CONGLOMERABLE NATURAL EXTENSION

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Abstract. At the foundations of probability theory lies a question that has been open since de Finetti framed it in 1930: whether or not an uncertainty model should be required to be conglomerable. Conglomerability is related to accepting infinitely many conditional bets. Walley is one of the authors who have argued in favor of conglomerability, while de Finetti rejected the idea. In this paper we study the extension of the conglomerability condition to two types of uncertainty models that are more general than the ones envisaged by de Finetti: sets of desirable gambles and coherent lower previsions. We focus in particular on the weakest (i.e., the least-committal) of those extensions, which we call the *conglomerable natural extension*. The weakest extension that does not take conglomerability into account is simply called the natural extension. We show that taking the natural extension of assessments after imposing conglomerability—the procedure adopted in Walley's theory—does not yield, in general, the conglomerable natural extension (but it does so in the case of the marginal extension). Iterating this process of imposing conglomerability and taking the natural extension produces a sequence of models that approach the conglomerable natural extension, although it is not known, at this point, whether this sequence converges to it. We give sufficient conditions for this to happen in some special cases, and study the differences between working with coherent sets of desirable gambles and coherent lower previsions. Our results indicate that it is necessary to rethink the foundations of Walley's theory of coherent lower previsions for infinite partitions of conditioning events.

1. Introduction

Consider an experiment whose non-empty set of possible outcomes is the so-called *possibility space* Ω. Suppose you are offered a *gamble* f: a bounded real-valued function on Ω. It represents an uncertain reward as it depends on the outcome of the experiment. You find out that, whatever you might observe, as expressed by an event B in a certain partition $\mathcal{B}$ of Ω, you would accept f conditional on B. Does this imply that you should *unconditionally* accept f?

This question can be conveniently addressed using the notion of *desirability*, which leads to a very general way of dealing with uncertainty. Common rationality axioms for desirability—these are also called *coherence* axioms—, such as those in [21, Section 3.7] or [23], imply that f should indeed be accepted, if $\mathcal{B}$ is finite. When $\mathcal{B}$ is infinite, some authors have proposed to impose the above requirement through an additional axiom of so-called *conglomerability*. In fact, conglomerability (suitably reformulated in a more recognisably probabilistic manner) is a foundational axiom for Walley's theory of coherent lower previsions when the conditioning partition is infinite. We recall here that a *coherent lower prevision* is a lower envelope of *linear previsions*, each of which is the expectation functional associated with an additive probability.

The notion of conglomerability was originally introduced by de Finetti [4, 6] as a property that a finitely additive—but not countably additive—probability may or may not satisfy. In fact, de Finetti was also the first to reject the idea that

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conglomerability should be imposed. The concept was studied later by Dubins [9], who established a connection with the notion of *disintegrability*. Conglomerability has also been studied by Seidenfeld, Schervisch and Kadane in a number of papers [16, 17, 18], and by Doria [8]. In particular, in [16] it is shown that countable additivity is necessary, but for a few pathological cases, for *full* conglomerability, that is, for conglomerability with respect to all possible partitions of Ω.[1]

Imposing conglomerability, even with respect to only a single partition $\mathcal{B}$, comes at the expense of mathematical properties that might be considered undesirable: for example, a conglomerable coherent lower prevision may not be the lower envelope of conglomerable linear previsions. Perhaps also because of this, the idea of conglomerability was rejected in some extensions of de Finetti's work, like Williams's [23] (see also [15]).

In this paper, we do not wish to take any philosophical position on whether it is reasonable to require conglomerability with respect to a partition we envisage conditioning on.[2] But we do think that requiring full conglomerability, rather than requiring conglomerability only for the partitions that are actually used for updating beliefs, is rather more questionable: in specifying beliefs, it seems to be useful, and sometimes even essential, to envisage beforehand which partitions we will want to condition on. Automatic conditioning can indeed lead to problems: see [19] for a clear exposition of this point of view. This is also the approach taken by De Cooman and Hermans with their 'cut conglomerability' in [2]. Our aim here is to perform a mathematical study of the impact of conglomerability on the possible extensions of an initial set of desirability, or probabilistic, assessments. The focus is, in particular, on what we call the *conglomerable natural extension*: loosely speaking, this is the weakest (i.e., least-committal) conglomerable and coherent model that extends given assessments. A related concept is the *natural extension*, which is defined as the weakest coherent extension, and where conglomerability is not imposed.

We start in Section 2 by introducing some basic notions: desirability, along with its characterising axioms; coherent lower previsions induced by a set of desirable gambles, and the set of desirable gambles induced by some coherent lower previsions. Moreover, we introduce conglomerability in a few different forms: for desirable gambles, in the traditional form and in a weaker variant; for coherent lower previsions, in the traditional way and in a strengthened form. We show how these notions are related, which allows us to transform problems written for one type of model into the other.

In Section 3 we focus on desirability. We show that the conglomerable natural extension $\mathcal{F}$, provided that it exists, of a set $\mathcal{R}$ of desirable gambles with respect to to a partition $\mathcal{B}$, is the intersection of all conglomerable sets of desirable gambles including $\mathcal{R}$. Moreover, we relate $\mathcal{F}$ to the natural extension: we start from $\mathcal{R}$, close it with respect to conglomerability, and take its natural extension, obtaining $\mathcal{E}_1$; we iterate this process, yielding $\mathcal{E}_2, \ldots, \mathcal{E}_n, \ldots$. We show that $\mathcal{E}_n \subseteq \mathcal{F}$ for all n, and that the sequence stabilises (becomes constant) if and only if one if its elements coincides with $\mathcal{F}$. We provide sufficient conditions for this to happen, as well as a few examples to illustrate the situation. One of them, in particular, shows that taking the closure with respect to conglomerability may extend a non-conglomerable set of desirable gambles beyond its topological border.

In Section 4 we study the conglomerable natural extension $\underline{F}$ of a coherent lower prevision $\underline{P}$ with respect to a partition $\mathcal{B}$. Here, too, we consider a sequence: we

[1]See also [10, Section 6 and Theorem A1] and [20, Example 3] for some interesting examples showing that countable additivity is not always sufficient for full conglomerability, and [21, Section 6.9] for a discussion of this matter within Walley's theory.

[2]This is also called *partial conglomerability*. In this paper by conglomerability we just mean partial conglomerability.

start from $\underline{P}$, compute its conditional natural extension $\underline{E}(\cdot|\mathcal{B})$, and then the natural extension of the two of them together, $\underline{E}_1$; we iterate the process, yielding $\underline{E}_2, \dots, \underline{E}_n, \dots$. We show that $\underline{E}_n \leq \underline{F}$ for all n, and again that the sequence stabilises if and only if one of its elements coincides with $\underline{F}$. Then we provide what is arguably the most important result of this paper: we show in Example 5 that $\underline{E}_1$ may not equal $\underline{F}$. The importance of this example stems from the fact that, when it comes to the natural extension (as well as to coherence), Walley's theory is implicitly based on stopping at the first element of the sequence: $\underline{E}_1$. We show that this is not enough to fully capture the implications of conglomerability. This raises the need to rethink the foundations of Walley's theory when a model is based on an infinite conditioning partition. We also give sufficient conditions for $\underline{E}_1 = \underline{F}$.

In Section 5 we relate the results obtained for sets of desirable gambles and coherent lower previsions: we start from a set of desirable gambles $\mathcal{R}$ and deduce from this a coherent lower prevision $\underline{P}$; we create the sequences of sets of desirable gambles, on the one hand, and coherent lower previsions, on the other. We explore the relationship between the elements of the sequences. This allows us, in Example 7, to exploit Example 5 and show that also $\mathcal{E}_1$ need not coincide with $\mathcal{F}$: this means that one-step conglomerability is not enough for sets of desirable gambles either. We give sufficient conditions for $\mathcal{E}_1 = \mathcal{F}$, as well as for the two sequences to be made out of equivalent models.

Section 6 deepens the connection between conglomerability and coherence. Coherence is arguably the most important notion both for desirable gambles and lower previsions. Loosely speaking, a coherent model is one that is self-consistent. We investigate to what extent the conglomerability of a set of gambles implies that it is coherent with the conditional set of gambles it induces; moreover we show that any coherent pair of conditional and unconditional lower previsions can be derived from a single conglomerable set of desirable gambles.

Finally, in Section 7 we focus on the problem where more than one partition is considered. We focus in particular on the important case where information is represented in a hierarchical way through the *marginal extension* (see [21, Theorem 6.7.2], [12]), which is a generalisation of the law of iterated expectation to sets of desirable gambles. We show that in this case $\mathcal{E}_1 = \mathcal{F}$: one-step conglomerability yields the conglomerable natural extension. We provide our concluding views and some remarks in Section 8.

2. Introduction to imprecise probabilities

2.1. **Coherent lower previsions.** Let us introduce the basics of the theory of coherent lower previsions that we use in this paper. We refer to [21] for an in-depth study, and to [11] for a survey.

Consider a possibility space Ω. A *gamble* is a bounded map $f\colon \Omega \to \mathbb{R}$. The set of all gambles is denoted by $\mathcal{L}(\Omega)$, or simply by $\mathcal{L}$ when there is no ambiguity about the possibility space we are working with. In particular, we use $f \lneq 0$ to denote a gamble $f \leq 0, f \neq 0$ (and we will refer to this as a *negative* gamble), and $f \gneq 0$ to denote a gamble $f \geq 0, f \neq 0$ (this will be called a *positive* gamble). We use the notation $\mathcal{L}^+(\Omega)$, or simply $\mathcal{L}^+$, to refer to the set of positive gambles.

A *lower prevision* $\underline{P}$ is a real-valued functional defined on some set of gambles $\mathcal{K} \subseteq \mathcal{L}$. When the domain $\mathcal{K}$ of $\underline{P}$ is a linear space—closed under point-wise addition and multiplication by real numbers—$\underline{P}$ is called *coherent* when it satisfies the following conditions:

C1. $\underline{P}(f) \geq \inf f$ for all gambles $f \in \mathcal{K}$;
C2. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ for all gambles $f \in \mathcal{K}$ and all positive real λ;
C3. $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g)$ for all gambles $f, g \in \mathcal{K}$.

Given a partition $\mathcal{B}$ of Ω, a *conditional lower prevision* on $\mathcal{L}$ is a functional $\underline{P}(\cdot|\mathcal{B}) := \sum_{B\in\mathcal{B}} B\underline{P}(\cdot|B)$ such that for every set $B \in \mathcal{B}$, $\underline{P}(\cdot|B)$ is a lower prevision on $\mathcal{L}$. Note the use, in this case, of B to denote the indicator function of event $B \in \mathcal{B}$; we shall use this notation repeatedly in the paper. $\underline{P}(\cdot|\mathcal{B})$ is called *separately coherent* when $\underline{P}(\cdot|B)$ is coherent and $\underline{P}(B|B) = 1$ for every $B \in \mathcal{B}$. For every gamble f, $\underline{P}(f|\mathcal{B})$ is a gamble on Ω that is constant on the elements of $\mathcal{B}$; such gambles are called $\mathcal{B}$*-measurable.*

For every lower prevision $\underline{P}$ and every conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, we use the notations:

$$G_{\underline{P}}(f) := f - \underline{P}(f),\ G_{\underline{P}}(f|B) := B(f - \underline{P}(f|B)),$$
$$G_{\underline{P}}(f|\mathcal{B}) := f - \underline{P}(f|\mathcal{B}) = \sum_{B\in\mathcal{B}} G_{\underline{P}}(f|B).$$

If we consider a coherent lower prevision $\underline{P}$ on $\mathcal{L}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$, they are called *coherent*[3] if and only if for every gamble f and every $B \in \mathcal{B}$, $\underline{P}(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ and

$$\underline{P}(G_{\underline{P}}(f|B)) = 0. \tag{GBR}$$

This second condition is called the Generalised Bayes Rule, and if $\underline{P}(B) > 0$ it can be used to uniquely determine the value $\underline{P}(f|B)$: in that case there is only one value satisfying (GBR) with respect to $\underline{P}$. If $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ satisfy (GBR), we also say that they are *Williams coherent* [23].[4]

One particular case of coherent $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are the *vacuous* unconditional and conditional lower previsions, given by

$$\underline{P}(f) = \inf_{\omega\in\Omega} f(\omega) \text{ and } \underline{P}(f|B) = \inf_{\omega\in B} f(\omega) \text{ for all } f \in \mathcal{L} \text{ and all } B \in \mathcal{B}.$$

A particular case of coherent lower previsions is that of *linear* previsions. A linear prevision is a functional $P\colon \mathcal{L} \to \mathbb{R}$ satisfying conditions C1 and C2, and

$$P(f + g) = P(f) + P(g) \text{ for all } f, g \in \mathcal{L}.$$

Its restriction to events is a finitely additive probability, and P the corresponding expectation operator.[5] The set of all linear previsions is denoted by $\mathbb{P}$. Given a coherent lower prevision $\underline{P}$ on $\mathcal{K}$, we define its associated *credal set* as

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}\colon (\forall f \in \mathcal{K})P(f) \geq \underline{P}(f)\}.$$

Using this set, we can define the *natural extension* of a coherent lower prevision $\underline{P}$ from its domain $\mathcal{K}$ to $\mathcal{L}$: it is simply the lower envelope of $\mathcal{M}(\underline{P})$, and it corresponds to the smallest coherent lower prevision on $\mathcal{L}$ that dominates $\underline{P}$ on $\mathcal{K}$.

Similarly, a *conditional linear prevision* is a functional $P(\cdot|\mathcal{B})$ on $\mathcal{L}$ such that $P(B|B) = 1$ and $P(\cdot|B)$ is a linear prevision for every $B \in \mathcal{B}$.

2.2. **Sets of desirable gambles.** The above theory can be generalised using *sets of desirable gambles.* We consider a set of gambles $\mathcal{Q}$ whose desirability we have evaluated, resulting in a subset $\mathcal{R} \subseteq \mathcal{Q}$ of desirable gambles. For now, we are going to focus on the simplest case where $\mathcal{Q}$ coincides with $\mathcal{L}$.

Let $\mathcal{R}$ be a set of gambles. We consider the following rationality axioms for desirability:

D1. $\mathcal{L}^+ \subseteq \mathcal{R}$.
D2. $0 \notin \mathcal{R}$.

[3]See [21, Section 6.3.2] for a definition of coherence on more general domains.

[4]Williams coherence is not constrained to gambles whose conditioning events form a partition of the sure event; this is one of the reasons why it does not necessarily satisfy conglomerability, as we shall see later on.

[5]The expectation is obtained by taking the Dunford integral [1].

D3. $f \in \mathcal{R}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{R}$.
D4. $f, g \in \mathcal{R} \Rightarrow f + g \in \mathcal{R}$.

A set of desirable gambles satisfying these four axioms is called *coherent* relative to $\mathcal{L}$, or simply coherent.

Given a set of desirable gambles $\mathcal{R}$, we define

$$\operatorname{posi}(\mathcal{R}) := \left\{ \sum_{k=1}^{n} \lambda_k f_k \colon f_k \in \mathcal{R}, \lambda_k > 0, n \geq 1 \right\}.$$

We call $\mathcal{R}$ a *convex cone* if it is closed under positive linear combinations, meaning that $\operatorname{posi}(\mathcal{R}) = \mathcal{R}$. This is equivalent to $\mathcal{R}$ satisfying conditions D3 and D4. The set $\operatorname{posi}(\mathcal{R} \cup \mathcal{L}^+)$ is called the *natural extension* of $\mathcal{R}$, and corresponds to its smallest coherent superset—provided $\mathcal{R}$ is included in some coherent set.

Moreover, given a partition $\mathcal{B}$ of Ω, $\mathcal{R}$ is called *$\mathcal{B}$-conglomerable* when it also satisfies the following axiom:

D5. $((f \in \mathcal{L})(\forall B \in \mathcal{B})(Bf \in \mathcal{R} \cup \{0\})) \Rightarrow f \in \mathcal{R} \cup \{0\}$.

This axiom D5 is a consequence of D4 when $\mathcal{B}$ is finite.

Similarly, we can define a notion of $\mathcal{B}$-conglomerability for coherent lower previsions:

Definition 1. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, and $\mathcal{B}$ a partition of Ω. $\underline{P}$ is called *$\mathcal{B}$-conglomerable* when the following condition holds:

wBC. if $f \in \mathcal{L}$ and B_n, $n \in N$, are distinct sets in $\mathcal{B}$ such that $\underline{P}(B_n) > 0$ and $\underline{P}(B_n f) \geq 0$ for all $n \in N$, then $\underline{P}(\sum_{n \in N} B_n f) \geq 0$.[6]

Again, wBC holds trivially when N is finite, and in particular when the partition $\mathcal{B}$ is finite, because of the super-additivity C3 of coherent lower previsions.

Let us establish the relation between the different concepts for lower previsions and for sets of desirable gambles. In order to do this, we introduce two additional concepts for sets of desirable gambles. A set $\mathcal{R}$ is called a *coherent set of strictly desirable gambles* when it is coherent and moreover

$$(\forall f \in \mathcal{R} \setminus \mathcal{L}^+)(\exists \varepsilon > 0) \text{ such that } f - \varepsilon \in \mathcal{R},$$

and it is called a *coherent set of almost-desirable gambles* when it satisfies axioms D1, D3 and D4, as well as

D2'. $\sup f < 0 \Rightarrow f \notin \mathcal{R}$;

and

D6. $((\forall \varepsilon > 0)(f + \varepsilon \in \mathcal{R})) \Rightarrow f \in \mathcal{R}$.

A coherent set of almost-desirable gambles is not a coherent set of desirable gambles: axioms D1 and D6 imply that any set of almost-desirable gambles includes the zero gamble, and as a consequence it violates D2.

Given a coherent lower prevision $\underline{P}$, we define its associated set of *strictly desirable gambles* by

$$\underline{\mathcal{R}} := \mathcal{L}^+ \cup \{f \in \mathcal{L} \colon \underline{P}(f) > 0\}, \tag{1}$$

and its associated coherent set of *almost-desirable gambles* by

$$\overline{\mathcal{R}} := \{f \in \mathcal{L} \colon \underline{P}(f) \geq 0\}. \tag{2}$$

[6] Here N is any index set, but because of the condition that $\underline{P}(B_n) > 0$ for all $n \in N$, we can effectively restrict ourselves to countable N; however, this is not the case with the notion of *strong conglomerability* to be introduced later on, and more generally the partition $\mathcal{B}$ is not necessarily countable.

It is not difficult to show that $\underline{\mathcal{R}}$ satisfies the axioms D1–D4 considered above, and that $\overline{\mathcal{R}}$ is a cone that includes all non-negative gambles. Moreover, $\underline{\mathcal{R}} \subseteq \overline{\mathcal{R}}$, and $\overline{\mathcal{R}}$ contains all non-negative gambles and is closed under dominance.

Conversely, given a coherent set of gambles $\mathcal{R}$, we can define a lower prevision by

$$\underline{P}(f) := \sup\{\mu \colon f - \mu \in \mathcal{R}\} \text{ for all } f \in \mathcal{L}. \tag{3}$$

It follows from [14, Theorem 6] that $\underline{P}$ is a coherent lower prevision.[7] Moreover, if we consider the sets $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ given by Eqs. (1) and (2), it follows from [21, Theorem 3.8.1] that

$$\sup\{\mu \colon f - \mu \in \underline{\mathcal{R}}\} = \underline{P}(f) = \sup\left\{\mu \colon f - \mu \in \overline{\mathcal{R}}\right\}. \tag{4}$$

Hence, there is a one-to-one correspondence between coherent sets of strictly desirable gambles and coherent lower previsions, and as a consequence also with closed and convex sets of linear previsions.[8]

Any set $\mathcal{R}$ such that $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$ induces the same lower prevision $\underline{P}$ by means of (3) [21, Theorem 3.8.1]. If in particular $\mathcal{R}$ is a *maximal* coherent set of desirable gambles, meaning that it satisfies D1–D4 and moreover

$$f \notin \mathcal{R} \text{ and } f \neq 0 \Rightarrow -f \in \mathcal{R},$$

then the coherent lower prevision it induces via Eq. (3) is a linear prevision.

The set $\overline{\mathcal{R}}$ is the closure of $\underline{\mathcal{R}}$ (and as a consequence also of any $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$) in the topology of uniform convergence [14, Proposition 4]:

$$\overline{\mathcal{R}} = \{f \in \mathcal{L} \colon (\forall \varepsilon > 0)(f + \varepsilon \in \mathcal{R})\},$$

and on the other hand:

$$\underline{\mathcal{R}} = \mathcal{L}^+ \cup \{f \in \mathcal{R} \colon f - \varepsilon \in \mathcal{R} \text{ for some } \varepsilon > 0\},$$

for any $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$.

Hence, any coherent lower prevision is in correspondence with an infinite class of coherent sets of desirable gambles: they are all the coherent $\mathcal{R}$ such that $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$. Similarly to Eq. (3), given a coherent set of gambles $\mathcal{R}$ and a partition $\mathcal{B}$ of Ω, we can induce a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$ by

$$\underline{P}(f|B) := \sup\{\mu \colon B(f - \mu) \in \mathcal{R}\} \text{ for all } f \in \mathcal{L}, B \in \mathcal{B}. \tag{5}$$

2.3. **The behavioural interpretation.** The above concepts can be given a behavioural interpretation, in terms of buying and selling prices [7, 21]. Given a gamble f, its lower prevision $\underline{P}(f)$ can be seen as a supremum desirable *buying* price for f, in the sense that for every $\mu < \underline{P}(f)$ the transaction $f - \mu$ is desirable.

When this supremum acceptable buying price coincides with the infimum acceptable selling price for f, which is $\inf\{\mu \colon \mu - f \in \mathcal{R}\} = -\underline{P}(-f)$, this common value can be seen as a fair price for f, and if we can establish fair prices for all gambles, we determine a linear prevision. A similar interpretation can be provided for the conditional lower previsions: $\underline{P}(f|B)$ is the supremum price we would (currently) give for f, if we observed the event B.

The rationality of our buying and selling prices can be verified by means of axioms D1–D4: condition D1, for instance, means that a transaction that can never make us lose utiles, and possibly make us gain some, should be desirable; D3 means that a change in the linear utility scale should not affect the set of gambles we consider desirable. All these axioms together imply that by combining a finite

[7] This follows under general assumptions from [22, Proposition 1], where some slightly weaker desirability axioms are considered.

