Computing the Conglomerable Natural Extension

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Abstract

Given a coherent lower prevision $\underline{P}$, we consider the problem of computing the smallest coherent lower prevision $\underline{F} \geq \underline{P}$ that is conglomerable, in case it exists. $\underline{F}$ is called the conglomerable natural extension. Past work has showed that $\underline{F}$ can be approximated by an increasing sequence $(\underline{E}_n)_{n\in\mathbb{N}}$ of coherent lower previsions. We close an open problem by showing that this sequence can be made of infinitely many distinct elements. Moreover, we give sufficient conditions, of quite broad applicability, to make sure that the point-wise limit of the sequence is $\underline{F}$ in case $\underline{P}$ is the lower envelope of finitely many linear previsions. In addition, we study the question of the existence of $\underline{F}$ and its relationship with the notion of marginal extension.

Keywords. Coherent lower previsions, conglomerability, conglomerable natural extension, natural extension, marginal extension.

1 Introduction

When the possibility space Ω is infinite and you express your beliefs through a coherent lower prevision $\underline{P}$, you may want to consider a partition $\mathcal{B}$ of Ω made of infinitely many conditioning events. In this case it may happen that $\underline{P}$ is not coherent, in Walley's sense, with any lower prevision conditional on $\mathcal{B}$; we say that $\underline{P}$ is not conglomerable.[1]

Conglomerability is a concern for Walley's theory, because its failure makes it impossible to update $\underline{P}$. More generally speaking, conglomerability should arguably be a rationality requirement for a probabilistic model under a dynamic interpretation of conditioning that relates present and future commitments, as detailed in [9].[2]

If we endorse conglomerability as a rationality requirement and consider a non-conglomerable coherent lower prevision $\underline{P}$, it becomes interesting to consider the *conglomerable natural extension* of $\underline{P}$, if it exists: that is, the weakest conglomerable coherent lower prevision $\underline{F}$ that extends $\underline{P}$. Thus, it plays the analogous role that the natural extension of a lower prevision (which avoids sure loss) plays with respect to coherence. Some recent work [5] has showed that $\underline{F}$ can be approximated though a sequence of coherent lower previsions $(\underline{E}_n)_{n\in\mathbb{N}}$ such that $\underline{P} \leq \underline{E}_1 \leq \underline{E}_2 \leq \cdots \leq \underline{E}_i \leq \cdots \leq \underline{F}$. It is known already that if the sequence becomes stable, that is, if $\underline{E}_{i-1} = \underline{E}_i$ for some i, then $\underline{E}_i = \underline{F}$; and, conversely, if the sequence breaks down, which means that $\underline{E}_i$ cannot be produced for some i, then $\underline{F}$ does not exist.

However, some fundamental questions have been left open with regard to the sequence $(\underline{E}_n)_n$. One of them is whether or not it may be infinite—without ever becoming stable. If that is the case, then the next question is whether or not the point-wise limit $\underline{Q}$ of the sequence equals $\underline{F}$. In fact, in principle it could be the case that $\underline{Q}$ is not conglomerable while $\underline{F}$ exists; this would mean that you should re-start a new sequence from $\underline{Q}$ in order to get to $\underline{F}$ (and possibly another, and another, and another, etc.).

After some introductory concepts we give in Section 2, we start a preliminary analysis in Section 3: we show that some basic procedures, like taking point-wise limits, or convex combinations, of conglomerable models do not preserve conglomerability in general. In Section 4 we discuss the question of the existence of $\underline{F}$ and its relationship with some pre-existing concepts about coherent lower previsions. In particular, Example 3 yields one more negative, and yet important, result: that $\underline{F}$ may not exist even when $\underline{P}$ avoids partial loss with its *conditional natural extension* $\underline{P}(\cdot|\mathcal{B})$, i.e., the model obtained by conditioning $\underline{P}$ in the least-committal way.

In Section 5 we close the first question mentioned above: we construct in Example 4 a model $\underline{P}$ whose related sequence $(\underline{E}_n)_n$ is infinite. In this case the limit $\underline{Q}$ of the sequence equals $\underline{F}$, which does not allow us to close the second question, which remains thus open.

In Section 6 we deepen the study, preliminarily started in [5], on the relationship between *marginal extension* and the conglomerable natural extension. We consider in particular

[1] We consider the case of a fixed partition $\mathcal{B}$ in this paper, so we do not deal with *full conglomerability*. See [6, Sections 6.8 and 6.9] for more details on the latter notion.

[2] On the other hand, there are also works on conditional previsions where conglomerability is not imposed [1, 3, 7]; we refer to [6, Section 6.8] for some discussion on this topic.

the relationship between $(\underline{E}_n)_n$ and the sequence $(\underline{M}_n)_n$, where $\underline{M}_n := \underline{E}_{n-1}(\underline{E}_{n-1}(\cdot|\mathcal{B}))$ is the marginal extension of $\underline{E}_{n-1}$ and its conditional natural extension $\underline{E}_{n-1}(\cdot|\mathcal{B})$. It turns out that $(\underline{M}_n)_n$ is also an increasing sequence of coherent lower previsions that is dominated by $\underline{F}$; however we show in Example 5 that the point-wise limit $\underline{Q}'$ of the sequence $(\underline{M}_n)_n$ may differ from $\underline{F}$. In addition, by detailing the relationships among $\underline{P}$, $\underline{Q}$, $\underline{Q}'$ and $\underline{F}$ we deduce in Proposition 8 that if $(\underline{E}_n(\cdot|\mathcal{B}))_n$ converges uniformly to the conditional natural extension $\underline{Q}(\cdot|\mathcal{B})$ of $\underline{Q}$, then $\underline{Q} = \underline{F}$.

