1) >$

$\overline{P}(A|B) - \underline{P}(A|B)$. Outside the case $A \Rightarrow B_1$, formulae less direct than (27), (28) apply, but analogous conclusions may be drawn.

## 6. Conclusions

The pari-mutuel model represents a simple and natural way of eliciting upper and lower probabilities, and can be extended in more directions, thanks to the availability of standard procedures for 2-monotone and 2-alternating previsions. We computed explicitly its natural extension $\overline{E}$ starting from a PMM assignment on a lattice of events, generalizing the approach in [16], which is anyway discussed, focusing on comparing the different formulae available for $\overline{E}$. While a naive extension, considered in insurance premium pricing, does not seem to be a valuable alternative to the natural extension, being generally not coherent, the various formulae for the natural extension have a notable meaning in risk measurement. In fact, they correspond to known measures of risk or generalize them. We discussed also how to use the natural extension when conditioning, characterising dilation and investigating imprecision increase for the PMM.

A tempting new direction would, in a sense, merge our analysis in the conditional and unconditional framework, studying the natural extension to conditional gambles. Here a difficulty arises: available generalizations of equations (21), studied in [12], are lower/upper bounds for the natural extension and might not be attained, even when $\overline{P}$ is 2-alternating. In other words, the available procedures seem to give weaker results.

This and the considerations in Section 5.3 on how to limit dilation or imprecision increase might motivate investigating coherent extensions of the PMM alternative to the natural extension.


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## A. Proofs of the main results

*Proof of Proposition 6.* By Proposition 3 (a), $\overline{E}(B) = \overline{P}^*(B) = \inf\{\min\{(1+\delta)P(A), 1\} : A \in S^+, B \Rightarrow A\}$. Defining $L_B = \{A \in S^+ : B \Rightarrow A, (1+\delta)P(A) < 1\}$, $L_B = \varnothing$ iff $(1+\delta)\tilde{P}^*(B) \geq 1$.

Two cases may occur: if $L_B = \varnothing$, that is if $(1+\delta)\tilde{P}^*(B) \geq 1$, then $\overline{E}(B) = 1$; if $L_B \neq \varnothing$, that is if $(1+\delta)\tilde{P}^*(B) < 1$, $\overline{E}(B) = \inf\{(1+\delta)P(A) : A \in L_B\} = (1+\delta)\inf\{P(A) : A \in L_B\} = (1+\delta)\tilde{P}^*(B)$. In summary, equation (15) holds. □

*Proof of Proposition 7.* Apply (4) and Proposition 3, (a) substituting $\overline{P}^* = \overline{E}$ with its expression in equation (15), getting

$$\overline{E}(X) = \inf X + \int_{\inf X}^{\sup X} \min\{(1+\delta)\tilde{P}^*(X > x), 1\}dx. \tag{29}$$

From here, the derivation of (16) is similar to that just sketched in [16], Section 3.2.5, to obtain (6). We detail the proof here for the sake of completeness. Since $x < x^u_\tau$ (alternatively $x > x^u_\tau$) implies $\tilde{P}^*(X > x) \geq 1 - \tau = \frac{1}{1+\delta}$ (alternatively $\tilde{P}^*(X > x) < \frac{1}{1+\delta}$), we get from (29)

$$\begin{aligned}\overline{E}(X) = \inf X + \int_{\inf X}^{x^u_\tau} dx + (1+\delta)\int_{x^u_\tau}^{\sup X} \tilde{P}^*(X > x)dx = \\ x^u_\tau + (1+\delta)\int_{x^u_\tau}^{\sup X} \tilde{P}^*(X > x)dx.\end{aligned} \tag{30}$$

We prove now that $\overline{E}_P((X - x^u_\tau)^+) = \int_{x^u_\tau}^{\sup X} \tilde{P}^*(X > x)dx$. For this, we apply Proposition 3 to the coherent, 2-alternating (upper) probability $\tilde{P}^*$. Since $\inf(X - x^u_\tau)^+ = 0$ and $\sup(X - x^u_\tau)^+ = \sup X - x^u_\tau$, as ensues using also a property analogous to Proposition 4 b), and since, for $x \geq 0$, $(X - x^u_\tau)^+ > x$ iff $X - x^u_\tau > x$, we get

$$\overline{E}_P((X - x^u_\tau)^+) = \int_0^{\sup X - x^u_\tau} \tilde{P}^*(X - x^u_\tau > x)dx = \int_{x^u_\tau}^{\sup X} \tilde{P}^*(X > x)dx,$$

where integration by substitution is employed in the last equality. □

*Proof of Proposition 8.* Noting that $(X - x^u_\tau)^+ = (X - x^u_\tau)I_{X > x^u_\tau}$ and by subadditivity of coherent upper previsions and, at the second equality, the

GBR,[6,7] $\overline{E}_P((X-x^u_\tau)^+) = \overline{E}_P((X-x^u_\tau)I_{X>x^u_\tau}) \leq \overline{E}_P(I_{X>x^u_\tau}(X-\overline{E}_P(X|X > x^u_\tau))) + \overline{E}_P(I_{X>x^u_\tau}(\overline{E}_P(X|X > x^u_\tau) - x^u_\tau)) = \overline{E}_P(I_{X>x^u_\tau}(\overline{E}_P(X|X > x^u_\tau) - x^u_\tau)) = \overline{E}_P(I_{X>x^u_\tau})(\overline{E}_P(X|X > x^u_\tau) - x^u_\tau) \stackrel{\text{def}}{=} \lambda$.