[8] But the correspondence does not hold for open sets of previsions, in the sense that a gamble may be strictly desirable for all P in the interior of $\mathcal{M}(\underline{P})$ but not for $\underline{P}$; interestingly, we do have a correspondence like this when we work with almost-desirable gambles.

number of acceptable transactions a Dutch book cannot be built against us, and moreover that our supremum buying prices are the result of some thorough reflection, in the sense that one cannot force a change in our prices by taking into account the implications of any *finite* number of our desirable gambles.

Now, the reason why conglomerability is controversial is because, unlike coherence, it involves the combination of an infinite number of transactions:[9] it means that the infinite sum of desirable gambles that depend on different elements of a partition should be desirable. This is called the *conglomerative principle* in [21] and implies, for instance, that the gamble $G_{\underline{P}}(f|\mathcal{B})+\varepsilon$ should be desirable for all $\varepsilon>0$. This is rejected by authors such as Williams, for whom the gamble $G_{\underline{P}}(f|\mathcal{B})$ is only almost-desirable when f has a finite support in $\mathcal{B}$, i.e., when there is only a finite number of elements of $\mathcal{B}$ on which f is non-zero.

Walley's position is to support conglomerability, and for this reason his definition of coherence for conditional and unconditional lower previsions is based on the conglomerative principle. On the other hand, for linear previsions, conglomerability with respect to all partitions is very strongly related to countable additivity, and de Finetti and others have argued [4] that in some cases countable additivity can give rise to unreasonable conclusions.

We also mention here a property of conglomerability that might be undesirable to at least some people: it is not compatible with the sensitivity analysis interpretation, in the sense that a conglomerable set of gambles is not necessarily the intersection of conglomerable maximal supersets (see Example 1 later on), and related to this, that coherent conditional and unconditional lower previsions are not necessarily the lower envelope of a set of coherent conditional and unconditional linear previsions (see [21, Examples 6.6.9,6.6.10] for examples).

2.4. **Connection between the conglomerability conditions.** We establish a conglomerability condition for sets of desirable gambles that is equivalent to the conglomerability of the associated lower prevision.

Definition 2. A set of desirable gambles $\mathcal{R}$ is called *weakly $\mathcal{B}$-conglomerable* if and only if $\underline{\mathcal{R}}$ is $\mathcal{B}$-conglomerable.

For a set of strictly desirable gambles, conglomerability and weak conglomerability coincide. Let us now show that this weak $\mathcal{B}$-conglomerability is indeed weaker than $\mathcal{B}$-conglomerability. In order to do so, we first establish the following lemma:

Lemma 1. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, and consider $B\subseteq\Omega$. Then $\underline{P}(B)=0\Rightarrow(\forall f\in\mathcal{L})\underline{P}(Bf)\leq 0$.*

Proof. If $\sup f\leq 0$, then the coherence of $\underline{P}$ implies that $\underline{P}(Bf)\leq 0$. On the other hand, if $\sup f>0$ then the monotonicity of coherent lower previsions implies that $\underline{P}(Bf)\leq\underline{P}(B\sup f)=\underline{P}(B)\sup f=0$. □

Theorem 2. *Let $\mathcal{R}$ be a coherent set of desirable gambles. Then $\mathcal{R}$ is weakly $\mathcal{B}$-conglomerable if and only if*

wD5. $((f\in\mathcal{L})(\forall B\in\mathcal{B})(Bf\in\underline{\mathcal{R}}\cup\{0\}))\Rightarrow f\in\overline{\mathcal{R}}$.

Proof. Since it is clear that condition D5 implies wD5 when applied to $\underline{\mathcal{R}}$, it suffices to prove the converse. Assume that wD5 holds, and consider a non-zero gamble f such that $Bf\in\underline{\mathcal{R}}\cup\{0\}$ for every $B\in\mathcal{B}$. We need to show that $f\in\underline{\mathcal{R}}$.

Let $\underline{P}$ be the coherent lower prevision induced by $\mathcal{R}$. If $f\in\mathcal{L}^+$ then it follows immediately that $f\in\underline{\mathcal{R}}$; if $f\notin\mathcal{L}^+$, then there is some $B\in\mathcal{B}$ such that $Bf\not\geq 0$. Since $Bf\in\underline{\mathcal{R}}$, it follows that $\underline{P}(Bf)>0$, so there is some $\varepsilon>0$ such that

$$\underline{P}(B(f-\varepsilon))\geq\underline{P}(Bf-\varepsilon)>0.$$

[9]But see also [24] for a recent finitary interpretation.

Let $g := f - B\varepsilon$. Then for every $B' \in \mathcal{B}$ it follows that $B'g \in \underline{\mathcal{R}} \cup \{0\}$. Applying wD5, we deduce that $g \in \overline{\mathcal{R}}$, and therefore $\underline{P}(g) \geq 0$. As a consequence,

$$\underline{P}(f) = \underline{P}(g + B\varepsilon) \geq \underline{P}(g) + \underline{P}(B\varepsilon) \geq \varepsilon \underline{P}(B) > 0,$$

taking into account that $\underline{P}(B) > 0$ because of Lemma 1. Hence $\underline{P}(f) > 0$ and therefore $f \in \underline{\mathcal{R}}$. We conclude that this set satisfies D5. □

By comparing conditions D5 and wD5, we see that if a coherent set of gambles $\mathcal{R}$ is $\mathcal{B}$-conglomerable, it is in particular weakly $\mathcal{B}$-conglomerable. When $\mathcal{R}$ is a coherent set of strictly desirable gambles, Theorem 2 implies that it is conglomerable if and only if

$$((f \in \mathcal{L})(\forall B \in \mathcal{B})(Bf \in \mathcal{R} \cup \{0\})) \Rightarrow f \in \overline{\mathcal{R}}.$$

Next we show that the weak $\mathcal{B}$-conglomerability of a coherent set of desirable gambles is equivalent to the $\mathcal{B}$-conglomerability of the coherent lower prevision it induces.

Theorem 3. *Let $\mathcal{R}$ be a coherent set of desirable gambles, and let $\underline{P}$ be the coherent lower prevision it induces by means of Eq.* (3)*. Then $\underline{P}$ satisfies* wBC *if and only if $\mathcal{R}$ is weakly $\mathcal{B}$-conglomerable.*

Proof. Let us show that $\underline{P}$ satisfies wBC if $\mathcal{R}$ satisfies wD5. Consider a gamble f such that $\underline{P}(B_n f) \geq 0$ for some distinct sets $B_n \subseteq \mathcal{B}$, with $\underline{P}(B_n) > 0$ for all $n \in N$. Then for any fixed $\varepsilon > 0$ and any $n \in N$ it holds that $\underline{P}(B_n(f+\varepsilon)) \geq \underline{P}(B_n f) + \varepsilon \underline{P}(B_n) > 0$, whence $B_n(f+\varepsilon) \in \underline{\mathcal{R}}$. Applying wD5, we deduce that $\sum_n B_n(f+\varepsilon) \in \overline{\mathcal{R}}$ for every $\varepsilon > 0$. Since this set is closed under uniform convergence, we deduce that $\sum_n B_n f \in \overline{\mathcal{R}}$, and therefore $\underline{P}(\sum_n B_n f) \geq 0$.

Conversely, assume that $\underline{P}$ satisfies wBC, and that for some $f \in \mathcal{L}$, $Bf \in \underline{\mathcal{R}} \cup \{0\}$ for all $B \in \mathcal{B}$. This implies that $\underline{P}(Bf) \geq 0$ for all $B \in \mathcal{B}$, using Eq. (2) and the inclusion $\underline{\mathcal{R}} \cup \{0\} \subseteq \overline{\mathcal{R}}$. If $\underline{P}(B) = 0$, then Lemma 1 implies that $\underline{P}(Bf) \leq 0$, and as a consequence $\underline{P}(Bf) = 0$. Taking into account Eq. (1), we deduce that if $Bf \in \underline{\mathcal{R}} \cup \{0\}$ then also $Bf \geq 0$.

Let $\mathcal{B}' := \{B \in \mathcal{B} \colon \underline{P}(B) > 0\}$. If $\mathcal{B}'$ is empty, then $Bf \geq 0$ for all $B \in \mathcal{B}$ and hence $f \in \mathcal{L}^+ \subseteq \overline{\mathcal{R}}$. Let us consider the case that $\mathcal{B}'$ is not empty. By wBC, $\underline{P}(\sum_{B \in \mathcal{B}'} Bf) \geq 0$. Thus

$$\underline{P}(f) = \underline{P}\Bigg(\sum_{B \in \mathcal{B}'} Bf + \sum_{B \in \mathcal{B} \setminus \mathcal{B}'} Bf\Bigg) \geq \underline{P}\Bigg(\sum_{B \in \mathcal{B}'} Bf\Bigg) + \underline{P}\Bigg(\sum_{B \in \mathcal{B} \setminus \mathcal{B}'} Bf\Bigg) \geq 0,$$

where the first inequality follows from C3 and the second from C1. As a consequence $\underline{P}(f) \geq 0$ and then Eq. (2) implies that $f \in \overline{\mathcal{R}}$. □

This result, together with Theorem 2, implies that a coherent lower prevision $\underline{P}$ is $\mathcal{B}$-conglomerable if and only if its associated set of strictly desirable gambles is $\mathcal{B}$-conglomerable.

On the other hand, when we consider a coherent set of almost-desirable gambles $\overline{\mathcal{R}}$, condition D5 is equivalent to

D5'. $((f \in \mathcal{L})(\forall B \in \mathcal{B})(Bf \in \overline{\mathcal{R}})) \Rightarrow f \in \overline{\mathcal{R}}$.

We next show that condition D5 can also be related to a notion of conglomerability for coherent lower previsions:

Definition 3. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, and $\mathcal{B}$ a partition of Ω. $\underline{P}$ is called *strongly $\mathcal{B}$-conglomerable* when the following condition holds:

BC. $((\forall B \in \mathcal{B})(\underline{P}(Bf) \geq 0)) \Rightarrow \underline{P}(f) \geq 0$.

Theorem 4. *Let $\underline{P}$ be a coherent lower prevision, and let $\overline{\mathcal{R}}$ be its associated set of almost-desirable gambles, given by Eq.* (2)*. Then $\underline{P}$ is strongly $\mathcal{B}$-conglomerable if and only if $\overline{\mathcal{R}}$ satisfies* D5*. Conversely, a coherent set of almost-desirable gambles satisfies* D5 *if and only if the coherent lower prevision $\underline{P}$ it induces satisfies* BC*.*

Proof. Since there is a one-to-one correspondence between coherent lower previsions and coherent sets of almost-desirable gambles, it suffices to prove the first of the two equivalences. But this is immediate once we remark that a gamble g belongs to $\overline{\mathcal{R}}$ if and only if $\underline{P}(g) \geq 0$, and that in particular this holds for the gambles Bf, for all $f \in \mathcal{L}$ and $B \in \mathcal{B}$. □

By comparing conditions BC and wBC, we see that if a coherent lower prevision is strongly $\mathcal{B}$-conglomerable, then it is also $\mathcal{B}$-conglomerable.

Theorems 3 and 4 lead to one of the most important points we make in this paper: that the usual correspondence between sets of desirable gambles and coherent lower previsions does not extend to conglomerability: the usual notion of conglomerability for sets of desirable gambles, given by D5, is stronger (more restrictive) than the one for coherent lower previsions, given in Definition 1. Nevertheless, we still maintain the one-to-one correspondence between coherent lower previsions and sets of strictly desirable gambles when we add conglomerability, because for the latter the notions of weak and strong conglomerability are equivalent.

3. Conglomerability for sets of desirable gambles

Let us consider a set of gambles $\mathcal{R}$, and look for the smallest superset $\mathcal{F}$ (if it exists) that satisfies D1–D5 with respect to a fixed partition $\mathcal{B}$. We call this set the *conglomerable natural extension* of $\mathcal{R}$. A first characterisation of this set is given in the following proposition. We use the notation $\mathbb{D}_{\mathrm{C}}(\mathcal{B})$ for the set of all conglomerable coherent sets of desirable gambles—satisfying D1–D5—, and $\mathbb{D}_{\mathrm{wC}}(\mathcal{B})$ for the set of all weakly conglomerable coherent sets of desirable gambles—satisfying D1–D4 and wD5—, on Ω.

Proposition 5. *If there is some coherent set of gambles that includes $\mathcal{R}$ and satisfies* D5 *(respectively* wD5*) then the* conglomerable *(respectively* weakly conglomerable*) natural extension of $\mathcal{R}$ is given by:*

$$\mathcal{F} := \bigcap \{\mathcal{D} \in \mathbb{D}_{\mathrm{C}}(\mathcal{B}) \colon \mathcal{R} \subseteq \mathcal{D}\} \;\; \textit{respectively } \mathcal{F} := \bigcap \{\mathcal{D} \in \mathbb{D}_{\mathrm{wC}}(\mathcal{B}) \colon \mathcal{R} \subseteq \mathcal{D}\}\,.$$

Proof. It suffices to show that the sets $\mathbb{D}_{\mathrm{C}}(\mathcal{B})$ and $\mathbb{D}_{\mathrm{wC}}(\mathcal{B})$ are closed under arbitrary non-empty intersections. It was shown elsewhere [3] that D1–D4 are preserved under taking such intersections. We now show that this also holds for D5 and wD5.

Consider an arbitrary non-empty family $\mathcal{R}_i$, $i \in I$ of sets of desirable gambles that satisfy D5. We show that $\mathcal{R} := \bigcap_{i \in I} \mathcal{R}_i$ satisfies D5 too. Suppose that $Bf \in \mathcal{R} \cup \{0\}$ for all $B \in \mathcal{B}$, then also $Bf \in \mathcal{R}_i \cup \{0\}$ for all $B \in \mathcal{B}$ and all $i \in I$, and therefore $f \in \mathcal{R}_i \cup \{0\}$ for all $i \in I$. Hence indeed $f \in \mathcal{R} \cup \{0\}$.

Next, consider an arbitrary non-empty family $\mathcal{R}_i$, $i \in I$ of sets of desirable gambles that satisfy wD5. We show that $\mathcal{R} := \bigcap_{i \in I} \mathcal{R}_i$ satisfies wD5 too. Suppose that $Bf \in \underline{\mathcal{R}} \cup \{0\}$ for all $B \in \mathcal{B}$; then for any $B \in \mathcal{B}$ either $Bf \in \mathcal{L}^+ \cup \{0\}$ (whence $Bf \in \underline{\mathcal{R}_i} \cup \{0\}$ for all $i \in I$) or there is some $\varepsilon > 0$ such that $Bf - \varepsilon \in \mathcal{R}$ (whence $Bf - \varepsilon \in \mathcal{R}_i$ for all $i \in I$, and therefore $Bf \in \underline{\mathcal{R}_i}$ for all $i \in I$). Hence, also $Bf \in \underline{\mathcal{R}_i} \cup \{0\}$ for all $B \in \mathcal{B}$ and all $i \in I$, and therefore $f \in \overline{\mathcal{R}}_i$ for all $i \in I$. This implies that $f + \varepsilon \in \mathcal{R}_i$ for all $i \in I$ and all $\varepsilon > 0$, and therefore $f + \varepsilon \in \mathcal{R}$ for all $\varepsilon > 0$. Hence indeed $f \in \overline{\mathcal{R}}$. □

From now on, we will assume that $\mathcal{R}$ satisfies conditions D1–D4, we denote this by writing $\mathcal{R} \in \mathbb{D}$. Condition D2 is necessary for the existence of a conglomerable natural extension, and properties D1, D3 and D4 can be satisfied by considering the natural extension $\operatorname{posi}(\mathcal{R} \cup \mathcal{L}^{+})$.

A coherent set of desirable gambles is always the intersection of all its maximal supersets [3]. However, this property does not necessary keep on holding if we add the conglomerability requirement: the existence of a superset of $\mathcal{R}$ that satisfies D1–D5 does not guarantee that there is a maximal superset of $\mathcal{R}$ that satisfies these axioms. Our example is just a reformulation of [21, Example 6.6.9]:

Example 1. Let Ω be the set of integers without zero, and consider the partition $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$ given by $B_n := \{-n, n\}$, where $\mathbb{N}$ denotes the set of natural numbers without zero. Let P_1 be a linear prevision on $\mathcal{L}$ satisfying $P_1(\{n\}) = 1/2^{n+1}$ and $P_1(\{-n\}) = 0$ for all $n \in \mathbb{N}$ and $P_1(\mathbb{I}_{-\mathbb{N}}) = 1/2$, where $\mathbb{I}$ denotes the indicator gamble. Consider also a linear prevision P_2 satisfying $P_2(\{-n\}) = 1/3^n$ and $P_2(\{n\}) = 0$ for all $n \in \mathbb{N}$, and $P_2(\mathbb{I}_{\mathbb{N}}) = 1/2$. Let $\underline{P} := \min\{P_1, P_2\}$.

Let $\underline{\mathcal{R}}$ be the set of strictly-desirable gambles associated with $\underline{P}$, given by Eq. (1). This set satisfies axioms D1–D4. To see that it also satisfies D5, note that if a gamble $0 \neq f$ satisfies that $B_n f$ belongs to $\underline{\mathcal{R}} \cup \{0\}$ for every n, then either $\underline{P}(B_n f) > 0$ or $B_n f \geq 0$. But since $\underline{P}(B_n f) > 0$ implies that both $P_1(B_n f) > 0$ and $P_2(B_n f) > 0$, and this in turn means that $f(-n)$ and $f(n)$ are non-negative, we also deduce that $\underline{P}(B_n f) > 0$ implies that $B_n f \gneq 0$. As a consequence, if $B_n f \in \underline{\mathcal{R}} \cup \{0\}$ for every $B_n \in \mathcal{B}$, then $f \geq 0$, and since it is different from the zero gamble we deduce that $f \in \underline{\mathcal{R}}$.

Let us show now that there is no maximal superset of $\underline{\mathcal{R}}$ satisfying wD5, and as a consequence neither D5. Assume ex absurdo that $\mathcal{D}$ is such a set. Let P be its associated linear prevision, determined by Eq. (3). Since $\underline{\mathcal{R}} \subseteq \mathcal{D}$, we deduce that P dominates $\underline{P}$. But Walley has shown in [21, Example 6.6.9] that no dominating linear prevision satisfies wBC. Using Theorem 3, we deduce that $\mathcal{D}$ does not satisfy wD5, and as a consequence it does not satisfy D5 either. ⧫

Remark 1. Theorem 3 allows us to deduce that the conglomerable natural extension does not exist when $\mathcal{R}$ induces a linear prevision P, using Eq. (3), that is not conglomerable: if the conglomerable natural extension $\mathcal{F}$ did exist, then since $\mathcal{R} \subseteq \mathcal{F}$, the latter should induce a coherent lower prevision $\underline{P}$ that dominates P, and it is a consequence of C1 and C3 that this can only happen when $\underline{P} = P$.

Now, since $\mathcal{F}$ is conglomerable, it is also weakly conglomerable, and applying Theorem 3 we deduce that P is conglomerable, a contradiction.

On the other hand, if $\mathcal{R}$ induces a conglomerable linear prevision P, the conglomerable natural extension of $\mathcal{R}$ may exist—for instance if $\mathcal{R}$ is the set of strictly desirable gambles associated to P then Theorem 3 implies that $\mathcal{R}$ is conglomerable—or it may not. To see an example of the latter case, consider for instance $\Omega := \mathbb{N}$, $B_n := \{2n, 2n-1\}$, $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$ and a linear prevision P satisfying

$$P(\{n\}) = 0 \text{ for all } n \in \mathbb{N} \text{ and } P(\{2n : n \in \mathbb{N}\}) = \frac{1}{2}.$$

Then $P(B_n) = 0$ for every $n \in \mathbb{N}$, so P is trivially $\mathcal{B}$-conglomerable. On the other hand, the set

$$\mathcal{R} := \{f : P(f) > 0\} \cup \{f : P(f) = 0 \text{ and } f(\min\{n : f(n) \neq 0\}) > 0\}$$

is a maximal set of gambles that lies between $\{f : P(f) > 0\}$ and $\{f : P(f) \geq 0\}$, so it induces the linear prevision P. To see that it is not conglomerable, note that the gamble $f := \mathbb{I}_{\{\text{odd}\}} - 2\mathbb{I}_{\{\text{even}\}}$ does not belong to $\mathcal{R}$ because $P(f) < 0$ even if $B_n f$ belongs to $\mathcal{R}$ for all n. Since from its definition a maximal set has a conglomerable natural extension if and only if it is itself conglomerable, we deduce that in this case the conglomerable natural extension $\mathcal{F}$ does not exist. ⧫

Our next goal is to find more practically constructive ways of expressing the (weakly) conglomerable natural extension $\mathcal{F}$. To get some intuition for how to proceed, look at the example of the natural extension of a (not necessarily coherent) assessment $\mathcal{R}$: we first use the axiom D1 to turn $\mathcal{R}$ into $\mathcal{R}\cup\mathcal{L}^+$, and then use the productive coherence axioms D3 and D4 successively to add gambles to this set. In this case, it so happens that after applying D3 and D4 only once, we arrive at $\operatorname{posi}(\mathcal{R}\cup\mathcal{L}^+)$, which satisfies D1, D3 and D4. If this set of gambles satisfies D2, it is clearly the smallest coherent set to do so; if it does not, then $\mathcal{R}$ has no coherent extension.

This suggests that, in order to find the conglomerable natural extension $\mathcal{F}$, we could use a similar procedure. We start out with the coherent, but not necessarily conglomerable, set $\mathcal{R}$, and we use the productive axiom D5 to add gambles to it, making it conglomerable. The problem now is that, unlike D3 and D4 separately, D3–D4 and D5 do not play well together: the result of using D5 is a set of desirable gambles that is no longer necessarily coherent—it need not satisfy D4. So we use D3 and D4—or the posi operator—again, which now leads us to a set of desirable gambles that is no longer conglomerable, and so on.