In Section 7 we focus on the special case where $\underline{P}$ is dominated by a set of linear previsions with finitely many extreme points. This allows us to deduce two new simple conditions, which seem to be quite broadly applicable, that make sure that $(\underline{E}_n(\cdot|\mathcal{B}))_n$ converges uniformly to $\underline{Q}(\cdot|\mathcal{B})$, and hence, through Proposition 8, that $\underline{Q} = \underline{F}$. This analysis shows in particular that, when $\underline{P}$ is the lower envelope of two linear previsions, there is a procedure to determine whether $\underline{F}$ exists, and in this case we always have that $\underline{Q} = \underline{F}$.

We report our summary views in Section 8. Due to limitations of space, proofs have been omitted.

2 Introduction to Imprecise Probabilities

Let us introduce the basics of the theory of coherent lower previsions that we use in this paper. We refer to [6] for an in-depth study, and for a behavioural interpretation of the following notions in terms of buying and selling prices.

Consider a possibility space Ω. A *gamble* is a bounded map $f\colon \Omega \to \mathbb{R}$. The set of all gambles is denoted by $\mathcal{L}(\Omega)$, or simply by $\mathcal{L}$ when there is no ambiguity about the possibility space we are working with.

A *lower prevision* $\underline{P}$ is a real-valued functional defined on some set of gambles $\mathcal{K} \subseteq \mathcal{L}$. When the domain $\mathcal{K}$ of $\underline{P}$ is a linear space—closed under point-wise addition and multiplication by real numbers—$\underline{P}$ is called *coherent* when it satisfies the following conditions:

C1. $\underline{P}(f) \geq \inf f \; \forall f \in \mathcal{K}$;

C2. $\underline{P}(\lambda f) = \lambda \underline{P}(f) \; \forall f \in \mathcal{K}, \lambda \geq 0$;

C3. $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g) \; \forall f, g \in \mathcal{K}$.

Given a partition[3] $\mathcal{B}$ of Ω, a *conditional lower prevision* on $\mathcal{L}$ is a functional $\underline{P}(\cdot|\mathcal{B}) := \sum_{B\in\mathcal{B}} B\underline{P}(\cdot|B)$ such that for every set $B \in \mathcal{B}$, $\underline{P}(\cdot|B)$ is a lower prevision on $\mathcal{L}$. $\underline{P}(\cdot|\mathcal{B})$ is called *separately coherent* when $\underline{P}(\cdot|B)$ is coherent and $\underline{P}(B|B) = 1$ for every $B \in \mathcal{B}$. For every gamble f, $\underline{P}(f|\mathcal{B})$ is the gamble on Ω that takes the value $\underline{P}(f|B)$ on $\omega \in B$, and this for every $B \in \mathcal{B}$.

[3] See also [7] for an alternative approach where the conditioning is made on a class of events that do not necessarily form a partition.

For every lower prevision $\underline{P}$ and every conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, we use the notations: $G_{\underline{P}}(f) := f - \underline{P}(f)$, $G_{\underline{P}}(f|B) := B(f - \underline{P}(f|B))$ and $G_{\underline{P}}(f|\mathcal{B}) := f - \underline{P}(f|\mathcal{B}) = \sum_{B\in\mathcal{B}} G_{\underline{P}}(f|B)$. If we consider a coherent lower prevision $\underline{P}$ on $\mathcal{L}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$, they are called *coherent*[4] if and only if for every gamble f and every $B \in \mathcal{B}$,

$$\underline{P}(G_{\underline{P}}(f|\mathcal{B})) \geq 0, \tag{CNG}$$

$$\underline{P}(G_{\underline{P}}(f|B)) = 0. \tag{GBR}$$

This second condition is called the *generalised Bayes rule*, and if $\underline{P}(B) > 0$ it can be used to uniquely determine the value $\underline{P}(f|B)$: in that case there is only one value satisfying (GBR) with respect to $\underline{P}$. On the other hand, (CNG) is a *conglomerability* condition based on the behavioral idea that $G_{\underline{P}}(f|\mathcal{B})$ is a combination of (possibly infinitely many) acceptable transactions, and should be then an acceptable transaction, too.

A particular case of coherent $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ is that made of the *vacuous* unconditional and conditional lower previsions, given by $\underline{P}(f) = \inf_{\omega\in\Omega} f(\omega)$ and $\underline{P}(f|B) = \inf_{\omega\in B} f(\omega)$ for all $f \in \mathcal{L}$ and all $B \in \mathcal{B}$.

On the other hand, a coherent lower prevision $\underline{P}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$ are said to *avoid partial loss* (APL) when

$$\sup \left[G_{\underline{P}}(f) + G_{\underline{P}}(g|\mathcal{B}) \right] \geq 0 \tag{1}$$

for every pair of gambles $f, g \in \mathcal{L}$. Eq. (1) holds whenever $\underline{P}(\cdot|\mathcal{B})$ is the vacuous conditional lower prevision irrespective of the coherent lower prevision $\underline{P}$, because in that case $G_{\underline{P}}(f|\mathcal{B}) \geq 0$ for any gamble f.