Using also the definition of $\varepsilon^u$ and $\lambda$, we get further $x^u_\tau + (1+\delta)\lambda = x^u_\tau(1-(1+\delta)\overline{E}_P(X > x^u_\tau)) + (1+\delta)(\lambda + x^u_\tau\overline{E}_P(X > x^u_\tau)) = \varepsilon^u x^u_\tau + (1+\delta)(\overline{E}_P(X > x^u_\tau)(\overline{E}_P(X|X > x^u_\tau) - x^u_\tau) + x^u_\tau\overline{E}_P(X > x^u_\tau)) = \varepsilon^u x^u_\tau + (1-\varepsilon^u)\overline{E}_P(X|X > x^u_\tau)$.

Finally, by (16) and the expressions above, $\overline{E}(X) = x^u_\tau + (1+\delta)\overline{E}_P((X-x^u_\tau)^+) \leq x^u_\tau + (1+\delta)\lambda = \varepsilon^u x^u_\tau + (1-\varepsilon^u)\overline{E}_P(X|X > x^u_\tau)$. □

*Proof of Proposition 10.* To prove (24), we use Table 1 to choose the appropriate values of $\underline{E}$, $\underline{P}$ and substitute them into $\underline{E}(A|B) \leq \underline{P}(A)$.

If $\tau < P(A \wedge B)$, $\underline{E}(A|B) \leq \underline{P}(A)$ becomes $\frac{P(A\wedge B)-\tau}{P(B)-\tau} \leq \frac{P(A)-\tau}{1-\tau}$, which, with simple calculations and recalling the well-known inclusion-exclusion principle $P(A\vee B)+P(A\wedge B) = P(A)+P(B)$, is easily seen to be equivalent to $P(A\wedge B)-P(A)P(B) \leq \tau P(A^c\wedge B^c)$. When $P(A^c\wedge B^c) = 0$, this inequality is satisfied for any value of $\tau$, since, in this case, $P(A\wedge B) - P(A)P(B) = P(A)+P(B)-1-P(A)P(B) = (1-P(B))(P(A)-1) \leq 0$ (use the inclusion-exclusion principle again in the first equality). Otherwise, it is clearly satisfied for $\tau \geq \frac{P(A\wedge B)-P(A)P(B)}{P(A^c\wedge B^c)}$.

When $\tau \geq P(A \wedge B)$, the inequality $\underline{E}(A|B) \leq \underline{P}(A)$ is trivially satisfied. To conclude the proof of (24), we observe that, when $P(A^c \wedge B^c) > 0$, some algebraic calculations show that $\frac{P(A\wedge B)-P(A)P(B)}{P(A^c\wedge B^c)} \leq P(A \wedge B)$.

The proof of (25) is analogous. □

*Proof of Proposition 13.* We start by establishing (27). When $\overline{P}$ is a W-coherent upper prevision, $X$ a gamble, $C$ an event and $I_C$ its indicator, the following inequality holds:

$$\text{If } \overline{P}(X|B \wedge C) > 0, \overline{P}(I_C X|B) \leq \overline{P}(C|B)\overline{P}(X|B \wedge C). \tag{31}$$

Inequality (31) can be proven in a way quite analogue to the proof of the opposite inequality for lower previsions given in [12], Proposition 3.1 (a). Putting $C = B_1$, $X = A$, (31) specialises into $\overline{P}(A|B) \leq \overline{P}(B_1|B)\overline{P}(A|B_1)$, which implies (27).

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[6]Recall also that the natural extension $\overline{E}_P$ always exists with W-coherence, cf. [11].

[7]When $\underline{E}_P(X > x^u_\tau) = 0$, there is an infinite number of possible values satisfying the GBR. In this case, it can be proved that any of them can replace $\overline{E}_P(X|X > x^u_\tau)$ in Proposition 8.

Now we consider (28). We use the definition of W-coherence (Definition 1 in Section 2) to show that (28) is necessary to ensure $\sup \underline{G}|B \geq 0$, for a specific $\underline{G}$ obtained from betting on $A|B$, $A|B_1$ with stakes $1, -1$ respectively. In fact, using $I_B = I_{B_1} + I_B I_{B_1^c}$ at the second equality,

$$\begin{aligned}\underline{G} &= I_B(A - \underline{P}(A|B)) - I_{B_1}(A - \underline{P}(A|B_1)) \\ &= I_{B_1}(A - \underline{P}(A|B)) + I_B I_{B_1^c}(A - \underline{P}(A|B)) - I_{B_1}(A - \underline{P}(A|B_1)) \\ &= I_{B_1}(\underline{P}(A|B_1) - \underline{P}(A|B)) + I_B I_{B_1^c}(A - \underline{P}(A|B)).\end{aligned}$$

Then, $\underline{G}|B$ has only two possible values: $\underline{G}|B = \underline{P}(A|B_1) - \underline{P}(A|B)$ when $B_1$ and $B$ are true, $\underline{G}|B = -\underline{P}(A|B) \leq 0$ when $B_1$ is false and $B$ is true.

Hence (28) holds, trivially when $\underline{P}(A|B) = 0$, to guarantee that $\sup \underline{G}|B \geq 0$, when $\underline{P}(A|B) > 0$. □

## References


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