We are, in other words, led to define the following sequence of sets of desirable gambles:

$$\begin{aligned}
\mathcal{R}^{\mathcal{B}} &\coloneqq \{0\neq f\in\mathcal{L}\colon (\forall B\in\mathcal{B})Bf\in\mathcal{R}\cup\{0\}\} && (6)\\
\mathcal{E}_1 &\coloneqq \operatorname{posi}(\mathcal{R}\cup\mathcal{R}^{\mathcal{B}})\\
&\vdots\\
\mathcal{E}_n^{\mathcal{B}} &\coloneqq \{0\neq f\in\mathcal{L}\colon (\forall B\in\mathcal{B})Bf\in\mathcal{E}_n\cup\{0\}\}, \quad n\geq 1\\
\mathcal{E}_{n+1} &\coloneqq \operatorname{posi}(\mathcal{E}_n\cup\mathcal{E}_n^{\mathcal{B}}), \quad n\geq 1. && (7)
\end{aligned}$$

In order to make the notation more uniform, we will sometimes use $\mathcal{E}_0\coloneqq\mathcal{R}$ and $\mathcal{E}_0^{\mathcal{B}}\coloneqq\mathcal{R}^{\mathcal{B}}$.

Lemma 6. *Let $\mathcal{F}'$ be a superset of $\mathcal{R}$ that satisfies* D1–D5. *Then $\mathcal{E}_n\subseteq\mathcal{E}_{n+1}\subseteq\mathcal{F}'$ for all $n\in\mathbb{N}\cup\{0\}$.*

Proof. We proceed by induction on n. That $\mathcal{E}_n\subseteq\mathcal{E}_{n+1}$ follows trivially from the definition. If $\mathcal{F}'$ is a superset of $\mathcal{R}$ that satisfies D1–D5, then it must include $\mathcal{R}^{\mathcal{B}}$—because it satisfies D5—and therefore also $\mathcal{E}_1$—because it satisfies D3–D4. Now, assume that $\mathcal{E}_n$ is included in $\mathcal{F}'$. Then condition D5 implies that also $\mathcal{E}_n^{\mathcal{B}}\subseteq\mathcal{F}'$, and then D3–D4 imply that $\mathcal{E}_{n+1}$ is included in $\mathcal{F}'$. □

It follows that the conglomerable natural extension of $\mathcal{R}$, if it exists, must include the limit $\bigcup_n\mathcal{E}_n$ of the converging sequence $\mathcal{E}_n$. We next investigate which desirability axioms are satisfied by the $\mathcal{E}_n$ and $\mathcal{E}_n^{\mathcal{B}}$.

Proposition 7. *Let $\mathcal{R}$ be a coherent set of desirable gambles, and assume that there is some superset $\mathcal{F}'$ of $\mathcal{R}$ satisfying* D1–D5.

(i) *For every $n\in\mathbb{N}\cup\{0\}$, $\mathcal{E}_n$ satisfies* D1–D4.
(ii) *For every $n\in\mathbb{N}\cup\{0\}$, $\mathcal{E}_n^{\mathcal{B}}$ satisfies* D1–D5.

Proof. We begin with the first statement. Consider any $n\in\mathbb{N}\cup\{0\}$. It follows from the definition of $\mathcal{E}_n$ and $\mathcal{E}_n^{\mathcal{B}}$ that $\mathcal{R}\coloneqq\mathcal{E}_0\subseteq\mathcal{E}_1\subseteq\mathcal{E}_2\subseteq\dots$ and that $\mathcal{R}^{\mathcal{B}}\coloneqq\mathcal{E}_0^{\mathcal{B}}\subseteq\mathcal{E}_1^{\mathcal{B}}\subseteq\mathcal{E}_2^{\mathcal{B}}\subseteq\dots$.

Let us first prove that for every $n\in\mathbb{N}\cup\{0\}$, if $\mathcal{E}_n$ satisfies D3, respectively D4, then so does $\mathcal{E}_n^{\mathcal{B}}$:

D3: Consider $f\in\mathcal{E}_n^{\mathcal{B}}$ and $\lambda>0$. Then $f\neq 0$ and for every $B\in\mathcal{B}$, $Bf\in\mathcal{E}_n\cup\{0\}$, so $B(\lambda f)=\lambda(Bf)\in\mathcal{E}_n\cup\{0\}$, and therefore $\lambda f\in\mathcal{E}_n^{\mathcal{B}}$.

D4: Similarly, if $f, g \in \mathcal{E}_n^{\mathcal{B}}$, then for every $B \in \mathcal{B}$, both Bf and Bg belong to $\mathcal{E}_n \cup \{0\}$, whence $Bf + Bg = B(f+g) \in \mathcal{E}_n \cup \{0\}$, and therefore $f + g \in \mathcal{E}_n^{\mathcal{B}}$: we also have that $f + g \neq 0$ because otherwise there would be some $B \in \mathcal{B}$ such that $0 \neq Bf \in \mathcal{E}_n, Bg = -Bf \in \mathcal{E}_n$ and this would mean that $0 \in \mathcal{E}_n \subseteq \mathcal{F}'$, a contradiction.

We proceed to show that $\mathcal{E}_n$ and $\mathcal{E}_n^{\mathcal{B}}$ satisfy the different desirability axioms.

D1: Since $\mathcal{L}^+$ is included in $\mathcal{R} \subseteq \mathcal{E}_n$, it belongs to $\mathcal{E}_n$, and consequently also to $\mathcal{E}_n^{\mathcal{B}}$, for any $n \geq 0$.

D2: If there is some superset $\mathcal{F}'$ of $\mathcal{R}$ satisfying D1–D5, it follows from Lemma 6 that $0 \notin \mathcal{E}_{n+1} = \operatorname{posi}(\mathcal{E}_n \cup \mathcal{E}_n^{\mathcal{B}})$ for all $n \geq 0$, and consequently $\mathcal{E}_n$ and $\mathcal{E}_n^{\mathcal{B}}$ satisfy D2.

On the other hand, it follows from Eq. (7) that $\mathcal{E}_n$ is a convex cone for all $n \geq 0$, and therefore it satisfies D3 and D4. Applying the first part of the proof, we deduce that so does $\mathcal{E}_n^{\mathcal{B}}$.

Finally, to see that $\mathcal{E}_n^{\mathcal{B}}$ also satisfies D5, consider a non-zero gamble f such that $Bf \in \mathcal{E}_n^{\mathcal{B}} \cup \{0\}$ for all $B \in \mathcal{B}$. This implies that $Bf \in \mathcal{E}_n \cup \{0\}$ for all $B \in \mathcal{B}$, and as a consequence $f \in \mathcal{E}_n^{\mathcal{B}}$. □

It follows from this result that, when the conglomerable natural extension of $\mathcal{R}$ exists, we can equivalently express $\mathcal{E}_n$ as

$$\mathcal{E}_n = \left\{\mu_1 f_1 + \mu_2 f_2 \colon f_1 \in \mathcal{E}_{n-1}, f_2 \in \mathcal{E}_{n-1}^{\mathcal{B}}, \mu_1, \mu_2 \in \{0,1\}, \max\{\mu_1, \mu_2\} = 1\right\}$$

and also

$$\mathcal{E}_n = \left\{f + g \colon f \in \mathcal{E}_{n-1} \cup \{0\}, g \in \mathcal{E}_{n-1}^{\mathcal{B}} \cup \{0\}\right\} \setminus \{0\}$$

for any $n \geq 1$.

The intuition has been all along that when the sequence $\mathcal{E}_n$ breaks off (becomes constant) at some point, we have reached the conglomerable natural extension $\mathcal{F}$. Using Proposition 7, we can now confirm this.

Proposition 8. *The following conditions are equivalent for any $n \in \mathbb{N} \cup \{0\}$:*

(i) $\mathcal{E}_n^{\mathcal{B}} \subseteq \mathcal{E}_n$;
(ii) $\mathcal{E}_n$ *satisfies* D5;
(iii) $\mathcal{F} = \mathcal{E}_n$.

Proof. We give a circular proof.

If $\mathcal{E}_n^{\mathcal{B}}$ is included in $\mathcal{E}_n$, then given a non-zero gamble f such that $Bf \in \mathcal{E}_n \cup \{0\}$ for all $B \in \mathcal{B}$, it follows from the definition of $\mathcal{E}_n^{\mathcal{B}}$ that $f \in \mathcal{E}_n^{\mathcal{B}}$, and therefore $f \in \mathcal{E}_n$. This implies that $\mathcal{E}_n$ satisfies D5.

Secondly, if $\mathcal{E}_n$ satisfies D5 then it follows from Proposition 7 that it is a superset of $\mathcal{R}$ that satisfies conditions D1–D5. As a consequence, it must include the smallest such superset, and therefore $\mathcal{F} \subseteq \mathcal{E}_n$. The converse inclusion follows from Lemma 6.

Finally, if $\mathcal{F} = \mathcal{E}_n$, we deduce that $\mathcal{E}_n^{\mathcal{B}} \subseteq \mathcal{E}_n$ from $\mathcal{E}_n^{\mathcal{B}} \subseteq \mathcal{E}_{n+1} \subseteq \mathcal{F}$. □

This simple result has a couple of interesting consequences. On the one hand, if $\mathcal{E}_n$ is not the conglomerable natural extension of $\mathcal{R}$, then there must be some gamble f in $\mathcal{E}_n^{\mathcal{B}} \setminus \mathcal{E}_n$, and as a consequence $\mathcal{E}_n$ is a proper subset of $\mathcal{E}_{n+1}$. In other words, the sequence $\mathcal{E}_n$ does not stabilise unless we get to the conglomerable natural extension. On the other hand, if $\mathcal{E}_n^{\mathcal{B}} = \mathcal{E}_{n+1}^{\mathcal{B}}$ for some n then $\mathcal{E}_{n+1}^{\mathcal{B}}$ is included in $\mathcal{E}_{n+1}$, and Proposition 8 implies that $\mathcal{E}_{n+1}$ is the conglomerable natural extension of $\mathcal{R}$. This means that the sequence $\mathcal{E}_n^{\mathcal{B}}$ also stabilises when we get to the conglomerable natural extension, and only then: it is strictly increasing before that.

We go a bit further and provide a sufficient condition for $\mathcal{E}_1$ to coincide with $\mathcal{F}$:

Proposition 9. *Let $\mathcal{R}$ be a coherent set of desirable gambles.*

(i) $\mathcal{R}^{\mathcal{B}}$ *satisfies* D1–D5, *even if $\mathcal{R}$ has no conglomerable natural extension.*

(ii) $\mathcal{R}^{\mathcal{B}} = \mathcal{F} \Leftrightarrow \mathcal{R} \subseteq \mathcal{R}^{\mathcal{B}} \Leftrightarrow (\forall B \in \mathcal{B})(f \in \mathcal{R} \Rightarrow Bf \in \mathcal{R} \cup \{0\})$. *As a consequence, under any of these equivalent conditions,* $\mathcal{E}_1 = \mathcal{R}^{\mathcal{B}} = \mathcal{F}$.

(iii) *If there is some superset* $\mathcal{Q}$ *of* $\mathcal{R}$ *satisfying* D1–D5 *and such that* $\mathcal{Q}^{\mathcal{B}} = \mathcal{R}^{\mathcal{B}}$*, then* $\mathcal{E}_1 = \mathcal{F}$.

Proof. The first statement is a consequence of [24, Proposition 1].

To prove the second statement, observe that from the definition of $\mathcal{R}^{\mathcal{B}}$, it includes $\mathcal{R}$ if and only if $Bf \in \mathcal{R} \cup \{0\}$ for all $f \in \mathcal{R}$ and $B \in \mathcal{B}$. On the other hand, if $\mathcal{R} \subseteq \mathcal{R}^{\mathcal{B}}$, then we deduce that $\mathcal{F} \subseteq \mathcal{R}^{\mathcal{B}}$, since the first statement tells us that $\mathcal{R}^{\mathcal{B}}$ satisfies D1–D5. Hence $\mathcal{F} = \mathcal{R}^{\mathcal{B}}$, and the converse implication is trivial. Since $\mathcal{R}^{\mathcal{B}} \subseteq \mathcal{E}_1 \subseteq \mathcal{F}$, we deduce that if $\mathcal{R}$ is included in $\mathcal{R}^{\mathcal{B}}$ then $\mathcal{E}_1 = \mathcal{F} = \mathcal{R}^{\mathcal{B}}$.

For the last statement, note that $\mathcal{R} \subseteq \mathcal{Q}$ implies that $\mathcal{E}_1 \subseteq \mathcal{Q}_1 = \mathcal{Q}$, where the last equality holds because $\mathcal{Q}$ satisfies D5, and is therefore equal to its own conglomerable natural extension. As a consequence, we also have $\mathcal{E}_1^{\mathcal{B}} \subseteq \mathcal{Q}^{\mathcal{B}} = \mathcal{R}^{\mathcal{B}}$, and since we always have the converse inclusion, we deduce that $\mathcal{E}_1^{\mathcal{B}} = \mathcal{R}^{\mathcal{B}}$. But then $\mathcal{E}_1^{\mathcal{B}}$ is included in $\mathcal{E}_1$, and applying Proposition 8 we deduce that $\mathcal{E}_1$ is the conglomerable natural extension of $\mathcal{R}$. □

We present an example showing that the inclusion $\mathcal{R} \subseteq \mathcal{R}^{\mathcal{B}}$ does not imply that $\mathcal{R} = \mathcal{R}^{\mathcal{B}}$, or, equivalently, that we may have $\mathcal{R} \subsetneq \mathcal{E}_1 = \mathcal{F}$:

Example 2. Consider $\Omega := \mathbb{N}$, $B_n := \{2n-1, 2n\}$ and $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$. Let $\mathcal{R}$ be the set of gambles given by

$$\left\{f \colon (\exists n \in \mathbb{N}) f\mathbb{I}_{[n,\infty)} \in \mathcal{L}^+,\ (\forall n \in \mathbb{N})(\min\{f(2n) + f(2n-1), f(2n)\} \geq 0)\right\}, \quad (8)$$

where we use the notation '$[n,\infty)$' to denote the set $\{n, n+1, \dots\}$. Then it is easy to see that $\mathcal{R}$ satisfies D1–D4. To see that $\mathcal{R}$ is included in $\mathcal{R}^{\mathcal{B}}$, note that given a gamble $f \in \mathcal{R}$ and $B_n \in \mathcal{B}$, $B_n(f(2m)+f(2m-1)) \geq 0$ and $B_n(f(2m)) \geq 0$ for every natural number m. Moreover, if $B_n f = 0$, then automatically $B_n f \in \mathcal{R} \cup \{0\}$. If $B_n f \neq 0$ then $f(2n) > 0$, whence $B_n f \in \mathcal{R}$ because $B_n f \mathbb{I}_{[2n,\infty)} \in \mathcal{L}^+$, or $f(2n-1) > 0$ and whence $B_n f \in \mathcal{R}$ because $B_n f \mathbb{I}_{[2n-1,\infty)} \in \mathcal{L}^+$.

However, $\mathcal{R}$ does not satisfy D5, and as a consequence it does not coincide with $\mathcal{R}^{\mathcal{B}}$: the non-zero gamble g given by

$$g(2n) := 1 \text{ and } g(2n-1) := -1 \text{ for all } n \in \mathbb{N} \quad (9)$$

does not belong to $\mathcal{R}$ because it does not become positive eventually: there is no natural number n such that $g\mathbb{I}_{[n,\infty)} \in \mathcal{L}^+$. On the other hand, for every natural number n, the non-zero gamble $B_n g$ does belong to $\mathcal{R}$ since $B_n g \mathbb{I}_{[2n,\infty)} \in \mathcal{L}^+$, and therefore $g \in \mathcal{R}^{\mathcal{B}}$. ♦

This example allows us also to show that conglomerability and weak conglomerability of gambles are not equivalent:

Example 3. Let the coherent set $\mathcal{R}$ of desirable gambles be given by Eq. (8). We have already shown in Example 2 that it does not satisfy D5. To see that it satisfies wD5, note that given a gamble f and $B_n \in \mathcal{B}$, $B_n f$ belongs to $\underline{\mathcal{R}} \cup \{0\}$ if and only if $B_n f \geq 0$, because there is no $\delta > 0$ such that $B_n f - \delta \in \mathcal{R}$: $B_n f - \delta$ does not become positive eventually. As a consequence, $(\forall B_n \in \mathcal{B}) B_n f \in \underline{\mathcal{R}} \cup \{0\}$ implies that $f \geq 0$ and therefore $f \in \overline{\mathcal{R}}$. ♦

The same example shows us something else that is quite interesting: it may happen that every $B_n f$ lies in the 'boundary' $\overline{\mathcal{R}} \setminus \underline{\mathcal{R}}$ (as well as every B_n) and at the same time $f = \sum_n B_n f$ lies outside $\overline{\mathcal{R}}$:

Example 4. Let $\mathcal{R}$ by given by Eq. (8) and let g be the gamble defined in Eq. (9). Taking into account the comments in Example 3, there is no $\delta > 0$ such that $B_n g - \delta \in \mathcal{R}$, because this gamble does not become positive eventually. On the other hand, we know that $B_n g \in \mathcal{R}$. This means that $B_n g \in \mathcal{R} \setminus \underline{\mathcal{R}} \subseteq \overline{\mathcal{R}} \setminus \underline{\mathcal{R}}$ for all $B_n \in \mathcal{B}$.

Now, to show that $g \notin \overline{\mathcal{R}}$, we consider any $0 < \delta < 1$ and show that $g + \delta$ does not belong to $\mathcal{R}$. In fact, $g(2n-1) + \delta < 0$ for all $n \geq 1$, so g cannot become positive eventually.

Observe that $g + 1 \in \mathcal{L}^+$, so $g + 1 \in \mathcal{R}$. This means that for the associated lower prevision: $\underline{P}(g) = \sup\left\{\mu\colon g - \mu \in \overline{\mathcal{R}}\right\} = \sup\left\{\mu\colon g - \mu \in \mathcal{R}\right\} = -1$. ♦

Stated differently, this means that even if the gambles that violate D5 are only on the border of $\mathcal{R}$, taking the closure of $\mathcal{R}$ with respect to D5 may not ony affect the border (or the interior) of $\mathcal{R}$, but may require to enlarge the set beyond its borders.

On the other hand, when $\underline{\mathcal{R}}$ is a set of strictly desirable gambles, the inclusion $\underline{\mathcal{R}} \subseteq \underline{\mathcal{R}}^{\mathcal{B}}$ only holds in trivial cases, as we see from the following proposition:

Proposition 10. *Let $\underline{P}$ be a coherent lower prevision and let $\underline{\mathcal{R}}$ be its associated set of strictly desirable gambles. Then $\underline{\mathcal{R}} \subseteq \underline{\mathcal{R}}^{\mathcal{B}}$ if and only if $\underline{\mathcal{R}} = \underline{\mathcal{R}}^{\mathcal{B}} = \mathcal{L}^+$.*

Proof. The 'if' part is trivial, so we concentrate on the other implication. From Proposition 9, the inclusion $\underline{\mathcal{R}} \subseteq \underline{\mathcal{R}}^{\mathcal{B}}$ is equivalent to

$$f \in \underline{\mathcal{R}} \text{ and } B \in \mathcal{B} \Rightarrow Bf \in \underline{\mathcal{R}} \cup \{0\}. \tag{10}$$

Assume for the lower prevision $\underline{P}$ associated with $\underline{\mathcal{R}}$ that there is some $B_0 \in \mathcal{B}$ such that $0 < \underline{P}(B_0) \leq 1$. Then it follows from Eq. (4) that there is some $\mu > 0$ such that the gamble $B_0 - \mu$ belongs to $\underline{\mathcal{R}}$. But this gamble is equal to $-\mu < 0$ on every $B \neq B_0$, a contradiction with (10).

As a consequence, $\underline{P}(B) = 0$ for all $B \in \mathcal{B}$. Consider any $f \in \underline{\mathcal{R}}$ then Lemma 1 tells us that $\underline{P}(Bf) \leq 0$, so we infer from Eq. (10) that $Bf \in \mathcal{L}^+ \cup \{0\}$ for all $B \in \mathcal{B}$, so $f \in \mathcal{L}^+$. Hence $\underline{\mathcal{R}} = \mathcal{L}^+$, and then we deduce from Eq. (6) that $\mathcal{R}^{\mathcal{B}} = \mathcal{L}^+$. □

4. Conglomerability for coherent lower previsions

We investigate, for lower previsions, the relationship between the natural extension and the conglomerable natural extension. To this end, we consider a coherent lower prevision $\underline{P}$ on $\mathcal{K}$ that is not $\mathcal{B}$-conglomerable, and denote by $\underline{F}$ its $\mathcal{B}$-conglomerable natural extension.

Definition 4. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{K}$. Its *conglomerable natural extension* is the smallest coherent lower prevision $\underline{F}$ on $\mathcal{L}$ that dominates $\underline{P}$ and is conglomerable.

There may not be any dominating conglomerable coherent lower prevision, and then the conglomerable natural extension will not exist. On the other hand, if there is a dominating conglomerable model then there is a conglomerable natural extension, because conglomerability is preserved by taking lower envelopes.

We can assume without loss of generality that the domain of $\underline{P}$ is the set $\mathcal{L}$ of all gambles: otherwise, it suffices to consider the natural extension $\underline{E}$ of $\underline{P}$ to $\mathcal{L}$. To see that the conglomerable natural extensions of $\underline{P}$ and $\underline{E}$ coincide, denote them by $\underline{F}_1$ and $\underline{F}_2$, respectively. Trivially $\underline{F}_2 \geq \underline{F}_1$. Conversely, $\underline{F}_1$ is by definition a ($\mathcal{B}$-conglomerable) coherent lower prevision that dominates $\underline{P}$ on $\mathcal{K}$, and which as a consequence dominates also the natural extension $\underline{E}$—which is the smallest dominating coherent lower prevision. Hence $\underline{F}_1 \geq \underline{F}_2$, and therefore they are equal.