A particular case of coherent lower previsions is that of *linear* previsions. A linear prevision is a functional $P\colon \mathcal{L} \to \mathbb{R}$ satisfying conditions C1 and C2, and condition C3 with equality for all gambles $f, g \in \mathcal{L}$. Its restriction to $\mathcal{P}(\Omega)$, the powerset of Ω, is a finitely additive probability, and P is the corresponding expectation operator. The set of all linear previsions is denoted by $\mathbb{P}$. Given a lower prevision $\underline{P}$ on $\mathcal{K}$, its associated *credal set* is $\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}\colon (\forall f \in \mathcal{K}) P(f) \geq \underline{P}(f)\}$, and each P in $\mathcal{M}(\underline{P})$ is said to *dominate* $\underline{P}$. A lower prevision for which $\mathcal{M}(\underline{P}) \neq \emptyset$ is said to *avoid sure loss*. It is coherent if and only if $\underline{P} = \min \mathcal{M}(\underline{P})$. Similarly, a *conditional linear prevision* is a functional $P(\cdot|\mathcal{B})$ on $\mathcal{L}$ such that $P(B|B) = 1$ and $P(\cdot|B)$ is a linear prevision for every $B \in \mathcal{B}$.

Given a coherent lower prevision $\underline{P}$, we define by

$$\underline{P}(f|B) := \begin{cases} \inf_{\omega\in B} f(\omega) & \text{if } \underline{P}(B) = 0 \\ \min\{P(f|B) : P \in \mathcal{M}(\underline{P})\} & \text{otherwise} \end{cases} \tag{2}$$

[4] See [6, Section 6.3.2] for a definition of coherence on more general domains, and also [6, Theorem 6.5.3].

its *conditional natural extension*. $\underline{P}(f|B)$ is a separately coherent lower prevision, defined for every $B \in \mathcal{B}$ and every $f \in \mathcal{L}$, which always satisfies (GBR) with $\underline{P}$. Thus, $\underline{P}, \underline{P}(\cdot|B)$ are coherent if and only if (CNG) holds for every gamble f. When that is the case, we say that $\underline{P}$ is a *conglomerably coherent lower prevision*. We refer to [6, Sections 6.8 and 6.9] for a thorough study of conglomerability. For the purposes of this paper, the most important property is that a conglomerably coherent lower prevision is one that can be updated to a conditional lower prevision while satisfying Walley's notion of coherence, so it is essential if we want to use Walley's approach in the conditional case.

Conglomerability holds trivially whenever $\underline{P}(B) = 0$ for all but a finite number of conditioning events $B \in \mathcal{B}$. Moreover, (CNG) always holds whenever the *support* of the gamble f, which is given by $S(f) \coloneqq \{B \in \mathcal{B} : Bf \neq 0\}$ is finite. In particular, this means that conglomerability holds trivially for finite partitions.

3 Basic Properties of Conglomerability

Let us begin by making a preliminary study of conglomerably coherent lower previsions. Unlike the family of coherent lower previsions (see [6, Section 2.6]), the set of conglomerably coherent lower previsions is not closed under convex combinations or point-wise limits. We begin by focusing on this second property:

Example 1. Consider a partition $\mathcal{B}$ of Ω and two linear previsions P_1, P_2 on $\mathcal{L}$ such that P_1 is conglomerable and P_2 is not for a countable partition $\mathcal{B} \coloneqq \{B_n : n \in \mathbb{N}\}$ such that $P_1(B_n), P_2(B_n) > 0$ for all n. (In this paper $\mathbb{N}$ denotes the set of positive natural numbers.)

Define Q_n on $\mathcal{L}$ by $Q_n(f) \coloneqq P_2(f\mathbb{I}_{\cup_{i=1}^n B_i}) + P_1(f\mathbb{I}_{\cup_{i>n} B_i})$; it can easily be checked that Q_n is a linear prevision. Moreover, $Q_n(f|B_m)$ is equal to $P_2(f|B_m)$ if $m \leq n$ and to $P_1(f|B_m)$ if $m > n$, whence $Q_n(Q_n(f|\mathcal{B})) = Q_n(f)$.

This means that the linear prevision Q_n is conglomerable for every n. On the other hand, $\lim_n Q_n(f) = P_2(f)$ for every f, so the limit of the sequence $(Q_n)_n$ is not a conglomerable prevision.

The above comments also show that the coherence of an unconditional and a conditional lower prevision is not preserved by point-wise limits: since $Q_n(B_m) > 0$ for all $m, n \in \mathbb{N}$, we deduce that Q_n is coherent with its conditional natural extension $Q_n(\cdot|\mathcal{B})$, which is a linear prevision. However, the point-wise limit of the sequence $(Q_n)_n$, that is, the linear prevision P_2, is not coherent with its conditional natural extension $P_2(\cdot|\mathcal{B})$ because P_2 is not conglomerable. It also follows that $\lim_n Q_n(f|B_m) = P_2(f|B_m)$ for all $m \in \mathbb{N}, f \in \mathcal{L}$, whence $P_2(\cdot|\mathcal{B})$ is the limit of $Q_n(f|\mathcal{B})$. Thus $Q_n, Q_n(\cdot|\mathcal{B})$ are coherent for all n but their point-wise limits $P_2, P_2(\cdot|\mathcal{B})$ are not. ♦

Next, we investigate if the property of conglomerability is preserved by taking convex combinations. As discussed by Walley in [6, Theorem 6.9.1], a sufficient condition for a linear prevision P to be conglomerable is that it is countably additive on $\mathcal{B}$, in the sense that $\sum_{B\in\mathcal{B}} P(B) = 1$. This means in particular that a convex combination of two linear previsions P_1, P_2 that are countably additive on $\mathcal{B}$ will again be countably additive with respect to this partition, and as a consequence it will also be conglomerable.