Given a coherent lower prevision $\underline{P}$, Walley defines its *conditional natural extension* as:

$$\underline{P}(f|B) := \begin{cases} \sup\left\{\mu\colon \underline{P}(B(f-\mu)) \geq 0\right\} & \text{if } \underline{P}(B) > 0 \\ \inf_{\omega \in B} f(\omega) & \text{otherwise} \end{cases} \tag{11}$$

for every $f \in \mathcal{L}$ and $B \in \mathcal{B}$. When $\underline{P}(B) > 0$, $\underline{P}(f|B)$ corresponds to the unique value μ such that $\underline{P}(B(f-\mu)) = 0$, i.e., for which (GBR) is satisfied. A consequence of this is that if we consider two coherent lower previsions $\underline{P}_1$ and $\underline{P}_2$, and their conditional natural extensions $\underline{P}_1(\cdot|\mathcal{B})$ and $\underline{P}_2(\cdot|\mathcal{B})$ given by Eq. (11), then

$$\underline{P}_1 \geq \underline{P}_2 \Rightarrow \underline{P}_1(\cdot|\mathcal{B}) \geq \underline{P}_2(\cdot|\mathcal{B}). \tag{12}$$

It turns out that this conditional natural extension can be used to characterise the conglomerability of $\underline{P}$.

Proposition 11 ([21, Theorem 6.8.2])**.** *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$. The following statements are equivalent:*

(i) *$\underline{P}$ is $\mathcal{B}$-conglomerable.*
(ii) *$\underline{P}$ is coherent with some conditional lower prevision $\underline{P}'(\cdot|\mathcal{B})$.*
(iii) *$\underline{P}$ is coherent with its conditional natural extension $\underline{P}(\cdot|\mathcal{B})$.*

In [21, Section 6.6], Walley gives a number of examples of coherent lower previsions that are not $\mathcal{B}$-conglomerable. We next give a sufficient condition for conglomerability:

Proposition 12. *If the conditional natural extension $\underline{P}(\cdot|\mathcal{B})$ of $\underline{P}$ is vacuous, then $\underline{P}$ is $\mathcal{B}$-conglomerable, and so is any coherent lower prevision $\underline{Q} \leq \underline{P}$.*

Proof. From the definition of the conditional natural extension, $\underline{P}(G_{\underline{P}}(f|B)) = 0$ for every gamble f and every set $B \in \mathcal{B}$. On the other hand, if $\underline{P}(\cdot|\mathcal{B})$ is vacuous then it follows from the coherence of $\underline{P}$ that $\underline{P}(G_{\underline{P}}(f|\mathcal{B})) \geq 0$, because $G_{\underline{P}}(f|\mathcal{B})$ is then non-negative. Hence the discussion in Section 2.1 tells us that $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent, and applying Proposition 11, we deduce that $\underline{P}$ is conglomerable.

On the other hand, given $\underline{Q} \leq \underline{P}$, it follows from Eq. (12) that the conditional lower prevision $\underline{Q}(\cdot|\mathcal{B})$ derived from $\underline{Q}$ using natural extension must be dominated by $\underline{P}(\cdot|\mathcal{B})$, and applying separate coherence we deduce that $\underline{Q}(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$, because $\underline{P}(\cdot|\mathcal{B})$ is vacuous. Applying the first part, we deduce that $\underline{Q}$ and $\underline{Q}(\cdot|\mathcal{B})$ are coherent, and therefore $\underline{Q}$ is also $\mathcal{B}$-conglomerable. ∎

If $\underline{P}$ is not $\mathcal{B}$-conglomerable, it is not coherent with its conditional natural extension $\underline{P}(\cdot|\mathcal{B})$. We will use this fact to kickstart a procedure that generates a sequence of coherent lower previsions $\underline{E}_n$ that will get closer to the conglomerable natural extension $\underline{F}$, similarly to what we have done in the treatment of conglomerability for sets of desirable gambles in the previous section.

Consider, in this case, the natural extensions $\underline{E}$ and $\underline{E}(\cdot|\mathcal{B})$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, determined by [21, Theorem 8.1.5]:

$$\underline{E}(f) := \sup\left\{\mu\colon f-\mu \geq G_{\underline{P}}(g) + G_{\underline{P}}(h|\mathcal{B}) \text{ for some } g,h \in \mathcal{L}\right\}, \tag{13}$$

and

$$\underline{E}(f|B) := \begin{cases} \sup\{\mu\colon \underline{E}(B(f-\mu)) \geq 0\} & \text{if } \underline{E}(B) > 0 \\ \sup\left\{\mu\colon B(f-\mu) \geq G_{\underline{P}}(g|B) \text{ for some } g \in \mathcal{L}\right\} & \text{otherwise.} \end{cases}$$

The conditional lower prevision $\underline{E}(\cdot|\mathcal{B})$ coincides with the conditional natural extension of $\underline{E}$, i.e., it can be obtained using Eq. (11). To see this, note that if $\underline{E}(B) = 0$ then also $\underline{P}(B) = 0$. Applying Eq. (11), we deduce that $G_{\underline{P}}(g|B) \geq 0$ for every gamble g and therefore

$$\begin{aligned} &\sup\left\{\mu\colon B(f-\mu) \geq G_{\underline{P}}(g|B) \text{ for some } g \in \mathcal{L}\right\} \\ &\qquad\qquad = \sup\{\mu\colon B(f-\mu) \geq 0\} = \inf_{\omega\in B} f(\omega). \end{aligned}$$

Since $\underline{E} \geq \underline{P}$, we deduce from Eq. (12) that $\underline{E}(\cdot|\mathcal{B}) \geq \underline{P}(\cdot|\mathcal{B})$.

Proposition 13. *The natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ is dominated by the conglomerable natural extension $\underline{F}$ of $\underline{P}$, if it exists. They coincide if and only if $\underline{E}$ and $\underline{E}(\cdot|\mathcal{B})$ are coherent. Moreover,*

$$\mathcal{M}(\underline{E}) = \left\{P \in \mathcal{M}(\underline{P}) \colon (\forall f \in \mathcal{L}) P(G_{\underline{P}}(f|\mathcal{B})) \geq 0\right\}. \tag{14}$$

Proof. From [21, Theorem. 8.1.2(c)], $\underline{E}$ and $\underline{E}(\cdot|\mathcal{B})$ are respective lower bounds for any coherent pairs of lower and conditional lower previsions that dominate $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, respectively. Given the conglomerable natural extension $\underline{F} \geq \underline{P}$ and the conditional lower prevision $\underline{F}(\cdot|\mathcal{B})$ it defines by conditional natural extension—coherent with it—, we see that $\underline{F}(\cdot|\mathcal{B}) \geq \underline{P}(\cdot|\mathcal{B})$: it suffices to take into account that $\underline{F} \geq \underline{P}$ and apply Eq. (12). As a consequence, $\underline{F} \geq \underline{E}$ and $\underline{F}(\cdot|\mathcal{B}) \geq \underline{E}(\cdot|\mathcal{B})$. Now, $\underline{E} = \underline{F}$ if and only if $\underline{E}$ is $\mathcal{B}$-conglomerable, and by Proposition 11 this is equivalent to $\underline{E}$ and $\underline{E}(\cdot|\mathcal{B})$ being coherent.

We conclude by proving Eq. (14). For the direct inclusion, consider any $E \in \mathcal{M}(\underline{E})$. It follows from the definition of $\underline{E}$ that $E \geq \underline{E} \geq \underline{P}$, so $E \in \mathcal{M}(\underline{P})$. Moreover, $E(G_{\underline{P}}(f|\mathcal{B})) \geq \underline{E}(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for every gamble f, where the last inequality follows from Eq. (13). Conversely, consider any linear prevision P belonging to the set in the right-hand side of Eq. (14). If there were some gamble g such that $P(g) < \underline{E}(g)$, then there would be some $\varepsilon > 0$ and gambles f and h such that $g - P(g) - \varepsilon \geq G_{\underline{P}}(f) + G_{\underline{P}}(h|\mathcal{B})$, whence $P(g - P(g) - \varepsilon) = -\varepsilon \geq P(G_{\underline{P}}(f) + G_{\underline{P}}(h|\mathcal{B})) \geq 0$, a contradiction. □

The fact that the natural extensions $\underline{E}$ and $\underline{E}(\cdot|\mathcal{B})$ need not be coherent, and that consequently, the natural extension $\underline{E}$ need not coincide with the conglomerable natural extension $\underline{F}$, is an indication that, although Walley's treatment of coherence and natural extension is intended to adequately deal with conglomerability, it falls somewhat short of this aim.

We next provide another interesting characterisation of $\underline{E}$, by means of the so-called marginal extension:

Proposition 14. *Consider any coherent lower prevision $\underline{P}$ and any separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$. Define the marginal extension $\underline{M} := \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, and let $\underline{E}$ be the natural extension of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$. Then*

$$\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{M}). \tag{15}$$

As a consequence, if $\underline{M} \geq \underline{P}$, then $\underline{M}$ coincides with $\underline{E}$, and then $\underline{M}$ is the conglomerable natural extension of $\underline{P}$.

Proof. We begin with the direct inclusion in Eq. (15). Consider any linear prevision $P \in \mathcal{M}(\underline{E})$. Since $\underline{E} \geq \underline{P}$, $P \in \mathcal{M}(\underline{P})$. From Eq. (14), it satisfies $P(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for every gamble f. Since P is additive, we deduce that

$$P(f) = P(G_{\underline{P}}(f|\mathcal{B})) + P(\underline{P}(f|\mathcal{B})) \geq P(\underline{P}(f|\mathcal{B})) \geq \underline{P}(\underline{P}(f|\mathcal{B})) = \underline{M}(f)$$

for every gamble f. Hence $P \in \mathcal{M}(\underline{M})$.

Conversely, consider any linear prevision in $\mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{M})$. Then for every gamble f it holds that $P(G_{\underline{P}}(f|\mathcal{B})) \geq \underline{M}(G_{\underline{P}}(f|\mathcal{B})) \geq 0$, where the last inequality holds because $\underline{M}$ is the marginal extension of $\underline{P}(\cdot|\mathcal{B})$ and the restriction of $\underline{P}$ to the set of $\mathcal{B}$-measurable gambles, which by [21, Theorem 6.7.2] is coherent with $\underline{P}(\cdot|\mathcal{B})$. Using Eq. (14), we deduce that $P \in \mathcal{M}(\underline{E})$.

We turn to the second statement. If $\underline{M}$ dominates $\underline{P}$, then $\mathcal{M}(\underline{M}) \subseteq \mathcal{M}(\underline{E})$ and therefore $\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{M})$ by Eq. (15). Hence $\underline{E} = \underline{M}$, or equivalently, $\underline{E}$ is the marginal extension $\underline{M}$ of the restriction of $\underline{P}$ to the set of $\mathcal{B}$-measurable gambles with $\underline{P}(\cdot|\mathcal{B})$. Since $\underline{M}$ is coherent with $\underline{P}(\cdot|\mathcal{B})$, it is a $\mathcal{B}$-conglomerable model by Proposition 11, and therefore it must dominate the conglomerable natural extension

$\underline{F}$ of $\underline{P}$: $\underline{M} \geq \underline{F}$. But since we also have $\underline{M} = \underline{E} \leq \underline{F}$ by Proposition 13, we deduce that $\underline{M} = \underline{F}$. □

Let us give an example showing that $\underline{E}$ does not coincide with the conglomerable natural extension in general:

Example 5. Let us consider $\Omega \coloneqq \mathbb{N} \cup -\mathbb{N}$, where as before $\mathbb{N}$ is the set of natural numbers without zero, $B_n \coloneqq \{-n, n\}$ and $\mathcal{B}$ the partition of Ω given by $\mathcal{B} \coloneqq \{B_n \colon n \in \mathbb{N}\}$. Let P be any finitely additive probability on $\mathcal{P}(\mathbb{N})$ that satisfies $P(\{n\}) = 0$ for every n, and let us consider the linear previsions $P_1, \ldots, P_4$, where P_1 is determined by (the expectation associated with) the σ-additive probability measure with mass function:

$$P_1(\{n\}) \coloneqq P_1(\{-n\}) \coloneqq \frac{1}{2^{n+1}}, \quad n \in \mathbb{N}$$

and $P_2, \ldots, P_4$ are given by, for any $h \in \mathcal{L}(\Omega)$:

$$P_2(h) \coloneqq \frac{1}{2} \sum_{n \in \mathbb{N}} h(n) \frac{1}{2^n} + \frac{1}{2} P(h_2),$$

and

$$P_3(h) \coloneqq \frac{3}{4} P(h_1) + \frac{1}{4} P(h_2), \text{ and } P_4(h) \coloneqq \frac{1}{2} P_1(h) + \frac{1}{2} P_3(h),$$

where the gambles h_1 and h_2 on $\mathbb{N}$ are defined by $h_1(n) \coloneqq h(n)$ and $h_2(n) \coloneqq h(-n)$ for all $n \in \mathbb{N}$.

Consider the coherent lower prevision $\underline{P} \coloneqq \min\{P_1, P_2, P_4\}$. For every $B_n \in \mathcal{B}$,

$$\underline{P}(B_n) = \min \left\{ \frac{1}{2^n}, \frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} \right\} > 0.$$

As a consequence, for every gamble $h \in \mathcal{L}(\Omega)$, Theorem 6.4.2 in [21] guarantees that

$$\underline{P}(h|B_n) = \min\{P_1(h|B_n), P_2(h|B_n), P_4(h|B_n)\} = \min \left\{ \frac{h(n) + h(-n)}{2}, h(n) \right\}. \tag{16}$$

To see that $\underline{P}$ is not $\mathcal{B}$-conglomerable, consider the gamble f given by

$$f(n) \coloneqq 1 - \frac{1}{n} \text{ and } f(-n) \coloneqq -f(n) = \frac{1}{n} - 1, \quad n \in \mathbb{N}.$$

It follows from Eq. (16) that $\underline{P}(f|B_n) = 0$ for every $n \in \mathbb{N}$, so $G_{\underline{P}}(f|\mathcal{B}) = f$. Now

$$-1 = \inf f_2 \leq P(f_2) \leq P\left(\mathbb{I}_{[n,\infty)} \left(\frac{1}{n} - 1 \right) \right) = \left(\frac{1}{n} - 1 \right) P([n, \infty]) = \frac{1}{n} - 1 \tag{17}$$

for all $n \in \mathbb{N}$, where the last equality holds because $P(\{1, \ldots, n\}) = 0$ for all $n \in \mathbb{N}$. Hence $P(f_2) = -1$ and therefore

$$\underline{P}(G_{\underline{P}}(f|\mathcal{B})) = \underline{P}(f) \leq P_2(f) = \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}} \left(1 - \frac{1}{n} \right) - \frac{1}{2} < 0.$$

This implies that $\underline{P}$ is not coherent with the conditional $\underline{P}(\cdot|\mathcal{B})$, and therefore, indeed, $\underline{P}$ is not $\mathcal{B}$-conglomerable by Proposition 11.

This makes us look at the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, so we are going to apply Eq. (14) to determine $\mathcal{M}(\underline{E})$. First of all, for every linear prevision $Q \in \mathcal{M}(\underline{P})$, there are $\alpha_1, \alpha_2, \alpha_4 \in [0, 1]$ such that $\alpha_1 + \alpha_2 + \alpha_4 = 1$ and $Q = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_4 P_4$.

We need to check which of these convex combinations satisfies $Q(G_{\underline{P}}(h|\mathcal{B})) \geq 0$ for all gambles h, and therefore belongs to $\mathcal{M}(\underline{E})$. It is easy to infer from Eq. (16) that for all $n \in \mathbb{N}$:

$$G_{\underline{P}}(h|\mathcal{B})(n) = \max \left\{ 0, \frac{h(n) - h(-n)}{2} \right\}$$

and

$$G_{\underline{P}}(h|\mathcal{B})(-n) = \max\left\{h(-n) - h(n), \frac{h(-n) - h(n)}{2}\right\}.$$

As a consequence, $G_{\underline{P}}(h|B_n) \geq 0$ as soon as $h(n) \leq h(-n)$, and this implies that $G_{\underline{P}}(h|\mathcal{B}) \geq G_{\underline{P}}(h|\mathcal{B})\mathbb{I}_C$, where $C \coloneqq \bigcup\{B_n \colon h(n) \geq h(-n)\}$. We can therefore concentrate on the worst-case gambles h such that $h(n) \geq h(-n)$ for every $n \in \mathbb{N}$. Let $g \coloneqq G_{\underline{P}}(h|\mathcal{B})$, then $g_1 \geq 0$ and $g_2 = -g_1$, and as a consequence,

$$\begin{aligned}
Q(g) &= \alpha_1 P_1(g) + \alpha_2 P_2(g) + \alpha_4 P_4(g) \\
&= 0 + \alpha_2\left(P_1(g\mathbb{I}_{\mathbb{N}}) + \frac{1}{2}P(g_2)\right) + \alpha_4\left(0 + \frac{3}{8}P(g_1) + \frac{1}{8}P(g_2)\right) \\
&= \alpha_2 P_1(g\mathbb{I}_{\mathbb{N}}) + P(g_1)\frac{1}{4}(\alpha_4 - 2\alpha_2).
\end{aligned}$$

If $\alpha_4 \geq 2\alpha_2$, we deduce from the non-negativity of g_1, and the non-negativity of g on $\mathbb{N}$, that $Q(g) \geq 0$ and therefore $Q \in \mathcal{M}(\underline{E})$. On the other hand, if $\alpha_4 < 2\alpha_2$ (and therefore $\alpha_2 > 0$) there is some natural number m such that $\alpha_2/2^m < (2\alpha_2 - \alpha_4)/4$. We consider the gamble h_m given by

$$h_m(n) \coloneqq \mathbb{I}_{[m,\infty)}(n) = \begin{cases} 0 & \text{if } n < m \\ 1 & \text{if } n \geq m \end{cases} \text{ and } h_m(-n) \coloneqq -h_m(n) \text{ for all } n \in \mathbb{N}.$$

Then $g_m \coloneqq G_{\underline{P}}(h_m|\mathcal{B}) = h_m$, $P_1(g_m\mathbb{I}_{\mathbb{N}}) = P_1([m,\infty)) = 1/2^m$ and $P((g_m)_1) = 1$, using a course of reasoning similar to the one leading to Eq. (17). As a consequence,

$$Q(g_m) = \alpha_2 P_1(g_m\mathbb{I}_{\mathbb{N}}) + \frac{1}{4}P((g_m)_1)(\alpha_4 - 2\alpha_2) = \frac{1}{2^m}\alpha_2 + \frac{1}{4}(\alpha_4 - 2\alpha_2) < 0,$$

and therefore $Q \notin \mathcal{M}(\underline{E})$.

We conclude that $\underline{E}$ is the lower envelope of the set $\{P_1, P_4, 1/3 P_2 + 2/3 P_4\}$, and as a consequence its conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ is given by [again, see Theorem 6.4.2 in [21]]:

$$\underline{E}(h|B_n) = \min\left\{\frac{h(n) + h(-n)}{2}, \frac{2h(n) + h(-n)}{3}\right\} \text{ for all gambles } h \text{ on } \Omega. \quad (18)$$

To see that $\underline{E}$ is not $\mathcal{B}$-conglomerable, we use Proposition 11 and show that it is not coherent with $\underline{E}(\cdot|\mathcal{B})$. Consider any gamble h such that $h(n) \leq h(-n)$ for all $n \in \mathbb{N}$. Then Eq. (18) yields $\underline{E}(h|B_n) = (2h(n) + h(-n))/3$, and consequently

$$G_{\underline{E}}(h|B_n)(n) = \frac{1}{3}[h(n) - h(-n)] \text{ and } G_{\underline{E}}(h|B_n)(-n) = -\frac{2}{3}[h(n) - h(-n)].$$

So, if we let $g \coloneqq G_{\underline{E}}(h|\mathcal{B})$, then we obtain $g_2 = -2g_1 \geq 0$, whence

$$\begin{aligned}
P_4(g) &= \frac{1}{2}P_1(g) + \frac{3}{8}P(g_1) + \frac{1}{8}P(g_2) \\
&= \frac{1}{2}[P_1(g\mathbb{I}_{\mathbb{N}}) + P_1(g\mathbb{I}_{-\mathbb{N}})] + \frac{1}{8}P(g_1) = -\frac{1}{8}[4P_1(g\mathbb{I}_{\mathbb{N}}) - P(g_1)].
\end{aligned}$$

Let $h(n) = h(-n) \coloneqq 0$ for $n = 1, 2$ and $h(n) = -h(-n) \coloneqq -1$ for $n > 2$. Then $P_1(g\mathbb{I}_{\mathbb{N}}) = -1/12$ and $P(g_1) = -2/3$, and therefore $P(g_1) < 4P_1(g\mathbb{I}_{\mathbb{N}})$, so we get $P_4(G_{\underline{E}}(h|\mathcal{B})) < 0$, whence $\underline{E}(G_{\underline{E}}(h|\mathcal{B})) < 0$. This shows that $\underline{E}$ is not $\mathcal{B}$-conglomerable, and as a consequence it does not coincide with the conglomerable natural extension $\underline{F}$ of $\underline{P}$, which exists because $P_1 \geq \underline{P}$ is $\mathcal{B}$-conglomerable (because any σ-additive model is, see Theorem 6.9.1 in [21]). ⧫

On the other hand, we can give a number of sufficient conditions for the natural extension to be $\mathcal{B}$-conglomerable.

Proposition 15. *Consider a coherent lower prevision $\underline{P}$ that has a conglomerable natural extension $\underline{F}$. If there is some coherent lower prevision $\underline{Q} \geq \underline{P}$ that is coherent with the conditional natural extension $\underline{P}(\cdot|\mathcal{B})$ of $\underline{P}$, then the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ coincides with the conglomerable natural extension $\underline{F}$. As a consequence, if the conditional natural extension $\underline{P}(\cdot|\mathcal{B})$ of $\underline{P}$ is linear, then the conglomerable natural extension $\underline{F}$ coincides with the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$.*

Proof. Assume that there is some $\underline{Q} \geq \underline{P}$ such that $\underline{Q}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent. Then since $\underline{E}(\cdot|\mathcal{B}) \geq \underline{P}(\cdot|\mathcal{B})$ and the pair $\underline{E}$ and $\underline{E}(\cdot|\mathcal{B})$ constitute a lower bound for any pair of coherent lower and conditional lower previsions that dominate $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, we deduce that $\underline{P}(\cdot|\mathcal{B}) = \underline{E}(\cdot|\mathcal{B})$ and $\underline{E} \leq \underline{Q}$. To see that $\underline{E}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent, note on the one hand that, by Eq. (14), $\underline{E}(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for any gamble f, and in particular $\underline{E}(G_{\underline{P}}(f|B)) \geq 0$. On the other hand,

$$0 \leq \underline{E}(G_{\underline{P}}(f|B)) \leq \underline{Q}(G_{\underline{P}}(f|B)) = 0,$$

where the last inequality follows from the coherence of $\underline{Q}$ and $\underline{P}(\cdot|\mathcal{B})$. It follows that $\underline{E}$ and $P(\cdot|\mathcal{B})$ are coherent, so $\underline{E}$ is $\mathcal{B}$-conglomerable by Proposition 11, and as a consequence it is the $\mathcal{B}$-conglomerable natural extension $\underline{F}$ of $\underline{P}$.

Let us prove now the second statement. From Eq. (12), $\underline{F}(\cdot|\mathcal{B}) \geq \underline{P}(\cdot|\mathcal{B})$. Since this second functional is linear, it follows from separate coherence that $\underline{F}(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$: if otherwise $\underline{F}(f|B) > \underline{P}(f|B)$ for some gamble f and some $B \in \mathcal{B}$, then

$$\underline{F}(0|B) = \underline{F}(f - f|B) \geq \underline{F}(f|B) + \underline{F}(-f|B) > \underline{P}(f|B) + \underline{P}(-f|B) = 0,$$

a contradiction. Hence $\underline{F}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent. Applying the first statement, we deduce that $\underline{F}$ coincides with the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$. □

We can now continue with our procedure to generate a sequence of coherent lower previsions $\underline{E}_n$ that will get closer to the conglomerable natural extension $\underline{F}$, similarly to what we have done in the treatment of conglomerability for sets of desirable gambles in the previous section. When $\underline{P}$ is not $\mathcal{B}$-conglomerable, we can consider the natural extension $\underline{E}_1 := \underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$. If $\underline{E}_1$ is not $\mathcal{B}$-conglomerable, we can consider the natural extension $\underline{E}_2$ of $\underline{E}_1$ and its conditional natural extension $\underline{E}_1(\cdot|\mathcal{B})$, and so on. Our next result shows that the resulting sequence $\underline{E}_n$ of coherent lower previsions does not stabilise (become constant) unless we get to a $\mathcal{B}$-conglomerable coherent lower prevision:

Proposition 16. *If a coherent lower prevision $\underline{P}$ is not $\mathcal{B}$-conglomerable, then it does not coincide with the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$. On the other hand, if $\underline{E}(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$ then $\underline{E}$ is $\mathcal{B}$-conglomerable.*

Proof. If $\underline{P}$ is not $\mathcal{B}$-conglomerable, this means that it is not coherent with its conditional natural extension $\underline{P}(\cdot|\mathcal{B})$. Since $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ satisfy (GBR), this means that there is some gamble f such that $\underline{P}(G_{\underline{P}}(f|\mathcal{B})) < 0$. On the other hand, the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ satisfies $\underline{E}(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ because of Eq. (14), and as a consequence it cannot coincide with $\underline{P}$.

On the other hand, if $\underline{E}(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$, then since $\underline{E}(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for all f because of the definition of $\underline{E}$, we conclude that $\underline{E}$ is coherent with $\underline{E}(\cdot|\mathcal{B})$, and as a consequence it is $\mathcal{B}$-conglomerable. □

We can establish Proposition 12 as a consequence of this result: if $\underline{P}(\cdot|\mathcal{B})$ is vacuous, we deduce that $\underline{P}$ coincides with the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, and as a consequence it must be $\mathcal{B}$-conglomerable. We can also deduce the second statement of Proposition 15: if $P(\cdot|\mathcal{B})$ is linear then it necessarily coincides with $\underline{E}(\cdot|\mathcal{B})$, and this means that $\underline{E}$ is the conglomerable natural extension.

The sequence $\underline{E}_n$ is non-decreasing and dominated by the supremum operator, and it therefore converges point-wise to a coherent lower prevision $\underline{E}_\infty$, which by construction is dominated by the conglomerable natural extension $\underline{F}$ of $\underline{P}$: it suffices to use induction and to take into account that at any step n, $\underline{E}_{n+1}$ is a lower bound of any coherent extension of $\underline{E}_n$ and $\underline{E}_n(\cdot|\mathcal{B})$, and therefore it is bounded by the conglomerable natural extension $\underline{F}$. It is an open problem whether the two coherent lower previsions $\underline{E}_\infty$ and $\underline{F}$ coincide, or to find an example where $\underline{E}_n$ does not coincide with $\underline{F}_\infty$ for any $n \in \mathbb{N}$, i.e., where we cannot get to the conglomerable natural extension in a finite number of steps. But we can establish the following convergence result for the conditional natural extension $\underline{E}_\infty(\cdot|\mathcal{B})$ of $\underline{E}_\infty$:

Proposition 17. *For every gamble f, $\underline{E}_\infty(f|\mathcal{B}) = \lim_{n\to\infty} \underline{E}_n(f|\mathcal{B})$.*

Proof. Since $\underline{E}_n$ is an non-decreasing sequence of coherent lower previsions that converges towards $\underline{E}_\infty$, it follows from the definition of the conditional natural extension that for every gamble f the sequence $\underline{E}_n(f|\mathcal{B})$ is a bounded and non-decreasing sequence of gambles whose limit is dominated by $\underline{E}_\infty(f|\mathcal{B})$. To see that there is equality, consider an arbitrary set $B \in \mathcal{B}$. If $\underline{E}_\infty(B) = 0$ then $\underline{E}_\infty(f|B) = \inf_{\omega\in B} f(\omega) = \underline{E}_n(f|B)$ for all n. If $\underline{E}_\infty(B) > 0$, then there is some natural number m such that $\underline{E}_n(B) > 0$ for all $n \geq m$, taking into account that $\underline{E}_\infty(B) = \lim_{n\to\infty} \underline{E}_n(B)$. As a consequence, taking into account that due to (separate) coherence $\underline{E}_\infty(B(f-\mu))$ and $\underline{E}_n(B(f-\mu))$ are continuous and non-increasing functions of μ:

$$
\begin{aligned}
\underline{E}_\infty(f|B) &= \sup\{\mu\colon \underline{E}_\infty(B(f-\mu)) \geq 0\} = \sup\{\mu\colon \underline{E}_\infty(B(f-\mu)) > 0\} \\
&= \sup\{\mu\colon \underline{E}_n(B(f-\mu)) > 0 \text{ for some } n \in \mathbb{N}\} \\
&= \sup_{n\in\mathbb{N}} \sup\{\mu\colon \underline{E}_n(B(f-\mu)) > 0\} = \sup_{n\in\mathbb{N}} \sup\{\mu\colon \underline{E}_n(B(f-\mu)) \geq 0\} \\
&= \sup_{n\in\mathbb{N}} \underline{E}_n(f|B) = \lim_{n\to\infty} \underline{E}_n(f|B). \qquad \square
\end{aligned}
$$

$\underline{E}_\infty$ coincides with the conglomerable natural extension $\underline{F}$ if and only if $\underline{E}_\infty$ is $\mathcal{B}$-conglomerable, and this is equivalent to $\underline{E}_\infty(G_{\underline{E}_\infty}(f|\mathcal{B})) \geq 0$ for every gamble f. This holds for instance if $\underline{E}_\infty(\cdot|\mathcal{B})$ is the uniform limit of the sequence $\underline{E}_n(\cdot|\mathcal{B})$: it then follows from the coherence of $\underline{E}_\infty$ (e.g., see [21, Section 2.6.1(ℓ)]) that

$$\underline{E}_\infty(G_{\underline{E}_\infty}(f|\mathcal{B})) = \lim_{n\to\infty} \underline{E}_\infty(G_{\underline{E}_n}(f|\mathcal{B})) \geq \lim_{n\to\infty} \underline{E}_n(G_{\underline{E}_n}(f|\mathcal{B})) \geq 0,$$

using Eq. (14) for the last inequality. But we stress that this uniform convergence is a very strong requirement: assume for instance that we have a countable partition $\mathcal{B} = \{B_n\colon n \in \mathbb{N}\}$. If the following property holds

$$(\forall n \in \mathbb{N}, \varepsilon > 0)(\exists f_{n,\varepsilon} \in \mathcal{L})(\sup(|f_{n,\varepsilon}|) \leq 1, \underline{E}_\infty(f_{n,\varepsilon}|B_n) - \underline{E}_n(f_{n,\varepsilon}|B_n) > \varepsilon),$$

then the convergence is not uniform: consider, for instance, for any given ε, the gamble $f_\varepsilon := \sum_{n\in\mathbb{N}} B_n f_{n,\varepsilon}$.

5. Connection between the two approaches

In spite of the connection between sets of desirable gambles and coherent lower previsions we have summarised in Section 2.2, the correspondence does not extend towards the notion of conglomerable natural extension we have discussed in Sections 3 and 4. The reason is that in our definition of the conglomerable natural extension of a set of gambles we are using condition D5, while the conglomerable natural extension for coherent lower previsions is based on condition wBC which is equivalent to wD5, and which is therefore weaker than D5 in general. In this section, we explore the connection in detail.

Let $\mathcal{R}$ be a set of desirable gambles satisfying D1–D4, and let $\underline{P}$ be its associated coherent lower prevision, given by Eq. (3). If $\mathcal{R}$ does not satisfy D5, then we can consider the increasing sequence of sets of desirable gambles $\mathcal{E}_n$, defined by means of Eqs. (7). With each of these sets of desirable gambles we can associate a coherent lower prevision $\underline{P}_n$, again by means of Eq. (3). At the same time, we can consider the sequence $\underline{E}_n$ of coherent lower previsions derived from $\underline{P}$ in the manner discussed in Section 4: $\underline{E}_1$ is the natural extension of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, where $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{P}$; $\underline{E}_2$ is the natural extension of $\underline{E}_1$ and $\underline{E}_1(\cdot|\mathcal{B})$; and so on.

Let us investigate the relationship between the sequences $\underline{P}_n$ and $\underline{E}_n$:

Proposition 18. *For every gamble f, $\underline{E}_n(f) \leq \underline{P}_n(f)$.*

Proof. We use induction on n.

We begin with $n = 1$. Consider any gamble $f \in \mathcal{L}$ and $\mu < \underline{E}_1(f)$. Then there are gambles g, h such that

$$f - \mu \geq G_{\underline{P}}(g) + G_{\underline{P}}(h|\mathcal{B}).$$

As a consequence, given $\varepsilon > 0$,

$$f - \mu + \varepsilon \geq G_{\underline{P}}(g) + \frac{\varepsilon}{2} + G_{\underline{P}}(h|\mathcal{B}) + \frac{\varepsilon}{2}.$$

Since $\mathcal{R}$ is a coherent set of desirable gambles and $\underline{P}$ is derived from $\mathcal{R}$ using Eq. (3), we deduce that $G_{\underline{P}}(g) + \varepsilon/2 = g - (\underline{P}(g) - \varepsilon/2)$ belongs to $\mathcal{R}$. Similarly, for every $B \in \mathcal{B}$ the gamble $G_{\underline{P}}(h|B) + B\varepsilon/2$ also belongs to $\mathcal{R}$: if $\underline{P}(B) = 0$ this is a positive gamble, which belongs to $\mathcal{R}$ because this set satisfies D1; and if $\underline{P}(B) > 0$ then

$$\begin{aligned}\underline{P}(h|B) &= \sup\{\mu \colon \underline{P}(B(h-\mu)) \geq 0\} \\ &= \sup\{\mu \colon \underline{P}(B(h-\mu)) > 0\} = \sup\{\mu \colon B(h-\mu) \in \mathcal{R}\},\end{aligned}$$

so indeed $G_{\underline{P}}(h|B) + B\varepsilon/2 \in \mathcal{R}$. As a consequence, $G_{\underline{P}}(h|\mathcal{B}) + \varepsilon/2 \in \mathcal{R}^{\mathcal{B}}$, and therefore $f - \mu + \varepsilon \in \mathcal{E}_1$. This implies that $\underline{P}_1(f) \geq \mu - \varepsilon$, and since we can arrive at this conclusion for every $\mu < \underline{E}_1(f)$ and every $\varepsilon > 0$, we conclude that $\underline{P}_1(f) \geq \underline{E}_1(f)$.

Assume now that the result holds for $n-1$, and let us show that it also holds for n. From the induction hypothesis, it follows that $\underline{E}_{n-1} \leq \underline{P}_{n-1}$, and applying Eq. (12) we find that $\underline{E}_{n-1}(\cdot|\mathcal{B}) \leq \underline{P}_{n-1}(\cdot|\mathcal{B})$.

Using the same reasoning as in the case $n = 1$, $\underline{P}_n$ dominates the natural extension $\underline{Q}_n$ of $\underline{P}_{n-1}$ and $\underline{P}_{n-1}(\cdot|\mathcal{B})$. Since $\underline{P}_{n-1} \geq \underline{E}_{n-1}$ and $\underline{P}_{n-1}(\cdot|\mathcal{B}) \geq \underline{E}_{n-1}(\cdot|\mathcal{B})$, we deduce that $\underline{Q}_n$ in turn dominates the natural extension $\underline{E}_n$ of $\underline{E}_{n-1}$ and $\underline{E}_{n-1}(\cdot|\mathcal{B})$. Hence $\underline{E}_n \leq \underline{Q}_n \leq \underline{P}_n$. □

However, the coherent lower previsions $\underline{P}_n$ and $\underline{E}_n$ do not coincide in general, as the following counterexample shows:

Example 6. Consider the set of desirable gambles $\mathcal{R}$ from Example 2, and let $\underline{P}$ be its associated coherent lower prevision. We have shown in Example 3 that $\mathcal{R}$ satisfies wD5, and therefore Theorem 3 implies that $\underline{P}$ is $\mathcal{B}$-conglomerable, and in particular $\underline{E}_1(f) = \underline{P}(f)$ for every f. On the other hand, we have seen in Example 2 that $\mathcal{R}$ does not satisfy D5, and in particular that the gamble $g := \mathbb{I}_{\{\text{even}\}} - \mathbb{I}_{\{\text{odd}\}}$ belongs to $\mathcal{R}^{\mathcal{B}} \setminus \mathcal{R}$. Moreover, we have seen in Example 4 that $\underline{P}(g) = \sup\{\mu \colon g - \mu \in \mathcal{R}\} = -1$. From all this, we deduce that $\underline{P}_1(g) \geq 0 > -1 = \underline{P}(f) = \underline{E}_1(f)$, showing that $\underline{P}_1 \neq \underline{E}_1$: the inequality can be strict. ♦

The reason for this can be found in the following characterisation of $\underline{P}_n$:

Proposition 19. *$\underline{P}_n$ is the natural extension of $\underline{P}_{n-1}$ and $\underline{P}'_{n-1}(\cdot|\mathcal{B})$, where the conditional lower prevision $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ is derived from the set of desirable gambles $\mathcal{E}_{n-1}$ by Eq. (5).*

Proof. For every gamble f,

$$\begin{aligned}\underline{P}_n(f) &= \sup\{\mu\colon f-\mu \in \mathcal{E}_n\}\\ &= \sup\left\{\mu\colon f-\mu = \mu_1 g + \mu_2 h, g \in \mathcal{E}_{n-1}, h \in \mathcal{E}^{\mathcal{B}}_{n-1}, \mu_k \in \{0,1\}, \max_k \mu_k = 1\right\}.\end{aligned}$$

Consider $\mu < \underline{P}_n(f)$. Then there are $\mu_k \in \{0,1\}$ such that $\max_k \mu_k = 1$, $g \in \mathcal{E}_{n-1}$ and $h \in \mathcal{E}^{\mathcal{B}}_{n-1}$ such that $f - \mu = \mu_1 g + \mu_2 h$. $g \in \mathcal{E}_{n-1}$ implies $\underline{P}_{n-1}(g) \geq 0$, so $g \geq G_{\underline{P}_{n-1}}(g)$. $h \in \mathcal{E}^{\mathcal{B}}_{n-1}$ implies that $Bh \in \mathcal{E}_{n-1} \cup \{0\}$ for every $B \in \mathcal{B}$, whence $\underline{P}'_{n-1}(h|\mathcal{B}) \geq 0$. As a consequence

$$f - \mu \geq G_{\underline{P}_{n-1}}(\mu_1 g) + G_{\underline{P}'_{n-1}}(\mu_2 h|\mathcal{B}). \tag{19}$$

This means that the natural extension $\underline{E}'_n$ of $\underline{P}_{n-1}$ and $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ satisfies $\underline{E}'_n(f) \geq \mu$, and as a consequence $\underline{E}'_n(f) \geq \underline{P}_n(f)$.

Conversely, let $\mu < \underline{E}'_n(f)$. Then there are gambles g and h, and $\mu_1, \mu_2 \in \{0,1\}$ with at least one of them equal to 1 such that Eq. (19) holds. Given $\varepsilon > 0$ it follows from the definition of $\underline{P}_{n-1}$ $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ that $G_{\underline{P}_{n-1}}(\mu_1 g) + \frac{\varepsilon}{2} \in \mathcal{E}_{n-1}$ and similarly $B(\mu_2 h - \underline{P}'_{n-1}(\mu_2 h|B) + \frac{\varepsilon}{2})$ belongs to $\mathcal{E}_{n-1}$ for every $B \in \mathcal{B}$, or, equivalently, $G_{\underline{P}'_{n-1}}(\mu_2 h|\mathcal{B}) + \frac{\varepsilon}{2}$ belongs to $\mathcal{E}^{\mathcal{B}}_{n-1}$. But this means that $f - \mu + \varepsilon$ belongs to $\mathcal{E}_n$ for every $\varepsilon > 0$, and as a consequence $\underline{P}_n(f) \geq \mu$. We conclude that $\underline{P}_n(f) \geq \underline{E}'_n(f)$ and therefore they are equal. □

On the other hand, $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ satisfies (GBR) with respect to $\underline{P}_{n-1}$: given a gamble f and a set $B \in \mathcal{B}$, then for every $\varepsilon > 0$,

$$\underline{P}_{n-1}(G_{\underline{P}'_{n-1}}(f|B) + \varepsilon) \geq \underline{P}_{n-1}(B(f - \underline{P}'_{n-1}(f|B) + \varepsilon)) \geq 0,$$

so $\underline{P}_{n-1}(G_{\underline{P}'_{n-1}}(f|B)) \geq -\varepsilon$ for every $\varepsilon > 0$, whence $\underline{P}_{n-1}(G_{\underline{P}'_{n-1}}(f|B)) \geq 0$. And conversely, if there is some $\varepsilon > 0$ such that $\underline{P}_{n-1}(G_{\underline{P}'_{n-1}}(f|B)) \geq \varepsilon$, then the gamble $G_{\underline{P}'_{n-1}}(f|B) - \frac{\varepsilon}{2}$ must belong to $\mathcal{E}_{n-1}$, and therefore also the gamble $B(f - \underline{P}'_{n-1}(f|B) - \frac{\varepsilon}{2})$, which is greater, belongs to $\mathcal{E}_{n-1}$. But this means that we can increase the value $\underline{P}'_{n-1}(f|B)$ by $\frac{\varepsilon}{2} > 0$, a contradiction.

As a consequence, $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ can strictly dominate the conditional natural extension $\underline{P}_{n-1}(\cdot|\mathcal{B})$ of $\underline{P}_{n-1}$ only when some of the conditioning events have lower probability zero.

Using Proposition 18, we can now establish the following:

Proposition 20. *Let $\underline{\mathcal{R}}$ be a coherent set of strictly desirable gambles, and let $\underline{P}$ be its associated coherent lower prevision. Then $\underline{P}_1 = \underline{E}_1$. As a consequence, if $\mathcal{E}_1$ is the conglomerable natural extension of $\underline{\mathcal{R}}$, then $\underline{E}_1$ is the conglomerable natural extension of $\underline{P}$.*

Proof. By Proposition 18, it suffices to show that $\underline{E}_1(f) \geq \underline{P}_1(f)$ for every gamble f. Moreover, using the results from Section 2.2, $\underline{\mathcal{R}}$ must be the set of strictly desirable gambles associated to $\underline{P}$, so $\underline{\mathcal{R}} = \mathcal{L}^+ \cup \{f \in \mathcal{L}\colon \underline{P}(f) > 0\}$.

Since $\underline{P}_1$ is the coherent lower prevision associated to $\mathcal{E}_1$, for every $\varepsilon > 0$, the gamble $f - \underline{P}_1(f) + \varepsilon$ belongs to $\mathcal{E}_1$, and as a consequence there are gambles g and h such that $g \in \underline{\mathcal{R}}$ and $h \in \underline{\mathcal{R}}^{\mathcal{B}}$, and $\mu_1, \mu_2 \in \{0,1\}$ with $\max\{\mu_1, \mu_2\} = 1$ such that $f - \underline{P}_1(f) + \varepsilon = \mu_1 g + \mu_2 h$.

From the definition of $\underline{P}$, $g \in \underline{\mathcal{R}}$ implies that $\underline{P}(g) \geq 0$, whence $g \geq G_{\underline{P}}(g)$ and therefore $\mu_1 g \geq G_{\underline{P}}(\mu_1 g)$. On the other hand, if $h \in \underline{\mathcal{R}}^{\mathcal{B}}$ then $Bh \in \underline{\mathcal{R}} \cup \{0\}$ for every $B \in \mathcal{B}$. If $Bh = 0$, then trivially $\underline{P}(h|B) \geq 0$. If $Bh \in \underline{\mathcal{R}}$, there are two possibilities: if $\underline{P}(B) > 0$ then $Bh \in \underline{\mathcal{R}}$ implies that

$$0 \leq \sup\{\mu\colon B(h - \mu) \in \underline{\mathcal{R}}\} \leq \sup\{\mu\colon \underline{P}(B(h - \mu)) \geq 0\} = \underline{P}(h|B),$$

where the last equality follows from (GBR). The second possibility is that $\underline{P}(B) = 0$. Then Lemma 1 implies that $\underline{P}(Bh) \leq 0$, so Bh can only belong to $\underline{\mathcal{R}}$ if it is a non-negative gamble. Therefore $\underline{P}(h|B) = \sup\{\mu\colon B(h-\mu) \in \underline{\mathcal{R}}\} = \sup\{\mu\colon B(h-\mu) \gneq 0\} = \inf_{\omega\in B} h(\omega)$.