However, there are also conglomerable linear previsions P that are not countably additive on $\mathcal{B}$ [6, Examples 6.6.4, 6.6.5], and they can be used to show that conglomerability is not necessarily preserved by convex combinations:

Example 2. Consider $\Omega \coloneqq \mathbb{N} \cup -\mathbb{N}$, $B_n \coloneqq \{-n, n\}$ and the partition $\mathcal{B} \coloneqq \{B_n : n \in \mathbb{N}\}$. Let P_1, P_2 be two linear previsions whose restrictions to events satisfy

$$P_1(B_n) = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even,} \end{cases} \qquad P_1(\{2n\}_{n\in\mathbb{N}}) = \frac{1}{3},$$

$$P_2(B_n) = \begin{cases} \frac{1}{2^{n-1}} & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd,} \end{cases} \qquad P_2(\{2n-1\}_{n\in\mathbb{N}}) = \frac{1}{3};$$

that is, P_1 (resp., P_2) is countably additive on $\cup_{n\in\mathbb{N}} B_{2n-1}$ (resp., $\cup_{n\in\mathbb{N}} B_{2n}$) and purely finitely additive on $\cup_{n\in\mathbb{N}} B_{2n}$ (resp., $\cup_{n\in\mathbb{N}} B_{2n-1}$). Assume moreover that $P_1(\{n\}) = P_1(\{-n\})$ and $P_2(\{n\}) = P_2(\{-n\})$ for every n.

For any gamble f on Ω, it holds that $P_1(G_1(f|\mathcal{B})) \geq P_1(G_1(f\mathbb{I}_{\cup_{n\in\mathbb{N}} B_{2n-1}}|\mathcal{B}))$, taking into account that $\underline{P}_1(\cdot|B_{2n})$ is vacuous for every n and as a consequence $G_1(f\mathbb{I}_{\cup n\in\mathbb{N}} B_{2n}) \geq 0$. Moreover, if we consider the set $D \coloneqq \cup_{n\in\mathbb{N}} B_{2n}$ and the partition $\mathcal{B}' \coloneqq \{D\} \cup \{B_{2n-1} : n \in \mathbb{N}\}$ of Ω, it follows that $\sum_{B'\in\mathcal{B}'} P_1(B') = 1$. Applying [6, Theorem 6.9.1], it follows that P_1 is conglomerable with respect to $\mathcal{B}'$, and from this we deduce that $P_1(G_1(f\mathbb{I}_{\cup_{n\in\mathbb{N}} B_{2n-1}}|\mathcal{B})) = P_1(G_1(f\mathbb{I}_{\cup_{n\in\mathbb{N}} B_{2n-1}}|\mathcal{B}')) \geq 0$. As a consequence, P_1 is conglomerable. Similarly, so is P_2. However, if we consider the linear prevision $P \coloneqq 0.5P_1 + 0.5P_2$, it holds that $P(f|B_n) = \frac{f(n)+f(-n)}{2}$ $\forall n \in \mathbb{N}, f \in \mathcal{L}$. Given $f \coloneqq 2\mathbb{I}_{-\mathbb{N}}$, it follows that $P(f|B_n) = 1$ for every n, whence $P(G(f|\mathcal{B})) = \frac{1}{3} - \frac{2}{3} < 0$, since $P_1(\mathbb{N}) = P_2(\mathbb{N}) = \frac{2}{3}$ by construction. This shows that P is not conglomerable. ♦

4 On the Existence of the Conglomerable Natural Extension

The above preliminary results illustrate the fact that conglomerably coherent lower previsions do not share many of the properties of coherent lower previsions. Another instance of this is that a lower prevision $\underline{P}$ that avoids sure loss has always a smallest dominating coherent lower prevision, but it may not have a dominating conglomerably coherent lower prevision. This is easy to see by means of a linear prevision P that is not conglomerable: any conglomerably coherent lower prevision $\underline{F}$ that dominates P should also coincide with P, because of linearity, and as a consequence such an $\underline{F}$ does not exist.

Although in Section 3 we have showed that the limit of a sequence of conglomerable lower previsions may not be conglomerable, it follows from [6, Theorem 6.9.3] that the lower envelope of a family of conglomerable lower previsions is again conglomerable. Hence, if $\underline{P}$ has a dominating conglomerable model, then there is also a smallest dominating conglomerable model. We shall refer to it as the conglomerable natural extension of $\underline{P}$.

Definition 1. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$ and let $\mathcal{B}$ be a partition of Ω. The ($\mathcal{B}$-)*conglomerable natural extension* of $\underline{P}$ is the smallest coherent lower prevision $\underline{F} \geq \underline{P}$ that is conglomerable with respect to $\mathcal{B}$.

As we have showed before, the conglomerable natural extension of a lower prevision $\underline{P}$ may not exist. Taking this into account, it becomes interesting to provide sufficient conditions for its existence. We begin by investigating the relationships among a number of consistency notions from [6, Chapters 6 and 7]:

Proposition 1. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, $\mathcal{B}$ a partition of Ω, and $\underline{P}(\cdot|\mathcal{B})$ a separately coherent lower prevision. Consider the following possibilities:*

(a) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent.*

(b) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are dominated by coherent $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$.*

(c) *The conglomerable natural extension of $\underline{P}$ exists.*

(d) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are dominated by $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$ that avoid partial loss.*

(e) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss.*

Then (a)⇒(b)⇒(d)⇔(e) *and* (b)⇒(c). *If, in addition, $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{P}$, then* (c) ⇒ (b) *holds as well, and if in particular $\underline{P}$ is linear then we have also that* (b) ⇒ (a) *and* (d) ⇒ (b)*, so all of them are equivalent conditions.*