From all this we deduce that the conditional lower prevision associated to $\underline{\mathcal{R}}$ is the conditional natural extension $\underline{P}(\cdot|\mathcal{B})$ of $\underline{P}$, and in this case that $\underline{P}(h|\mathcal{B}) \geq 0$. This implies that $h \geq G_{\underline{P}}(h|\mathcal{B})$, and therefore $\mu_2 h \geq G_{\underline{P}}(\mu_2 h|\mathcal{B})$. As a consequence, $f - \underline{P}_1(f) + \varepsilon \geq G_{\underline{P}}(\mu_1 g) + G_{\underline{P}}(\mu_2 h|\mathcal{B})$. Hence $\underline{E}_1(f) \geq \underline{P}_1(f) - \varepsilon$, and since this holds for all $\varepsilon > 0$, we find that $\underline{E}_1(f) \geq \underline{P}_1(f)$.

For the second part, if $\mathcal{E}_1$ is the conglomerable natural extension of $\underline{\mathcal{R}}$, then it satisfies D5 and in particular wD5. We deduce from Theorem 3 that the coherent lower prevision $\underline{P}_1 = \underline{E}_1$ satisfies wBC, i.e., that it is $\mathcal{B}$-conglomerable. Since $\underline{E}_n$ is a lower bound of the conglomerable natural extension for every $n \in \mathbb{N}$, we find that in this case $\underline{E}_1$ is the conglomerable natural extension. □

We mention however that the number of steps necessary to compute the conglomerable natural extension can be different in the two cases, as Example 6 shows.

As a consequence of Proposition 20, if $\underline{E}_1$ is not $\mathcal{B}$-conglomerable, then $\mathcal{E}_1$ does not satisfy D5, provided we start from a set of strictly desirable gambles. This observation allows us to give another example where the sequence of sets of desirable gambles does not stabilise in the first step:

Example 7. Consider the coherent lower prevision $\underline{P}$ from Example 5 and let $\underline{\mathcal{R}}$ be its associated set of strictly desirable gambles. We have shown in Example 5 that the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ is not $\mathcal{B}$-conglomerable, and therefore it does not coincide with the conglomerable natural extension of $\underline{P}$. Applying Proposition 20, we deduce that $\mathcal{E}_1$ cannot be the conglomerable natural extension of $\underline{\mathcal{R}}$, and therefore the sequence $\mathcal{E}_n$ does not stabilise at the first step. ♦

We give another sufficient condition for the two sequences of coherent lower previsions to coincide:

Proposition 21. *If $\underline{P}(B) > 0$ for every $B \in \mathcal{B}$, then $\underline{P}_n(f) = \underline{E}_n(f)$ for all $f \in \mathcal{L}$.*

Proof. We use induction on n. We first give a proof for $n = 1$. From Proposition 19, $\underline{P}_1$ is the natural extension of $\underline{P}, \underline{P}'(\cdot|\mathcal{B})$, where $\underline{P}'(\cdot|\mathcal{B})$ is derived from $\mathcal{R}$ using Eq. (5). Since we have proven that $\underline{P}'(\cdot|\mathcal{B})$ satisfies (GBR) with respect to $\underline{P}$ and $\underline{P}(B) > 0$, it follows that $\underline{P}'(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$, and as a consequence $\underline{P}_1$ coincides with the natural extension $\underline{E}_1$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$.

Similarly, if the result holds for $n-1$, we know from Proposition 19 that $\underline{P}_n$ is the natural extension of $\underline{P}_{n-1}$ and $\underline{P}'_{n-1}(\cdot|\mathcal{B})$, where $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ is derived from $\mathcal{E}_{n-1}$ using Eq. (5). Since we have proved that $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ satisfies (GBR) with respect to $\underline{P}_{n-1} = \underline{E}_{n-1}$ and $\underline{P}_{n-1}(B) \geq \underline{P}(B) > 0$, it follows that $\underline{P}'_{n-1}(\cdot|\mathcal{B}) = \underline{E}_{n-1}(\cdot|\mathcal{B})$. As a result, $\underline{P}_n$ coincides with the natural extension $\underline{E}_n(f)$ of $\underline{E}_{n-1}, \underline{E}_{n-1}(\cdot|\mathcal{B})$. □

Corollary 22. *If $\underline{P}(B) > 0$ for all $B \in \mathcal{B}$ and $\mathcal{E}_n$ is the conglomerable natural extension of $\mathcal{R}$, then $\underline{E}_n$ is the conglomerable natural extension of $\underline{P}$.*

Proof. If $\mathcal{E}_n$ is the conglomerable natural extension of $\mathcal{R}$ then we have $\mathcal{E}_n = \mathcal{E}_{n+1}$, whence $\underline{P}_n = \underline{P}_{n+1}$, and, taking into account Proposition 21, also $\underline{E}_n = \underline{E}_{n+1}$. Now, applying Proposition 16 we deduce that $\underline{E}_n$ must be the conglomerable natural extension of $\underline{P}$. □

The condition $\underline{P}(B) > 0$ for every $B \in \mathcal{B}$ does not imply that the sequence stabilises at the first step, as Example 5 shows. On the other hand, the sequences $\underline{E}_n$ and $\mathcal{E}_n$ need not stabilise at the same time, as we can deduce from the following:

Example 8. Take $\Omega := \mathbb{N}$, $B_n := \{2n, 2n-1\}$, and let P be the countably additive probability defined by $P(\{2n\}) := P(\{2n-1\}) := \frac{1}{2^{n+1}}$. Consider the set of gambles

$$\mathcal{R} := \{f\colon P(f) > 0\} \cup \{f\colon P(f) = 0, \operatorname{supp}(f) \text{ finite, and } f(\min\operatorname{supp}(f)) > 0\},$$

where $\operatorname{supp}(f) := \{n \in \mathbb{N}\colon f(n) \neq 0\}$. Since P is countably additive on $\mathcal{B}$, it is $\mathcal{B}$-conglomerable by [21, Theorem 6.9.1]. Applying Theorem 3, we deduce that $\mathcal{R}$ is weakly conglomerable. However, it is not conglomerable because the gamble $\mathbb{I}_{\{\text{odd}\}} - \mathbb{I}_{\{\text{even}\}}$ belongs to $\mathcal{R}^{\mathcal{B}} \setminus \mathcal{R}$.

To see that the conglomerable natural extension of $\mathcal{R}$ exists, note that its superset

$$\mathcal{F}' := \{f\colon P(f) > 0\} \cup \{f\colon P(f) = 0 \text{ and } f(\min\operatorname{supp}(f)) > 0\}$$

is coherent and conglomerable.

D1: $\mathcal{L}^+ \subseteq \{f\colon P(f) > 0\} \subseteq \mathcal{F}'$.

D2: $\mathcal{F}'$ does not include the zero gamble by construction.

D3: Given a gamble f such that $P(f) > 0$, it follows that $P(\lambda f) > 0$ for every $\lambda > 0$; and given f such that $P(f) = 0$ and $f(\min\operatorname{supp}(f)) > 0$, it holds that $P(\lambda f) = 0$ and $\lambda f(\min\operatorname{supp}(\lambda f)) = \lambda f(\min\operatorname{supp}(f)) > 0$.

D4: Given $f, g \in \mathcal{F}'$, if either $P(f) > 0$ or $P(g) > 0$, we deduce that $P(f+g) > 0$; if $P(f) = P(g) = 0$, then $P(f+g) = 0$ and, taking into account that $f(\min\operatorname{supp}(f)) > 0$ and $g(\min\operatorname{supp}(g)) > 0$, we deduce the equality $\min\operatorname{supp}(f+g) = \min\{\min\operatorname{supp}(f), \min\operatorname{supp}(g)\}$. As a consequence, $(f+g)(\min\operatorname{supp}(f+g)) > 0$.

D5: Given $f \neq 0$ such that $B_n f \in \mathcal{F}' \cup \{0\}$ for every $B_n \in \mathcal{B}$, there are two possibilities: either there is some B_n such that $P(B_n f) > 0$, and the countable additivity of P implies that $P(f) > 0$ and $f \in \mathcal{F}'$; or $P(B_n f) = 0$ for every B, whence $P(f) = 0$. If we then consider the smallest n such that $B_n f$ is non-zero, we must have that $\min\operatorname{supp}(f) = \min\operatorname{supp}(B_n f)$, and $f(\min\operatorname{supp}(f)) = B_n f(\min\operatorname{supp}(B_n f)) > 0$. ♦

6. Conglomerability and coherence

In this section, we investigate in more detail the connections between the notions of conglomerability and coherence, first for sets of desirable gambles and later for coherent lower previsions.

6.1. Conglomerability and coherence for sets of desirable gambles.

Let $\underline{\mathcal{R}}$ be a coherent set of strictly desirable gambles, and let $\underline{\mathcal{R}}^{\mathcal{B}}$ be the set we can associate with it by means of D5. By Proposition 9, $\underline{\mathcal{R}}^{\mathcal{B}}$ satisfies D1– D5, and moreover $\underline{\mathcal{R}}$ is conglomerable when $\underline{\mathcal{R}}^{\mathcal{B}} \subseteq \underline{\mathcal{R}}$. The connection between this property and the coherence of $\underline{\mathcal{R}} \cup \underline{\mathcal{R}}^{\mathcal{B}}$ is given by the following theorem:

Theorem 23. *Let $\underline{\mathcal{R}}$ be a coherent set of strictly desirable gambles. Then each of the following statements implies the next:*

(i) $\underline{\mathcal{R}} = \underline{\mathcal{R}}^{\mathcal{B}}$.
(ii) $\underline{\mathcal{R}}$ *is conglomerable.*
(iii) $\mathcal{E}_1$ *is the conglomerable natural extension of* $\underline{\mathcal{R}}$.
(iv) $\underline{\mathcal{R}} \cup \underline{\mathcal{R}}^{\mathcal{B}}$ *is included in a coherent set.*

Proof. That the first statement implies the second is trivial. The second implies the third because if $\underline{\mathcal{R}}$ is conglomerable then $\underline{\mathcal{R}}^{\mathcal{B}} \subseteq \underline{\mathcal{R}}$ and therefore $\mathcal{E}_1 = \underline{\mathcal{R}}$. Finally, if $\mathcal{E}_1$ is the conglomerable natural extension of $\underline{\mathcal{R}}$ then it is in a particular a coherent superset of $\underline{\mathcal{R}} \cup \underline{\mathcal{R}}^{\mathcal{B}}$. □

Remark 2. None of the converse implications hold in general. Example 1 gives a conglomerable set of strictly desirable gambles that is different from $\mathcal{L}^+$ (note for instance that it includes the gamble $1 - \mathbb{I}_{\{1\}}$) and which, from Proposition 10, differs from $\underline{\mathcal{R}}^{\mathcal{B}}$. Example 7 gives an instance where the conglomerable natural extension of $\underline{\mathcal{R}}$ exists (and therefore $\underline{\mathcal{R}} \cup \underline{\mathcal{R}}^{\mathcal{B}}$ is included in a coherent set) but it is different from $\mathcal{E}_1$. In Example 9, we exhibit a coherent set of strictly desirable gambles that is not conglomerable but whose conglomerable natural extension is given by $\mathcal{E}_1$. ♦

Example 9. Let $\Omega := \mathbb{N} \cup -\mathbb{N}$, $B_n := \{-n, n\}$ and let $\mathcal{B}$ be the partition of Ω given by $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$. Consider the previsions P_1, P_2 defined in Example 5, and let $\underline{P} := \min\{P_1, P_2\}$. It follows from arguments similar to the ones on that example that $\underline{P}(B_n) > 0$ for all $n \in \mathbb{N}$, that

$$\underline{P}(f|B_n) = \min\left\{f(n), \frac{f(n) + f(-n)}{2}\right\},$$

and that $\underline{P}$ is not $\mathcal{B}$-conglomerable.

The set $\underline{\mathcal{R}}$ of strictly desirable gambles associated with $\underline{P}$ is included in the set $\{f : P_1(f) > 0\}$, whence also $\underline{\mathcal{R}}^{\mathcal{B}} \subseteq \{f : P_1(f) > 0\}$ (because P_1 is $\mathcal{B}$-conglomerable) and as a consequence $\mathcal{E}_1 \subseteq \{f : P_1(f) > 0\}$. To show the converse inclusion, it suffices to show that P_1 is the natural extension of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$: since $\underline{P}(B_n) > 0$ for all $n \in \mathbb{N}$, it follows from Proposition 21 that $\mathcal{E}_1$ induces the linear prevision P_1, and as a consequence it includes its set of strictly desirable gambles.

To determine the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, we apply Proposition 13. First of all, for every linear prevision $Q \in \mathcal{M}(\underline{P})$, there is some $\alpha \in [0, 1]$ such that $Q = \alpha P_1 + (1 - \alpha)P_2$. We are going to check that for all $\alpha \neq 1$ there is some gamble h such that $Q(h) < \underline{P}(\underline{P}(h|\mathcal{B}))$, which will mean that $Q \notin \mathcal{M}(\underline{E})$. Fix $\delta > 0$ and let h be given by $h(n) := 1 - \frac{1}{n} + \delta$ and $h(-n) := -1 + \frac{1}{n}$. Then $\underline{P}(h|B_n) = \delta/2 > 0$ for all $n \in \mathbb{N}$ implies that $\underline{P}(\underline{P}(h|\mathcal{B})) = \delta/2 > 0$, while

$$Q(h) = \alpha P_1(h) + (1 - \alpha)P_2(h) = \alpha\frac{\delta}{2} + (1 - \alpha)\frac{1}{2}\left[\sum_{n \in \mathbb{N}} \frac{1}{2^n}(1 + \delta - \frac{1}{n}) - 1\right] < 0$$

for δ small enough. Hence $\underline{E} = P_1$ and therefore $\mathcal{E}_1 = \{f : P_1(f) > 0\}$, which is conglomerable. ♦

Interestingly, it was proven in [24] that, given a coherent set of strictly desirable gambles $\underline{\mathcal{R}}$, the set $\underline{\mathcal{R}} \cup \underline{\mathcal{R}}^{\mathcal{B}}$ is coherent if and only if $\underline{\mathcal{R}}$ is conglomerable. Moreover, the coherence of $\underline{\mathcal{R}} \cup \underline{\mathcal{R}}^{\mathcal{B}}$ can be given a behavioural interpretation as the impossibility of making a Dutch book against us by combining our current beliefs, modelled by $\underline{\mathcal{R}}$, and the conditional beliefs $\underline{\mathcal{R}}^{\mathcal{B}}$, which only become effective after the observation of some element of the partition $\mathcal{B}$ (this notion is called *temporal coherence* in [24]). Theorem 23 shows that temporal coherence is an intermediate notion between the equality $\underline{\mathcal{R}} = \underline{\mathcal{R}}^{\mathcal{B}}$ (which, from Proposition 10 only holds when $\underline{\mathcal{R}} = \mathcal{L}^+$) and the conglomerable natural extension being attained in one step. On the other hand, when $\mathcal{R}$ is not a set of strictly desirable gambles, it is proven in [24] that the coherence of $\mathcal{R} \cup \mathcal{R}^{\mathcal{B}}$ is an intermediate notion between the conglomerability and the weak conglomerability of $\mathcal{R}$.

6.2. **Conglomerability and coherence of conditional lower previsions.** A particular case where the conglomerable natural extension of a set of gambles always exists is when we consider the set of gambles induced by a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$. For every $B \in \mathcal{B}$, let

$$\underline{\mathcal{R}}\rfloor B := \{f \in \mathcal{L}(B) : \underline{P}(f|B) > 0 \text{ or } f \gneq 0\}$$

be the coherent set of strictly desirable gambles on B associated to $\underline{P}(\cdot|B)$, and denote by

$$\underline{\mathcal{R}}|B := \{f \in \mathcal{L}(\Omega) : f = Bf, f|_B \in \underline{\mathcal{R}}\rfloor B\}$$

its extension to Ω, where $f|_B$ represents the restriction of f to B. Finally, let $\operatorname{posi}(\mathcal{L}^+ \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$ be the natural extension of $\bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B$. To see that this set is coherent, note that by definition $\underline{\mathcal{R}}|B$ does not include any gamble $f \leq 0$, and as a consequence neither does $\operatorname{posi}(\bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$; this implies that $\operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$ does not include the zero gamble, and as a consequence it is coherent by [14, Proposition 3(d)].

Since for any gamble f in this natural extension and any $B \in \mathcal{B}$ it follows that $Bf \in \operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B) \cup \{0\}$, we can apply Proposition 9 to deduce that the conglomerable natural extension of this set is given by

$$\mathcal{F} := \{0 \neq f \in \mathcal{L} \colon (\forall B \in \mathcal{B}) Bf \in \underline{\mathcal{R}}|B \cup \{0\}\}. \tag{20}$$

Then we can establish the following result:

Proposition 24. *Let $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ be a coherent lower prevision and a separately coherent conditional lower prevision on $\mathcal{L}$, and let $\underline{\mathcal{R}}$ and $\underline{\mathcal{R}}|B$ ($B \in \mathcal{B}$) be the respective sets of strictly desirable gambles they induce, and $\mathcal{F}$ be given by Eq.* (20)*. Then*

$$\underline{P} \text{ and } \underline{P}(\cdot|\mathcal{B}) \text{ coherent} \Rightarrow \underline{\mathcal{R}} \cup \mathcal{F} \text{ coherent.} \tag{21}$$

Moreover, if $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent then $\underline{\mathcal{R}}$ is conglomerable and $\underline{\mathcal{R}}^{\mathcal{B}} \subseteq \mathcal{F}$.

Proof. Assume ex absurdo that $\underline{\mathcal{R}} \cup \mathcal{F}$ is not coherent. Then there are gambles $f \in \underline{\mathcal{R}}$ and $g \in \mathcal{F}$ such that $f + g \notin \underline{\mathcal{R}}$, whence $\underline{P}(f+g) \leq 0$. We can assume without loss of generality that neither of these gambles is non-negative, or we would contradict the coherence of either $\underline{\mathcal{R}}$ or $\mathcal{F}$. Since $\underline{\mathcal{R}}$ is a set of strictly desirable gambles, $f \in \underline{\mathcal{R}}$ implies that $\underline{P}(f) > 0$. On the other hand, $g \in \mathcal{F}$ implies that $\underline{P}(g|\mathcal{B}) \geq 0$, so $g \geq G(g|\mathcal{B})$. As a consequence,

$$0 \geq \underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g) \Rightarrow 0 > -\underline{P}(f) \geq \underline{P}(g) \geq \underline{P}(G(g|\mathcal{B})),$$

and this contradicts that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent.

For the second part, apply [21, Theorem 6.8.2(a)] to deduce that $\underline{P}$ is conglomerable and Theorem 3 to conclude that so is $\underline{\mathcal{R}}$.

On the other hand, if $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent, then from [21, Theorem 6.8.2(a)] we infer that $\underline{P}(\cdot|\mathcal{B})$ dominates the conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ of $\underline{P}$, given by Eq. (11). For every $B \in \mathcal{B}$, [24, Lemma 1] implies that

$$\begin{aligned}
\underline{\mathcal{R}}^B &= \{f \in \mathcal{L} \colon f = Bf \in \underline{\mathcal{R}}\} \\
&= \{f \in \mathcal{L} \colon f = Bf \text{ and } [Bf \gneq 0 \text{ or } \underline{E}(f|B) > 0]\},
\end{aligned}$$

and since $\underline{E}(f|B) \leq \underline{P}(f|B)$ for every gamble f, we deduce that $\underline{\mathcal{R}}^B \subseteq \underline{\mathcal{R}}|B$. As a consequence, $\underline{\mathcal{R}}^{\mathcal{B}} \subseteq \mathcal{F}$. □

The converse to Eq. (21) does not hold:

Example 10. Consider $\Omega := \{1,2,3,4\}$, $B := \{1,2\}$ and $\mathcal{B} := \{B, B^c\}$. Let $\underline{P}$ be the vacuous lower prevision $\underline{P}$ on $\mathcal{L}$ and $P(\cdot|\mathcal{B})$ the linear conditional prevision given by

$$P(f|B) := \frac{f(1)+f(2)}{2} \text{ and } P(f|B^c) := \frac{f(3)+f(4)}{2}.$$

Let $\underline{\mathcal{R}}$ be the set of strictly desirable gambles associated to $\underline{P}$, and $\underline{\mathcal{R}}|B, \underline{\mathcal{R}}|B^c$ be the sets of gambles on Ω associated to $\underline{P}(\cdot|\mathcal{B})$. Then $\underline{\mathcal{R}} = \mathcal{L}^+$ and

$$\begin{aligned}
\underline{\mathcal{R}}|B &= \{f \in \mathcal{L}(\Omega) \colon f = Bf \text{ and } f(1)+f(2) > 0\}, \\
\underline{\mathcal{R}}|B^c &= \{f \in \mathcal{L}(\Omega) \colon f = B^c f \text{ and } f(3)+f(4) > 0\}.
\end{aligned}$$

Let $\mathcal{F}$ be given by Eq. (20). $\underline{\mathcal{R}} \cup \mathcal{F}$ is coherent because $\underline{\mathcal{R}} \subseteq \mathcal{F}$. To see that $\underline{P}$ and $P(\cdot|\mathcal{B})$ are not coherent, consider the gamble $f := (1,-1,1,-1)$. Then $P(f|\mathcal{B}) = 0$, whence $f - P(f|\mathcal{B}) = f$ and $\underline{P}(G(f|\mathcal{B})) = \underline{P}(f) = -1 < 0$. ♦

In Theorem 3, we have shown that the conglomerability of a set of strictly desirable gambles $\underline{\mathcal{R}}$ is equivalent to that of the coherent lower prevision $\underline{P}$ it induces, which in turn is equivalent to it being coherent with its conditional natural extension $\underline{E}(\cdot|\mathcal{B})$. More generally, if we consider a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ that is coherent with $\underline{P}$ and differs from $\underline{E}(\cdot|\mathcal{B})$, we may wonder if there is some coherent and conglomerable set of gambles from which we can induce both $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ by means of Eqs. (3) and (5), respectively. The following theorem answers this question, while also distinguishing the cases related to Walley's and Williams's notions of coherence for lower previsions:

Theorem 25. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$ and $\underline{P}(\cdot|\mathcal{B})$ a separately coherent conditional lower prevision on $\mathcal{L}$. Let $\underline{\mathcal{R}}$ be the set of strictly desirable gambles induced by $\underline{P}$, and $\underline{\mathcal{R}}|B$ ($B \in \mathcal{B}$) the sets of gambles associated to $\underline{P}(\cdot|\mathcal{B})$; let $\mathcal{F}$ be given by Eq.* (20).

(i) *$\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent if and only if they can be induced by a conglomerable coherent set. In that case, one such set is $\underline{\mathcal{R}} \cup \mathcal{F}$.*
(ii) *$\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are Williams coherent if and only if they can be induced by a coherent set. In that case, one such set is $\underline{\mathcal{R}} \cup \operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$.*

Proof. We begin with the first statement.

First of all, if $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent, then we know from Proposition 24 that $\underline{\mathcal{R}} \cup \mathcal{F}$ is coherent. To see that it is conglomerable, consider a non-zero gamble f such that $Bf \in \underline{\mathcal{R}} \cup \mathcal{F} \cup \{0\}$ for all B. Then

$$f = \sum_{B\in\mathcal{B}\colon Bf\in\underline{\mathcal{R}}} Bf + \sum_{B\in\mathcal{B}\colon Bf\notin\underline{\mathcal{R}}} Bf =: f_1 + f_2.$$

The coherence of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ implies that $\underline{\mathcal{R}}$ is conglomerable, and as a consequence the gamble f_1 belongs to $\underline{\mathcal{R}} \cup \{0\}$. On the other hand, $f_2 \in \mathcal{F} \cup \{0\}$ because this set $\mathcal{F}$ is conglomerable by definition. As a consequence, $f \in \operatorname{posi}(\underline{\mathcal{R}} \cup \mathcal{F} \cup \{0\}) = \operatorname{posi}(\underline{\mathcal{R}} \cup \mathcal{F}) \cup \{0\} = \underline{\mathcal{R}} \cup \mathcal{F} \cup \{0\}$, taking into account that $\underline{\mathcal{R}} \cup \mathcal{F}$ is coherent, by Proposition 24. Hence, it is also conglomerable.

Let $\underline{Q}$ be the coherent lower prevision induced by $\underline{\mathcal{R}}\cup\mathcal{F}$. Trivially $\underline{Q} \geq \underline{P}$. Assume ex absurdo that there is some gamble f such that $\underline{Q}(f) > \underline{P}(f)$, then there is some $\varepsilon > 0$ such that the gamble $g := f - \underline{P}(f) - \varepsilon$ belongs to $\underline{\mathcal{R}} \cup \mathcal{F}$. Since it cannot belong to $\underline{\mathcal{R}}$ because this is the set of strictly desirable gambles associated with $\underline{P}$ and $\underline{P}(f - \underline{P}(f) - \varepsilon) = -\varepsilon < 0$, it must belong to $\mathcal{F}$. Hence, for all $B \in \mathcal{B}$ either $Bg \geq 0$ or $\underline{P}(g|B) > 0$. We deduce that $\underline{P}(g|\mathcal{B}) \geq 0$, whence $g \geq G(g|\mathcal{B})$ and therefore $0 > \underline{P}(g) \geq \underline{P}(G(g|\mathcal{B}))$. This contradicts the coherence of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$. Hence $\underline{Q} = \underline{P}$.

Similarly, let $\underline{Q}(\cdot|\mathcal{B})$ be the conditional lower prevision associated with $\underline{\mathcal{R}} \cup \mathcal{F}$. Trivially $\underline{Q}(\cdot|\mathcal{B}) \geq \underline{P}(\cdot|\mathcal{B})$. Assume ex absurdo that there is some gamble f and some $B \in \mathcal{B}$ such that $\underline{Q}(f|B) > \underline{P}(f|B)$. Then there is some $\varepsilon > 0$ such that $B(f - \underline{P}(f|B) - \varepsilon) \in \underline{\mathcal{R}}$, and since $B(f - \underline{P}(f|B) - \varepsilon)$ cannot be non-negative or it would belong to $\mathcal{F}$, we deduce that $\underline{P}(B(f - \underline{P}(f|B) - \varepsilon)) > 0$. As a consequence, $\underline{P}(B(f - \underline{P}(f|B))) \geq \underline{P}(B(f - \underline{P}(f|B) - \varepsilon)) > 0$, and this contradicts that $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ satisfy (GBR).

Conversely, let $\mathcal{G}$ be a coherent and conglomerable set, and let $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ be the respective unconditional and conditional lower previsions associated with it. Consider any gamble f and any $B \in \mathcal{B}$. Then for every $\varepsilon > 0$ the gamble $G(f|B) + \varepsilon B$ belongs to $\mathcal{G}$, whence $\underline{P}(G(f|B) + \varepsilon B) \geq 0$ for all $\varepsilon > 0$ and therefore $\underline{P}(G(f|B)) \geq 0$, because the set of almost-desirable gambles associated to a coherent lower prevision is closed under uniform convergence. Now, if $\underline{P}(G(f|B)) > 0$ then there is some $\delta > 0$ such that

$$0 < \underline{P}(G(f|B) - \delta) \leq \underline{P}(G(f|B) - \delta B),$$

whence $G(f|B) - \delta B \in \mathcal{G}$ and therefore we can raise the value $\underline{P}(f|B)$ by δ, a contradiction.

On the other hand, if $G(f|B) + \varepsilon B$ belongs to $\mathcal{G}$ for every $B \in \mathcal{B}$, the conglomerability of $\mathcal{G}$ implies that $G(f|\mathcal{B}) + \varepsilon \in \mathcal{G}$ and therefore $\underline{P}(G(f|\mathcal{B}) + \varepsilon) \geq 0$. Since this holds for every $\varepsilon > 0$, we deduce that $\underline{P}(G(f|\mathcal{B})) \geq 0$ and therefore $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are coherent.

Next, we turn to the second statement.

Assume that $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are Williams coherent, and let us consider the set $\mathcal{G} := \underline{\mathcal{R}} \cup \operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$. We show that $\mathcal{G}$ is coherent. Since both $\underline{\mathcal{R}}$ and $\operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$ are coherent, it suffices to check that $f + g \in \mathcal{G}$ for every $f \in \underline{\mathcal{R}}$ and $g \in \operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$. Since $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are Williams coherent, we know that for every gamble f and every $B \in \mathcal{B}$, $\underline{P}(G(f|B)) = 0$. Applying the super-additivity of $\underline{P}$, we deduce that

$$\begin{aligned}\underline{P}(g) \geq \sum_{B\in\mathcal{B}\colon Bg\in\underline{\mathcal{R}}|B} \underline{P}(Bg) &\geq \sum_{B\in\mathcal{B}\colon Bg\in\underline{\mathcal{R}}|B} \underline{P}(B(g - \underline{P}(g|B))) \\ &= \sum_{B\in\mathcal{B}\colon Bg\in\underline{\mathcal{R}}|B} \underline{P}(G_{\underline{P}}(g|B)) \geq 0\end{aligned}$$

for every $g \in \operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$. To see that the first sum is finite, note that by definition of the posi operator the gamble g is a finite sum of gambles in $\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B$, because each of the sets in this union is a convex cone.

As a consequence, given that $f \in \underline{\mathcal{R}}$ and $g \in \operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$, there are two possibilities: either $f \in \mathcal{L}^+$, whence $f + g \in \operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$, or $\underline{P}(f) > 0$, and then $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g) > 0$, and therefore $f + g \in \underline{\mathcal{R}}$.

Now, let $\underline{Q}$ be the unconditional lower prevision induced by $\mathcal{G}$. Trivially, $\underline{Q} \geq \underline{P}$. Assume ex absurdo that there is some gamble f such that $\underline{Q}(f) > \underline{P}(f)$, then there is some $\varepsilon > 0$ such that the gamble $g := f - \underline{P}(f) - \varepsilon$ belongs to $\mathcal{G}$, so there are gambles $h_1 \in \underline{\mathcal{R}}$ and $h_2 \in \operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$ such that $g = h_1 + h_2$. As a consequence,

$$-\varepsilon = \underline{P}(g) = \underline{P}(h_1 + h_2) \geq \underline{P}(h_1) + \underline{P}(h_2) \geq \underline{P}(h_2) \geq 0,$$

a contradiction. Hence $\underline{Q} = \underline{P}$.

Similarly, let $\underline{Q}(\cdot|\mathcal{B})$ be the conditional lower prevision induced by $\mathcal{G}$. Trivially, $\underline{Q}(\cdot|\mathcal{B}) \geq \underline{P}(\cdot|\mathcal{B})$. Assume ex absurdo there is some gamble f and some $B \in \mathcal{B}$ such that $\underline{Q}(f|B) > \underline{P}(f|B)$. Then there is some $\varepsilon > 0$ such that $g := B(f - \underline{P}(f|B) - \varepsilon) \in \mathcal{G}$. If it belongs to $\operatorname{posi}(\mathcal{L}^+ \cup \bigcup_{B\in\mathcal{B}} \underline{\mathcal{R}}|B)$, then there must be some gamble h and some $\delta > 0$ such that $g \geq G(h|B) + B\delta$, whence $G(h|B) - G(f|B) \leq B(-\varepsilon - \delta)$, contradicting the separate coherence of $\underline{P}(\cdot|B)$. If instead it belongs to $\underline{\mathcal{R}}$, then either (i) $g \gneq 0$, which contradicts the definition of $\underline{P}(\cdot|B)$; or (ii) $\underline{P}(g) > 0$, whence $\underline{P}(G(f|B)) \geq \underline{P}(g) > 0$, also a contradiction, in this case with (GBR).

The converse proof follows the same lines as that of the first statement. □

7. Conglomerability for a number of partitions

To conclude the technical discussion, we turn to conglomerability with respect to a number of partitions, rather than just one. Consider a non-empty set $\mathbb{B}$ of partitions of Ω. This set need not be finite, although we will make this assumption in much of what follows.

We call a set of desirable gambles (weakly) $\mathbb{B}$-conglomerable if it is (weakly) conglomerable with respect to all partitions $\mathcal{B}$ in $\mathbb{B}$. We denote by $\mathbb{D}_{\mathrm{C}}(\mathbb{B})$ the set of all coherent sets of desirable gambles on Ω that satisfy D5 with respect to all partitions in $\mathbb{B}$, and similarly, by $\mathbb{D}_{\mathrm{wC}}(\mathbb{B})$ the set of all coherent sets of desirable gambles on Ω that satisfy wD5 with respect to all partitions in $\mathbb{B}$.

Clearly, like its counterpart for a single partition, (weak) $\mathbb{B}$-conglomerability is preserved under taking arbitrary intersections. This implies that if a set of desirable gambles is dominated by some coherent and $\mathbb{B}$-conglomerable set of desirable gambles, then there is a smallest such dominating set.

Definition 5. Consider a non-empty set $\mathbb{B}$ of partitions of Ω. If it exists, the (weakly) $\mathbb{B}$-*conglomerable natural extension* of a set $\mathcal{R}$ of desirable gambles on Ω, is its smallest coherent superset that is (weakly) conglomerable with respect to all $\mathcal{B}$ in $\mathbb{B}$.

Proposition 26. *If there is some coherent superset of $\mathcal{R}$ that satisfies condition* D5 *(respectively* wD5*) with respect to $\mathbb{B}$, then the smallest such superset is given by*

$$\mathcal{F} \coloneqq \bigcap \{\mathcal{D} \in \mathbb{D}_{\mathrm{C}}(\mathbb{B}) \colon \mathcal{R} \subseteq \mathcal{D}\} \text{ respectively } \mathcal{F} \coloneqq \bigcap \{\mathcal{D} \in \mathbb{D}_{\mathrm{wC}}(\mathbb{B}) \colon \mathcal{R} \subseteq \mathcal{D}\}.$$

From now on, we will concentrate on the case where $\mathbb{B} = \{\mathcal{B}_1, \dots, \mathcal{B}_m\}$ is a finite set of partitions of Ω. But we first show that conglomerability with respect to each of the partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$ is equivalent to conglomerability with respect to all the partitions that can be derived from them—we may refer to this notion as *cross-conglomerability.* Let us define

$$\mathbb{B}' \coloneqq \{\mathcal{B} \text{ partition} \colon (\forall B \in \mathcal{B})(\exists j \in \{1, \dots, m\}) B \in \mathcal{B}_j\}. \tag{22}$$

It should be remarked that, while $\mathbb{B}$ is finite, $\mathbb{B}'$ can be infinite.

Proposition 27. *Let $\mathcal{R}$ be a coherent set of desirable gambles.*

(i) *If $\mathcal{R}$ satisfies* D5 *with respect to any partition $\mathcal{B}_j$ in $\mathbb{B}$, then it also satisfies* D5 *with respect to any partition $\mathcal{B}$ in $\mathbb{B}'$: $\mathbb{D}_{\mathrm{C}}(\mathbb{B}) = \mathbb{D}_{\mathrm{C}}(\mathbb{B}')$.*

(ii) *If $\mathcal{R}$ satisfies* wD5 *with respect to any partition $\mathcal{B}_j$ in $\mathbb{B}$, then it also satisfies* wD5 *with respect to any partition $\mathcal{B}$ in $\mathbb{B}'$: $\mathbb{D}_{\mathrm{wC}}(\mathbb{B}) = \mathbb{D}_{\mathrm{wC}}(\mathbb{B}')$.*

Proof. Let us begin with the first statement. Consider a partition $\mathcal{B}$ in $\mathbb{B}'$, and let f be any gamble such that $Bf \in \mathcal{R} \cup \{0\}$ for all $B \in \mathcal{B}$. Define the collections of sets

$$\mathcal{A}_j \coloneqq \left\{ B \in \mathcal{B} \colon B \in \mathcal{B}_j \setminus \bigcup_{i=1}^{j-1} \mathcal{A}_i \right\},$$

for which $\bigcup_{j=1}^{m} \mathcal{A}_j = \mathcal{B}$, and the collection

$$\mathcal{A} \coloneqq \{\mathcal{A}_j : j = 1, \dots, m \text{ and } B_j f \neq 0 \text{ for some } B_j \in \mathcal{A}_j\}.$$

As a consequence,

$$f = \sum_{\mathcal{A}_j \in \mathcal{A}} \sum_{B_j \in \mathcal{A}_j} B_j f \in \mathcal{R},$$

taking into account that for every j the gamble $\sum_{B_j \in \mathcal{A}_j} B_j f \in \mathcal{R} \cup \{0\}$ because $\mathcal{R}$ satisfies D5 with respect to $\mathcal{B}_j$, and applying D4.

The second statement follows by applying the first to sets of strictly desirable gambles. □

7.1. **The Marginal Extension Theorem.** We now consider, for $i = 1, \dots, m$, a coherent set of desirable gambles $\mathcal{R}_i$ that is $\mathcal{B}_i$-conglomerable, and we want to determine the $\mathbb{B}$-conglomerable natural extension $\mathcal{F}$ of $\bigcup_{i=1}^{m} \mathcal{R}_i$, if it exists. Similarly to what happened in Section 3, when the conglomerable natural extension $\mathcal{F}$ exists, we can approximate it by means of a sequence of sets of desirable gambles. For the purposes of this section, it will suffice to consider the first element of this sequence. For every $\mathcal{R}_i$, consider, as before, the set

$$\mathcal{R}_i^{\mathcal{B}} \coloneqq \{0 \neq f \in \mathcal{L} \colon (\forall i \in \{1, \dots, m\}) B_i f \in \mathcal{R}_i \cup \{0\}\},$$

and let

$$\mathcal{D}_1 := \operatorname{posi}(\bigcup_{i=1}^{m}(\mathcal{R}_i \cup \mathcal{R}_i^{\mathcal{B}})).$$

When this set is coherent, it can be written equivalently as

$$\mathcal{D}_1 = \mathcal{L} \cap \left\{ \sum_{i=1}^{m}(f_i + g_i) \colon f_i \in \mathcal{R}_i \cup \{0\} \text{ and } (\forall B_i \in \mathcal{B}_i) B_i g_i \in \mathcal{R}_i \cup \{0\} \right\} \setminus \{0\}. \tag{23}$$

We will now prove that when the partitions are nested the sequence stabilises after one step: in this case $\mathcal{D}_1$ coincides with $\mathcal{F}$. This is a generalised version of the marginal extension theorem, which was originally established for coherent lower previsions and a single partition in [21, Theorem 6.7.2], and later extended to a finite number of partitions in [12]. In a different context, using different notations, a more general result than ours was also established (in a different manner) by De Cooman and Hermans [2, Theorem 3].

In order to do this, we need to introduce the notion of a coherent set of desirable gambles *relative to another subset*:

Definition 6. Let $\mathcal{Q}$ be a linear space of gambles containing constant gambles, and let $\mathcal{R} \subseteq \mathcal{Q}$. We say that $\mathcal{R}$ is coherent relative to $\mathcal{Q}$ if it satisfies axioms D2–D4 and D1'. $\mathcal{Q} \cap \mathcal{L}^+ \subseteq \mathcal{R}$.

Note that when $\mathcal{Q} = \mathcal{L}$, this becomes the usual coherence notion characterised by axioms D1–D4.

We shall also have recourse to the following simple lemma:

Lemma 28. *Let $\mathcal{R}$ be a set of gambles coherent relative to $\mathcal{Q}$. Then for every gamble $f \leq 0$, $f \notin \mathcal{R}$.*

Proof. That $0 \notin \mathcal{R}$ follows from D2. Assume ex absurdo that $0 \gneq f \in \mathcal{R}$. Then $f \in \mathcal{Q}$, and since this is a linear space also $-f \in \mathcal{Q}$, whence $0 \lneq -f \in \mathcal{R}$ by D1'. We deduce, applying D4 that $0 = f - f$ also belongs to $\mathcal{R}$. This is a contradiction. □

We begin by establishing our result for the least involved special case: one partition only. We consider a partition $\mathcal{B}$, and a set $\mathcal{R}_0$ that only contains $\mathcal{B}$-measurable gambles, meaning that they are constant on the elements of $\mathcal{B}$. We assume that the set of desirable gambles $\mathcal{R}$ is coherent relative to the set of $\mathcal{B}$-measurable gambles. It is trivially conglomerable with respect to the partition $\mathcal{B}_0 = \{\Omega\}$.

For each $B \in \mathcal{B}$, we also consider a coherent set of desirable gambles $\mathcal{R}\rfloor B$ on $\mathcal{L}(B)$. We can use these sets to construct the set of desirable gambles

$$\mathcal{R}_1 := \mathcal{L} \cap \left(\sum_{B \in \mathcal{B}} B(\mathcal{R}\rfloor B \cup \{0\}) \right) \setminus \{0\} = \mathcal{L} \cap \left\{ \sum_{B \in \mathcal{B}} B g_B \colon g_B \in \mathcal{R}\rfloor B \cup \{0\} \right\} \setminus \{0\}.$$

This coherent set of desirable gambles is the $\mathcal{B}$-conglomerable natural extension of the set of gambles $\bigcup_{B \in \mathcal{B}} \{B g_B \colon g_B \in \mathcal{R}\rfloor B\}$. Note that the gamble $B g_B$ is a gamble on Ω that is equal to g_B on B, and zero elsewhere. This is done to ensure all the gambles we will combine later on are defined on the same domain.

We are now looking for the smallest coherent set of desirable gambles that includes $\mathcal{R}_0 \cup \mathcal{R}_1$ and that is conglomerable with respect to $\mathcal{B}_0$ and $\mathcal{B}$. The following proposition solves this problem in slightly reformulated wording.

Proposition 29. *Let $\mathcal{R}_0$ be a set of $\mathcal{B}$-measurable desirable gambles that is coherent relative to the set of $\mathcal{B}$-measurable gambles. For each $B \in \mathcal{B}$, let $\mathcal{R}\rfloor B$ be a coherent set of desirable gambles on B. Then the $\mathcal{B}$-conglomerable natural extension of the set*

$$\mathcal{R}_0 \cup \{B g_B \colon g_B \in \mathcal{R}\rfloor B \text{ and } B \in \mathcal{B}\}$$

is given by

$$\mathcal{F} := \mathcal{L} \cap \left\{ f + \sum_{B \in \mathcal{B}} Bg_B : f \in \mathcal{R}_0 \cup \{0\} \text{ and } g_B \in \mathcal{R}\rfloor B \cup \{0\} \right\} \setminus \{0\}.$$

Proof. Let us show that $\mathcal{F}$ satisfies D1–D5.

D1: Consider $h \in \mathcal{L}^+$. Write it as $h = \sum_{B \in \mathcal{B}: Bh \neq 0} Bh = \sum_{B \in \mathcal{B}: Bh \neq 0} Bh|_B$, where the gamble $h|_B \in \mathcal{L}(B)$ is the restriction of h to the set B, defined by $h|_B(\omega) := h(\omega)$ for all $\omega \in B$. Since $h|_B \in \mathcal{L}^+(B)$, it belongs to the coherent set of desirable gambles $\mathcal{R}\rfloor B$. Therefore h belongs to $\mathcal{F}$.

D2: We know that $0 \notin \mathcal{F}$ by definition.

D3: Consider $h \in \mathcal{F}$ and $\lambda > 0$. We know that there are $f \in \mathcal{R}_0$ and $g_B \in \mathcal{R}\rfloor B \cup \{0\}$ such that $\lambda f = \lambda f + \sum_{B \in \mathcal{B}} B\lambda g_B$. But since the set of $\mathcal{B}$-measurable gambles is a linear space containing constant gambles, and $\mathcal{R}_0$ is coherent relative to it, $\lambda f \in \mathcal{R}_0 \cup \{0\}$. Similarly, $\lambda g_B \in \mathcal{R}\rfloor B \cup \{0\}$, since $\mathcal{R}\rfloor B$ is coherent. It follows that $\lambda h \in \mathcal{F}$, because $\lambda f \neq 0$.