Now, if we consider a coherent lower prevision $\underline{P}$, it follows that its conglomerable natural extension exists if and only if there is a coherent lower prevision $\underline{F} \geq \underline{P}$ that is conglomerable. Since conglomerability is equivalent to the coherence with the conditional natural extension, it follows that the conglomerable natural extension of $\underline{P}$ exists if and only if $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are dominated by coherent $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$, where $\underline{P}(\cdot|\mathcal{B})$ denotes the conditional natural extension of $\underline{P}$. We deduce from Proposition 1 that the following implications hold:

$$\underline{P} \text{ conglomerable } \Rightarrow \underline{F} \text{ exists} \Rightarrow \underline{P}, \underline{P}(\cdot|\mathcal{B}) \text{ APL}, \quad (3)$$

where $\underline{F}$ is the conglomerable natural extension of $\underline{P}$, introduced in Definition 1. Moreover, $\underline{F}$ exists if and only if $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ *avoid conglomerable partial loss*, in the sense of [4, Definition 21]. The converses of the implications in (3) do not hold in general: on the one hand, there are previsions $\underline{P}$ that are not conglomerable but whose conglomerable natural extension exists (one instance is that in Example 4 later on). Next we show that the converse of the second implication does not hold either. In other words, the conditions of avoiding partial loss and avoiding conglomerable partial loss are not equivalent in general. In order to build this example, we need to define the notion of unconditional natural extension:

Definition 2. Let $\underline{P}$ be a coherent lower prevision and $\underline{P}(\cdot|\mathcal{B})$ be a separately coherent conditional lower prevision on $\mathcal{L}$. Their *unconditional natural extension* $\underline{E}_1$ is given on f by the supremum α such that

$$f - \alpha \geq G_{\underline{P}}(g) + G_{\underline{P}}(h|\mathcal{B}) \text{ for some } g, h \in \mathcal{L}.$$

Then $\underline{E}_1$ is a coherent lower prevision on $\mathcal{L}$ if and only if $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss. Moreover, if $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{P}$ and $\underline{E}_1(\cdot|\mathcal{B})$ is that of $\underline{E}_1$, then any coherent $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$ that dominate $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ must also dominate $\underline{E}_1, \underline{E}_1(\cdot|\mathcal{B})$. Thus, the conglomerable natural extensions of $\underline{P}$ and $\underline{E}_1$ coincide.

Example 3. Consider $\Omega := \mathbb{N} \cup -\mathbb{N}$, $B_n := \{n, -n\}$ and $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$. Let P_1 be a σ-additive linear prevision on $\mathcal{L}$ determined by $P_1(n) := P_1(\{-n\}) := \frac{1}{2^{n+1}}$.

Let P be a finitely additive probability on $\mathcal{P}(\mathbb{N})$ satisfying $P(\{n\}) = 0$ for all n, $P(\{2n+1 : n \in \mathbb{N}\}) = 0$. We can use it to define a linear prevision P_2 on $\mathcal{L}$ whose restriction to events is the finitely additive probability given by $P_2(B) := \frac{3}{4}P(\Pi_1(B)) + \frac{1}{4}P(\Pi_2(B))$, where $\Pi_1(B) := B \cap \mathbb{N}$ and $\Pi_2(B) := -(B \cap -\mathbb{N})$. Define then the linear prevision $P_3 := \frac{1}{2}P_1 + \frac{1}{2}P_2$.

Let now P' be another finitely additive probability on $\mathcal{P}(\mathbb{N})$ such that $P'(\{n\}) = 0$ for all n, $P'(\{2n+1 : n \in \mathbb{N}\}) = 0.5$, so that $P'(\mathbb{I}_{\text{even}}) = 0.5$ too. Let P_4 be the linear prevision on $\mathcal{L}$ whose restriction to events is the finitely additive probability

$$P_4(B) := \frac{1}{4} \sum_{n \in B \cap \mathbb{N}} \frac{1}{2^n} + \frac{3}{4} P'(-(B \cap -\mathbb{N})).$$

Take $\underline{P} := \min\{P_3, P_4\}$. Then $\underline{P}(B_n) = \min\left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\} > 0 \;\forall n \in \mathbb{N}$, whence $\underline{P}(f|B_n) = \min\left\{\frac{f(n)+f(-n)}{2}, f(n)\right\} \forall f \in \mathcal{L}, n \in \mathbb{N}$.

Fix a gamble f and let $C := \cup_{n:f(n)<f(-n)} B_n$, so that $\underline{P}(f|B_n) = f(n)$ if $B_n \subseteq C$ and $\underline{P}(f|B_n) = \frac{f(n)+f(-n)}{2}$ otherwise. Then $G(f|\mathcal{B}) = G(f \cdot C|\mathcal{B}) + G(f \cdot C^c|\mathcal{B}) \geq G(f \cdot C^c|\mathcal{B})$ because $G(f|B_n) \geq 0$ if $B_n \subseteq C$.

Denote $P_\alpha := \alpha P_3 + (1-\alpha)P_4$. We are going to determine for which $\alpha \in [0,1]$ it holds that $P_\alpha(G(f|\mathcal{B})) \geq 0$ for all f. Taking into account the above observation, we can conclude that $P_\alpha(G(f|\mathcal{B})) \geq 0 \; \forall f \in \mathcal{L}$ if and only if $P_\alpha(G(f|\mathcal{B})) \geq 0 \, \forall f \in \mathcal{L}$ s.t. $f(n) \geq f(-n) \, \forall n$.