D4: Consider $h, h' \in \mathcal{F}$. Then $h + h' = f + f' + \sum_{B \in \mathcal{B}} B(g_B + g'_B)$, where $f, f' \in \mathcal{R}_0 \cup \{0\}$ and $g_B, g'_B \in \mathcal{R}\rfloor B \cup \{0\}$. For reasons analogous to the ones given above, $f + f' \in \mathcal{R}_0 \cup \{0\}$ and $g_B + g'_B \in \mathcal{R}\rfloor B \cup \{0\}$. From this, we obtain that $h + h' \in \mathcal{F} \cup \{0\}$. Assume ex absurdo that $h + h' = 0$; then either $0 = f + f'$ or $f + f' \neq 0$. In the first case, the coherence of $\mathcal{R}_0$ implies that $f = f' = 0$, and similarly since $g_B + g'_B = 0$ for every B we should have that $g_B = g'_B = 0$ for all B. But then $h = h' = 0$, a contradiction.

In the second case, $0 \neq f + f' = -\sum_{B \in \mathcal{B}} B(g_B + g'_B)$, and taking into account that $f + f'$ is $\mathcal{B}$-measurable, there must be some $B \in \mathcal{B}$ such that $B(f + f') \gneq 0$; otherwise it would follow that $f + f' \leq 0$, which is impossible because of Lemma 28. But for this B we obtain that $g_B + g'_B \lneq 0$, which is again impossible because of Lemma 28. This is a contradiction.

D5: Consider $h \in \mathcal{L}$ such that $Bh \in \mathcal{F} \cup \{0\}$ for all $B \in \mathcal{B}$. Fix any B such that $Bh \neq 0$, then $Bh = f + \sum_{B \in \mathcal{B}} Bg_B$. If $f = 0$, then $Bh = Bg_B$. If $f \neq 0$, then we consider any $B' \in \mathcal{B} \setminus \{B\}$. Bh is zero on B', and therefore $B'f + B'g_{B'} = 0$. Recalling that f is $\mathcal{B}$-measurable and therefore assumes the constant value $f(B')$ on B', this can only happen if $f(B') < 0$: otherwise, $\mathcal{R}\rfloor B'$ would contradict Lemma 28. Since we can repeat this reasoning for all $B' \neq B$, we deduce that f must assume a constant value $f(B) > 0$ on B, since otherwise $\mathcal{R}_0$ would contradict Lemma 28. Then $g_B + f(B) \in \mathcal{R}\rfloor B$, so that if we let $g'_B := g_B + f(B)$, we obtain that $Bh = Bg'_B$. As a consequence, $h = \sum_{B \in \mathcal{B}: Bh \neq 0} Bh = \sum_{B \in \mathcal{B}: Bh \neq 0} Bg'_B \in \mathcal{F}$.

Since on the other hand $\mathcal{F}$ is included in the set $\mathcal{D}_1$ given by Eq. (23) which is in turn included in the conglomerable natural extension, we deduce that $\mathcal{F}$ is the conglomerable natural extension of $\mathcal{R}_0 \cup \{Bg_B : g_B \in \mathcal{R}\rfloor B \text{ and } B \in \mathcal{B}\}$. $\square$

Let us extend this result to a finite number of partitions. We consider m partitions $\mathcal{B}_1, \ldots, \mathcal{B}_m$ of Ω that are successively finer: $\mathcal{B}_{i+1}$ is finer than $\mathcal{B}_i$ for $i = 1, \ldots, m-1$. $\mathcal{R}_0$ is a set of $\mathcal{B}_1$-measurable desirable gambles that is coherent relative to the set of all $\mathcal{B}_1$-measurable gambles. For each $i = 1, \ldots, m-1$ and each $B_i \in \mathcal{B}_i$, we consider the partition

$$\mathcal{B}_{i+1}\rfloor B_i := \{B_{i+1} \in \mathcal{B}_{i+1} : B_{i+1} \subseteq B_i\}$$

of B_i, and a set $\mathcal{R}_i\rfloor B_i$ of $\mathcal{B}_{i+1}\rfloor B_i$-measurable desirable gambles on B_i that is coherent relative to the set of all $\mathcal{B}_{i+1}\rfloor B_i$-measurable gambles. Finally, for each $B_m \in \mathcal{B}_m$, we consider a coherent set $\mathcal{R}_m\rfloor B_m$ of desirable gambles on B_m.

We can use the sets $\mathcal{R}_i\rfloor B_i$, $B_i \in \mathcal{B}_i$ to construct the set of desirable gambles

$$\begin{aligned}\mathcal{R}_i &\coloneqq \mathcal{L} \cap \left(\sum_{B_i \in \mathcal{B}_i} B_i(\mathcal{R}_i\rfloor B_i \cup \{0\})\right) \setminus \{0\}\\ &= \mathcal{L} \cap \left\{\sum_{B_i \in \mathcal{B}_i} B_i g_{B_i} \colon g_{B_i} \in \mathcal{R}_i\rfloor B_i \cup \{0\}\right\} \setminus \{0\}.\end{aligned}$$

This coherent set of desirable gambles is the $\mathcal{B}_i$-conglomerable natural extension of the set of gambles $\bigcup_{B_i \in \mathcal{B}_i} \{B_i g_{B_i} \colon g_{B_i} \in \mathcal{R}_i\rfloor B_i\}$.

We are now looking for the smallest coherent set of desirable gambles that includes $\mathcal{R}_0 \cup \bigcup_{i=1}^m \mathcal{R}_i$ and that is conglomerable with respect to $\mathcal{B}_0$, $\mathcal{B}_1$, …, $\mathcal{B}_m$.

Proposition 30. *Let $\mathcal{B}_1$, …, $\mathcal{B}_m$ be partitions of Ω such that $\mathcal{B}_{i+1}$ is finer than $\mathcal{B}_i$ for $i = 1, \dots, m-1$. Let $\mathcal{R}_0$ be a set of $\mathcal{B}_1$-measurable desirable gambles that is coherent relative to the set of all $\mathcal{B}_1$-measurable gambles. For each $i = 1, \dots, m-1$ and each $B_i \in \mathcal{B}_i$, let $\mathcal{R}_i\rfloor B_i$ be a set of $\mathcal{B}_{i+1}\rfloor B_i$-measurable desirable gambles on B_i that is coherent relative to the class of all $\mathcal{B}_{i+1}\rfloor B_i$-measurable gambles, For each $B_m \in \mathcal{B}_m$, let $\mathcal{R}_m\rfloor B_m$ be a coherent set of desirable gambles on B_m. Then the conglomerable natural extension of*

$$\mathcal{R}_0 \cup \bigcup_{i=1}^m \bigcup_{B_i \in \mathcal{B}_i} \{B_i g_{B_i} \colon g_{B_i} \in \mathcal{R}_i\rfloor B_i\}$$

is given by

$$\mathcal{F}_m \coloneqq \mathcal{L} \cap \left\{f_0 + \sum_{i=1}^m \sum_{B_i \in \mathcal{B}_i} B_i g_{B_i} \colon f_0 \in \mathcal{R}_0 \cup \{0\} \text{ and } g_{B_i} \in \mathcal{R}_i\rfloor B_i \cup \{0\}\right\} \setminus \{0\}.$$

Proof. To make the notation more uniform, we introduce the trivial partition $\mathcal{B}_0 \coloneqq \{\Omega\}$, and define $B_0 \coloneqq \Omega$, $\mathcal{R}_0\rfloor B_0 \coloneqq \mathcal{R}_0$ and $g_{B_0} \coloneqq f_0$.

We use induction on the number of partitions m. For $m = 1$, the result has already been established in Proposition 29. Assume therefore that the result holds for $m-1$, and let us prove that it also holds for m.

If we can prove that $\mathcal{F}_m$ satisfies D1–D5, then it is the conglomerable natural extension, because any superset of $\mathcal{R}_0 \cup \bigcup_{i=1}^m \bigcup_{B_i \in \mathcal{B}_i} \{B_i g_{B_i} \colon g_{B_i} \in \mathcal{R}_i\rfloor B_i\}$ that satisfies D1–D5 necessarily includes $\mathcal{F}_m$. Let us therefore show that $\mathcal{F}_m$ satisfies D1–D5.

D1: Consider any $h \in \mathcal{L}^+$. Write it as

$$h = \sum_{B_m \in \mathcal{B}_m \colon B_m h \neq 0} B_m h = \sum_{B_m \in \mathcal{B}_m \colon B_m h \neq 0} B_m g_{B_m},$$

where the gambles $g_{B_m} \in \mathcal{L}(B_m)$ are defined by $g_{B_m}(\omega) \coloneqq h(\omega)$ for all $\omega \in B_m$ and all B_m such that $B_m h \neq 0$. Since then $g_{B_m} \in \mathcal{L}^+(B_m)$, it belongs to $\mathcal{R}_m\rfloor B_m$, which is a coherent set of desirable gambles. This implies that $h \in \mathcal{F}_m$.

D2: We know that $0 \notin \mathcal{F}_m$ by definition.

D3: Consider $h \in \mathcal{F}_m$ and $\lambda > 0$. We know that $\lambda h = \sum_{i=0}^m \sum_{B_i \in \mathcal{B}_i} B_i \lambda g_{B_i}$. Since for $i = 0, \dots, m-1$ the set of $\mathcal{B}_{i+1}\rfloor B_i$-measurable gambles is a linear space containing all constant gambles, and since $\mathcal{R}_i\rfloor B_i$ is coherent relative to it, we know that $\lambda g_{B_i} \in \mathcal{R}_i\rfloor B_i \cup \{0\}$. Moreover, $\lambda g_{B_m} \in \mathcal{R}_m\rfloor B_m \cup \{0\}$, as $\mathcal{R}_m\rfloor B_m$ is coherent. Hence $\lambda h \in \mathcal{F}_m$.

D4: Consider $h, h' \in \mathcal{F}_m$. Then $h + h' = \sum_{i=0}^m \sum_{B_i \in \mathcal{B}_i} B_i(g_{B_i} + g'_{B_i})$, where $g_{B_i}, g'_{B_i} \in \mathcal{R}_i\rfloor B_i \cup \{0\}$. For reasons analogous to those mentioned above, $g_{B_i} + g'_{B_i} \in$

$\mathcal{R}_i\rfloor B_i \cup \{0\}$. Hence $h + h' \in \mathcal{F}_m \cup \{0\}$. Assume ex absurdo that $h + h' = 0$, then

$$f \coloneqq \sum_{i=0}^{m-1} \sum_{B_i \in \mathcal{B}_i} B_i(g_{B_i} + g'_{B_i}) = -\sum_{B_m \in \mathcal{B}_m} B_m(g_{B_m} + g'_{B_m}).$$

The gamble f belongs to the set $\mathcal{F}_{m-1}$ we would obtain by considering the sets $\mathcal{R}_0$, $\mathcal{R}_j\rfloor B_j$ for $j = 1, \dots, m-2$, and $\mathcal{R}'_{m-1}\rfloor B_{m-1}$, where $\mathcal{R}'_{m-1}\rfloor B_{m-1}$ is the natural extension of $\mathcal{R}_{m-1}\rfloor B_{m-1}$. Applying the induction hypothesis, we deduce that $f \neq 0$. Since moreover, f is $\mathcal{B}_m$-measurable, there must be some $B_m \in \mathcal{B}_m$ such that $B_m f \in \mathcal{L}^+$, otherwise $f \leq 0$ and $f \in \mathcal{F}_{m-1}$, which is coherent by the induction hypothesis, would contradict Lemma 28. But for this B_m we obtain that $g_{B_m} + g'_{B_m} \lneq 0$, so $\mathcal{R}_m\rfloor B_m$ violates Lemma 28. This is a contradiction.

D5: Consider $h \in \mathcal{L}$ such that $B_i h \in \mathcal{F}_m \cup \{0\}$ for all $B_i \in \mathcal{B}_i$ and all $i = 1, \dots, m$. We fix our attention for the time being on any one B_i for which $B_i h \neq 0$, and have by definition that $B_i h = \sum_{j=0}^{m} \sum_{B_j \in \mathcal{B}_j} B_j g_{B_j}$. Let

$$f \coloneqq \sum_{j=0}^{i-1} \sum_{B_j \in \mathcal{B}_j} B_j g_{B_j} \text{ and } g \coloneqq \sum_{j=i}^{m} \sum_{B_j \in \mathcal{B}_j} B_j g_{B_j}.$$

If $f = 0$, then we can express

$$B_i h = B_i g = \sum_{j=i}^{m} \sum_{B_j \in \mathcal{B}_j, B_j \subseteq B_i} B_j g_{B_j}.$$

If $f \neq 0$, look at any $B'_i \in \mathcal{B}_i$ such that $B'_i \neq B_i$. $B_i h$ is zero on B'_i, and hence $B'_i f + B'_i g = 0$. Now, recalling that f is $\mathcal{B}_i$-measurable, the constant value $f(B'_i)$ that f assumes on B'_i must be negative: $f(B'_i) < 0$. Otherwise, we would have that $0 \geq B'_i g \in \mathcal{F}_m$, and this contradicts Lemma 28. Since we can repeat this reasoning for all $B'_i \neq B_i$, we deduce that the constant value $f(B_i)$ that f assumes on B_i must be positive: $f(B_i) > 0$. Otherwise $\mathcal{F}_m$ would again contradict Lemma 28. But then $g_{B_i} + f(B_i) \in \mathcal{R}_i\rfloor B_i$, so that if we redefine $g_{B_i} \coloneqq g_{B_i} + f(B_i)$, we obtain that

$$B_i h = \sum_{j=i}^{m} \sum_{B_j \in \mathcal{B}_j, B_j \subseteq B_i} B_j g_{B_j}.$$

As a consequence,

$$\begin{aligned} h = \sum_{B_i \in \mathcal{B}_i \colon B_i h \neq 0} B_i h &= \sum_{B_i \in \mathcal{B}_i \colon B_i h \neq 0} \sum_{j=i}^{m} \sum_{B_j \in \mathcal{B}_j, B_j \subseteq B_i} B_j g_{B_j} \\ &= \sum_{j=i}^{m} \sum_{B_i \in \mathcal{B}_i \colon B_i h \neq 0} \sum_{B_j \in \mathcal{B}_j, B_j \subseteq B_i} B_j g_{B_j} \end{aligned}$$

and therefore $h \in \mathcal{F}_m$. □

7.2. **Conglomerability and weak coherence.** Finally, we turn to the notion of conglomerability for coherent lower previsions, this time with respect to a finite number of partitions. It is easy to relate this property to the notion of weak coherence:

Proposition 31. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, and let $\mathcal{B}_1$, ..., $\mathcal{B}_m$ be partitions of Ω. The following statements are equivalent:*

(i) *$\underline{P}$ is $\mathcal{B}_i$-conglomerable for $i = 1, \dots, m$.*
(ii) *$\underline{P}$ is $\mathcal{B}$-conglomerable for any partition $\mathcal{B}$ in the class $\mathbb{B}'$ defined by (22).*
(iii) *There are conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1)$, ..., $\underline{P}_m(\cdot|\mathcal{B}_m)$ weakly coherent with $\underline{P}$.*

Proof. First of all, $\underline{P}$ is $\mathcal{B}_i$-conglomerable if and only if it is coherent with its conditional natural extension $\underline{P}_i(\cdot|\mathcal{B}_i)$. As a consequence, it is $\mathcal{B}_i$-conglomerable for $i = 1, \dots, m$ if and only if it is pairwise coherent with the conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \dots, \underline{P}_m(\cdot|\mathcal{B}_m)$. Applying [13, Theorem 1], this is equivalent to $\underline{P}$, $\underline{P}_1(\cdot|\mathcal{B}_1)$, $\dots$, $\underline{P}_m(\cdot|\mathcal{B}_m)$ being weakly coherent. Hence, the first and third statements are equivalent.

To see that the first two statements are equivalent, note that, from Theorem 3, $\underline{P}$ is $\mathcal{B}_i$-conglomerable for $i = 1, \dots, m$ if and only if its associated set of strictly desirable gambles $\underline{\mathcal{R}}$ is $\mathcal{B}_i$-conglomerable for $i = 1, \dots, m$. Applying Proposition 27, this holds if and only if $\underline{\mathcal{R}}$ satisfies D5 with respect to any partition $\mathcal{B} \in \mathbb{B}'$, and using Theorem 3 again, this is equivalent to $\underline{P}$ being $\mathcal{B}$-conglomerable for any partition $\mathcal{B}$ in the class $\mathbb{B}'$. □

8. Conclusions

Some authors, amongst whom Peter Walley, have argued in favour of imposing conglomerability on uncertainty models, such as sets of desirable gambles, coherent lower previsions, or precise probabilities. We have studied the problem of extending a given uncertainty model into the weakest conglomerable model that logically follows from it: we have called this the conglomerable natural extension.

An intuitively natural approach to addressing the problem of constructing this conglomerable natural extension consists in imposing conglomerability on the model, and then taking the natural extension. In fact, this is the approach taken in Walley's theory. Unfortunately, our main finding in this paper shows that such an approach does not yield the conglomerable natural extension in general, even though it does so in the case of the marginal extension theorem. This has important implications for Walley's theory: it means that it does not fully take into account the implications of conglomerability. We have also shown that iterating the above-mentioned intuitive process yields models closer and closer to the conglomerable natural extension. The question whether the conglomerable natural extension is achieved in the limit, remains unanswered, however.

All this means that the foundations of Walley's theory of coherent lower previsions have to be reconsidered in the case where there are infinitely many events in the conditioning partition of the possibility space. Our results indicate that it may be necessary to modify his definition of natural extension of a conditional and an unconditional model (be it a set of desirable gambles or a coherent lower prevision) to make it truly and fully compatible with the notion of conglomerability. Related to this, it would be important to investigate possible modifications of his definition for the coherence of conditional and unconditional lower previsions, and whether these modifications allow us to obtain envelope theorems, thus also allowing a sensitivity analysis interpretation for coherence in the conditional case. In fact, on Walley's approach, the useful equivalence between coherent lower previsions and sets of probabilities breaks down in the conditional case for infinite possibility spaces. As a consequence of the results in this paper, it is no longer clear whether this is unavoidable (as seems to have been assumed before), or if it is merely due to the arguably inadequate treatment of conglomerability in Walley's theory.

More work should also be done to study the general case of multiple partitions that are not necessarily nested. Finally, in our definition of coherence for sets of desirable gambles we are assuming that the zero gamble is not desirable. This is in line with more recent work on desirability [3, 14]. Although we have not detailed it here, it is possible to show that our main finding, that the conglomerable natural extension is not attained after applying once coherence and conglomerability, still holds in the alternative approach where the zero gamble is desirable.

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References

[1] K. P. S. Bhaskara Rao and M. Bhaskara Rao. *Theory of Charges.* Academic Press, London, 1983.

[2] G. de Cooman and F. Hermans. Coherent immediate prediction: bridging two theories of imprecise probability. *Artificial Intelligence*, 172(11):1400–1427, 2008.

[3] G. de Cooman and E. Quaeghebeur. Exchangeability and sets of desirable gambles. *International Journal of Approximate Reasoning*, 53(3):362–394, 2012.

[4] B. de Finetti. Sulla proprietà conglomerativa della probabilità subordinate. *Rendiconti del Reale Instituto Lombardo*, 63:414–418, 1930.

[5] B. de Finetti. *Teoria delle Probabilità.* Einaudi, Turin, 1970.

[6] B. de Finetti. *Probability, Induction and Statistics.* Wiley, London, 1972.

[7] B. de Finetti. *Theory of Probability: A Critical Introductory Treatment.* John Wiley & Sons, Chichester, 1974–1975. English translation of [5], two volumes.

[8] S. Doria. Coherent upper and lower previsions defined by haussdorff outer and inner measures. In A. Rauth and E. Auer (Eds.), *Modeling, Design and Simulation of Systems with Uncertainties*, pp. 175–195. Springer, 2011.

[9] L. E. Dubins. Finitely additive conditional probabilities, conglomerability and disintegrations. *The Annals of Probability*, 3:88–99, 1975.

[10] J. Kadane, M. Schervisch, and T. Seidenfeld. Statistical implications of finitely additive probability. In P.K. Goel and A. Zellner (Eds.), *Bayesian inference and decision techniques*, pp. 59–76. Elsevier, Amsterdam, 1986.

[11] E. Miranda. A survey of the theory of coherent lower previsions. *International Journal of Approximate Reasoning*, 48(2):628–658, 2008.

[12] E. Miranda and G. de Cooman. Marginal extension in the theory of coherent lower previsions. *International Journal of Approximate Reasoning*, 46(1):188–225, 2007.

[13] E. Miranda and M. Zaffalon. Coherence graphs. *Artificial Intelligence*, 173(1):104–144, 2009.

[14] E. Miranda and M. Zaffalon. Notes on desirability and conditional lower previsions. *Annals of Mathematics and Artificial Intelligence*, 60(3–4):251–309, 2010.

[15] R. Pelessoni and P. Vicig. Williams coherence and beyond. *International Journal of Approximate Reasoning*, 50(4):612–626, 2009.

[16] M. Schervisch, T. Seidenfeld, and J. Kadane. The extent of non-conglomerability of finitely additive probabilities. *Zeitschrift fur Wahrscheinlichkeitstheorie und verwandte Gebiete*, 66:205–226, 1984.

[17] M. Schervisch, T. Seidenfeld, and J. Kadane. On the equivalence between conglomerability and disintegrability for unbounded random variables. *Annals of Probability*, 2010. Submitted for publication. Downloadable at *http://www.hss.cmu.edu/philosophy/faculty-seidenfeld.php*.

[18] T. Seidenfeld, M. Schervisch, and J.B. Kadane. Non-conglomerability for finite-valued finitely additive probability. *Sankhya*, 60(3):476–491, 1998.

[19] G. Shafer. Conditional probability. *International Statistical Review*, 53:261–277, 1985.

[20] T. Seidenfeld, M. Schervisch, and J. Kadane. Improper regular conditional distributions. *The Annals of Probability*, 29(4):1612–1624, 2001.

[21] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, London, 1991.

[22] P. Williams. Notes on conditional previsions. *International Journal of Approximate Reasoning*, 44:366–383, 2007. Revised journal version of [23].

[23] P. M. Williams. Notes on conditional previsions. Technical report, School of Mathematical and Physical Science, University of Sussex, UK, 1975. Reprinted in [22].

[24] M. Zaffalon and E. Miranda. Probability and time. 2011. Submitted for publication.

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