Take thus f s.t. $f(n) \geq f(-n)$ for all n (in this case C is empty). Then

$$\begin{cases} G(f|B_n)(n) = \frac{f(n)-f(-n)}{2} \geq 0, \\ G(f|B_n)(-n) = \frac{f(-n)-f(n)}{2} \leq 0. \end{cases} \tag{4}$$

If we denote $g \coloneqq G(f|\mathcal{B})$, it holds that $g(n) = -g(-n)$, whence $P_1(g\mathbb{I}_{\mathbb{N}}) + P_1(g\mathbb{I}_{-\mathbb{N}}) = 0$. On the other hand, $P_2(g\mathbb{I}_{\mathbb{N}}) + P_2(g\mathbb{I}_{-\mathbb{N}}) = \frac{3}{4}P(g^+) + \frac{1}{4}P(g^-)$, where

$$\begin{aligned} g^+ : \mathbb{N} &\to \mathbb{R} \\ n &\hookrightarrow g(n) \end{aligned} \quad \text{and} \quad \begin{aligned} g^- : \mathbb{N} &\to \mathbb{R} \\ n &\hookrightarrow g(-n) = -g(n). \end{aligned} \tag{5}$$

Thus $P_2(g\mathbb{I}_{\mathbb{N}}) + P_2(g\mathbb{I}_{-\mathbb{N}}) = \frac{1}{2}P(g^+) \geq 0$; as a consequence, $P_3(G(f|\mathcal{B})) \geq 0$ for every gamble f.

Now, if in particular we fix $n \in \mathbb{N}$ and let $f \coloneqq 2\mathbb{I}_{\{2n+1,2n+3,\dots\}}$, then, using (4) again, $G(f|\mathcal{B}) = \mathbb{I}_{\{2n+1,2n+3,\dots\}} - \mathbb{I}_{\{-2n-1,-2n-3,\dots\}}$ and $P_1(G(f|\mathcal{B})) = 0 = P_2(G(f|\mathcal{B}))$, because we have chosen P such that $P(\{2n+1 : n \in \mathbb{N}\}) = 0$. Thus, $P_3(G(f|\mathcal{B})) = 0$.

On the other hand, for this gamble f we obtain that $P_4(G(f|\mathcal{B})) = \sum_{k \geq n} \frac{1}{2^{(2k+1)+2}} - \frac{3}{8} < 0$ for n big enough.

This implies that $P_\alpha(G(f|\mathcal{B})) < 0$ for all $\alpha \neq 1$. As a consequence $\{P_\alpha : P_\alpha(G(f|\mathcal{B})) \geq 0 \, \forall f\} = P_3 = \underline{E}_1$, taking into account that $\mathcal{M}(\underline{P}) = \{P_\alpha : \alpha \in [0,1]\}$ and using [5, Proposition 13]. Since the natural extension $\underline{E}_1$ of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ exists, it follows that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ avoid partial loss. But P_3 is not conglomerable: given $g \coloneqq 2\mathbb{I}_{-\mathbb{N}}$, we can use the expression of $P_3(\cdot|B_n)$ (available from that of $\underline{P}(\cdot|B_n)$) to see that $P_3(g|B_n) = \frac{[2\mathbb{I}_{-\mathbb{N}}](n)+[2\mathbb{I}_{-\mathbb{N}}](-n)}{2} = \frac{2}{2} = 1$, so that $G_{P_3}(g|\mathcal{B}) = -\mathbb{I}_{\mathbb{N}} + \mathbb{I}_{-\mathbb{N}}$ and $P_3(G(g|\mathcal{B})) = -\frac{1}{4} < 0$. Thus $P_3, P_3(\cdot|\mathcal{B})$ do not avoid partial loss, and applying (3) we deduce that the conglomerable natural extension of P_3 does not exist. But since P_3 is the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$, the conglomerable natural extension of $\underline{P}$ coincides with that of P_3. Hence, the conglomerable natural extension of $\underline{P}$ does not exist either. ♦

We can get more, and different, results in the special case where the conditional natural extension of $\underline{P}$ is linear.

Proposition 2. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$ and assume that its conditional natural extension is a linear prevision $P(\cdot|\mathcal{B})$. Then:*

(a) *$\underline{P}, P(\cdot|\mathcal{B})$ avoid partial loss if and only if $\underline{P}, P(\cdot|\mathcal{B})$ avoid conglomerable partial loss.*[5]

(b) *$\underline{P}$ is conglomerable if and only if it is a lower envelope of conglomerable linear models.*

[5]This has essentially been showed already in [5, Proposition 15].

From [6, Theorem 6.9.3], a lower envelope of a family of conglomerable lower previsions is again a conglomerable lower prevision; the converse is not true: [6, Example 6.6.9] shows that it may be that $\underline{P}$ is a conglomerably coherent lower prevision but no dominating model is. One interesting particular case where an assessment of conglomerability is compatible with an envelope theorem is when we are dealing with marginal extension models [6, Theorem 6.7.4]: any marginal extension is a conglomerable model that is a lower envelope of a family of conglomerable linear previsions. Proposition 2 provides an instance of this case.

5 Approximation by a Sequence

In [5], it was devised a procedure to approximate the conglomerable natural extension (if it exists) of a coherent lower prevision $\underline{P}$: we consider the sequence of coherent lower previsions $(\underline{E}_n)_n$, where $\underline{E}_0 \coloneqq \underline{P}$ and for every $n \geq 1$, $\underline{E}_n$ is the (unconditional) natural extension of $\underline{E}_{n-1}, \underline{E}_{n-1}(\cdot|\mathcal{B})$, where $\underline{E}_{n-1}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{E}_{n-1}$, given by Eq. (2).

Proposition 3. *[5] Assume that the conglomerable natural extension $\underline{F}$ of $\underline{P}$ exists. Then:*

1. *$(\underline{E}_n)_n$ is an increasing sequence of coherent lower previsions, and $(\underline{E}_n(\cdot|\mathcal{B}))_n$ is an increasing sequence of separately coherent conditional lower previsions.*
2. *Given their point-wise limits $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$, it holds that $\underline{Q}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{Q}$.*
3. *$\underline{Q} \leq \underline{F}$, and $\underline{Q} = \underline{F} \Leftrightarrow \underline{Q}$ is conglomerable.*

Moreover, it was showed in [5, Example 5] that the sequence may not stabilise in the first step, or, in other words, that the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ does not always coincide with the conglomerable natural extension.

In terms of credal sets, we have the following:

Proposition 4. *[5, Propositions 13 and 14] Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, $\mathcal{B}$ a partition of Ω and $\underline{P}(\cdot|\mathcal{B})$ its conditional natural extension. Let $\underline{E}$ be the unconditional natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$. Then*

$$\begin{aligned} \mathcal{M}(\underline{E}) &= \{P \in \mathcal{M}(\underline{P}) : P(G_{\underline{P}}(f|\mathcal{B})) \geq 0 \, \forall f \in \mathcal{L}\} \\ &= \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{M}), \text{ where } \underline{M} \coloneqq \underline{P}(\underline{P}(\cdot|\mathcal{B})). \end{aligned}$$

In this section, we are going to study the above sequence in more detail. It follows that if the sequence stabilises in a finite number of steps, i.e., if $\underline{Q} = \underline{E}_n$ for some n, then $\underline{Q}$ is the conglomerable natural extension of $\underline{P}$. However, as we shall see later, it may happen that the sequence is infinite. In order to provide an example, we are going to give a tool first that will allow us to build sequences that can be made both conglomerable and non-conglomerable, depending on the choice of two parameters.

Proposition 5. *Let P_1 be a σ-additive probability on $\mathcal{L}(\mathbb{N})$ such that $P_1(\{n\}) > 0$ for all $n \in \mathbb{N}$; let P_2 be a finitely additive probability on $\mathcal{P}(\mathbb{N})$ such that $P_2(\{n\}) = 0$ for all $n \in \mathbb{N}$. We consider $\Omega := \mathbb{N} \cup -\mathbb{N}$ and $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, with $B_n := \{n, -n\}$. Given a gamble h in $\mathcal{L}(\Omega)$, we let h^+, h^- be derived from h as in Eq. (5). Consider $\alpha, \beta \in [0,1]$ and let Q_1, Q_2 on $\mathcal{L}(\Omega)$ be given by*

$$Q_1(h) := \alpha P_1(h^+) + (1-\alpha) P_1(h^-) \text{ and}$$
$$Q_2(h) := \beta P_2(h^+) + (1-\beta) P_2(h^-).$$

Consider also $\gamma \in (0,1)$ and let $Q := \gamma Q_1 + (1-\gamma) Q_2$. Then Q is conglomerable $\Leftrightarrow \alpha = \beta$.

We exploit Proposition 5 to show that the sequence $(\underline{E}_n)_n$ may not stabilise in a finite number of steps.

Example 4. Consider $\Omega := \mathbb{N} \cup -\mathbb{N}$, $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, with $B_n := \{n, -n\}$, and the linear previsions on $\mathcal{L}(\Omega)$

$$\begin{aligned} P_1(\{n\}) &:= P_1(\{-n\}) := \frac{1}{2^{n+1}} \text{ for all } n \in \mathbb{N} \\ P_2(h) &:= \frac{1}{2} \sum_n h(n) \frac{1}{2^n} + \frac{1}{2} P(h^-) \\ P_3(h) &:= \frac{3}{4} P(h^+) + \frac{1}{4} P(h^-) \\ P_4(h) &:= \frac{1}{2} P_1(h) + \frac{1}{2} P_3(h), \end{aligned}$$

where P is a finitely additive probability on $\mathbb{N}$ s.t. $P(\{n\}) = 0$ for all $n \in \mathbb{N}$ and h^+, h^- are determined by Eq. (5). Given $\alpha \in [0,1]$, we set $Q_\alpha := \alpha P_2 + (1-\alpha) P_4$. It follows that

$$\begin{aligned} Q_\alpha(h) &= \frac{1}{2} \left[\frac{1+\alpha}{2} \tilde{P}_1(h^+) + \frac{1-\alpha}{2} \tilde{P}_1(h^-) \right] \\ &+ \frac{1}{2} \left[\frac{3-3\alpha}{4} P(h^+) + \frac{1+3\alpha}{4} P(h^-) \right], \end{aligned}$$

where we denote by $\tilde{P}_1$ the linear prevision given by $\tilde{P}_1(\{n\}) := \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Proposition 5 yields:

$$Q_\alpha \text{ is conglomerable} \Leftrightarrow \frac{1+\alpha}{2} = \frac{3-3\alpha}{4} \Leftrightarrow \alpha = 0.2.$$

Let $\underline{P}$ be the lower envelope of the credal set $\{Q_\alpha : \alpha \in [a,b]\}$ for given a, b s.t. $0 < a < 0.2 < b < 1$. The conglomerable natural extension of $\underline{P}$ exists since $\underline{P} \leq Q_{0.2}$. We aim at analysing whether the sequence of coherent lower previsions $\underline{P}, \underline{E}_1, \underline{E}_2, \ldots$, originated by $\underline{P}$, yields the conglomerable natural extension in the limit and whether or not the sequence itself stabilises in a finite number of steps.

We start by detailing the form of the conditional natural extension of $\underline{P}$. Since $Q_\alpha(f|B_n) = \frac{1+\alpha}{2} f(n) + \frac{1-\alpha}{2} f(-n) \quad \forall f \in \mathcal{L}$ and $\underline{P}(B_n) > 0$, it follows from Eq. (2) and [6, Theorem 6.4.2] that for every gamble f,

$$\underline{P}(f|B_n) = \begin{cases} \frac{1+a}{2} f(n) + \frac{1-a}{2} f(-n) & \text{if } f(n) \geq f(-n) \\ \frac{1+b}{2} f(n) + \frac{1-b}{2} f(-n) & \text{if } f(n) \leq f(-n). \end{cases}$$

If we denote $A := \{n \in \mathbb{N} : f(n) \leq f(-n)\}$, then

$$\begin{cases} G_{\underline{P}}(f|B_n)(n) = \frac{1-b}{2}[f(n) - f(-n)] \leq 0 \\ G_{\underline{P}}(f|B_n)(-n) = \frac{1+b}{2}[f(-n) - f(n)] \geq 0 \end{cases}$$

whenever $n \in A$, and

$$\begin{cases} G_{\underline{P}}(f|B_n)(n) = \frac{1-a}{2}[f(n) - f(-n)] \geq 0 \\ G_{\underline{P}}(f|B_n)(-n) = \frac{1+a}{2}[f(-n) - f(n)] \leq 0 \end{cases}$$

when $n \notin A$. Now we would like to check for which values of α it is the case that $Q_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for all $f \in \mathcal{L}$, because from Proposition 4 we have that $\mathcal{M}(\underline{E}_1) = \{Q_\alpha : Q_\alpha(G_{\underline{P}}(f|\mathcal{B})) \geq 0 \text{ for all } f \in \mathcal{L}\}$.

Given a gamble f, its associated set $A = \{n \in \mathbb{N} : f(n) \leq f(-n)\}$, and $C := \cup_{n \in A} B_n$, it holds that $f = f\mathbb{I}_C + f\mathbb{I}_{C^c}$, whence $G_{\underline{P}}(f|\mathcal{B}) = G_{\underline{P}}(\mathbb{I}_C f|\mathcal{B}) + G_{\underline{P}}(\mathbb{I}_{C^c} f|\mathcal{B})$. Denote $g' := G_{\underline{P}}(\mathbb{I}_C f|\mathcal{B}), g'' := G_{\underline{P}}(\mathbb{I}_{C^c} f|\mathcal{B})$. We proceed to determine when $Q_\alpha(g') \geq 0, Q_\alpha(g'') \geq 0$.

- Let us consider $Q_\alpha(g')$. If $n \notin A$, then $g'(-n) = g'(n) = 0$; if $n \in A$, then $g'(-n) = \frac{1+b}{2}[f(-n) - f(n)]$ and $g'(n) = \frac{1-b}{2}[f(n) - f(-n)]$. As a consequence, $g'(-n) = -\frac{1+b}{1-b} g'(n) \geq 0$. Then:

$$P_2(g') = \sum_n g'(n) \frac{1}{2^{n+1}} + \frac{1}{2} P(g'^-) \text{ and}$$

$$P_4(g') = \sum_n g'(n) \frac{1}{2^{n+1}} \cdot \frac{-b}{1-b} + P(g'^-) \cdot \frac{1}{4} \cdot \frac{2b-1}{1+b}.$$

This implies that $Q_\alpha(g')$ is equal to

$$\underbrace{\underbrace{\sum_n g'(n) \frac{1}{2^{n+1}}}_{\leq 0} \cdot \underbrace{\frac{\alpha - b}{1-b}}_{\leq 0}}_{\geq 0} + \underbrace{P(g'^-) \cdot \frac{1}{4}}_{\geq 0} \cdot \frac{3\alpha + 2b - 1}{1+b},$$

so that $3\alpha + 2b - 1 \geq 0 \Rightarrow Q_\alpha(g') \geq 0$. On the other hand, if $3\alpha + 2b - 1 < 0$, we can always find g', by letting g'^- tend to 1 with $n \to \infty$, such that $P(g'^-) = 1$, using that P is a finitely additive probability that is not σ-additive. And this is compatible with making $\sum_n g'(n) \frac{1}{2^{n+1}}$ as small as we want by making the first m images equal to zero, where m is an arbitrary positive number: it holds that $\lim_m P(g' \mathbb{I}_{\cup_{n \geq m} B_n}) = P(g')$ while $\lim_m \sum_{n \geq m} g'(n) \frac{1}{2^{n-1}} = 0$. We conclude that we can always find some g' such that $Q_\alpha(g') < 0$ when $3\alpha + 2b - 1 < 0$.

- Let us focus on $Q_\alpha(g'')$. It holds that $g''(n) = -\frac{1-a}{1+a} g''(-n) \geq 0$. Then:

$$P_2(g'') = \sum_n g''(n) \frac{1}{2^{n+1}} + \frac{1}{2} P(g''^-) \text{ and}$$

$$P_4(g'') = \sum_n g''(n) \frac{1}{2^{n+1}} \cdot \frac{-a}{1-a} + P(g''^-) \cdot \frac{1}{4} \cdot \frac{2a-1}{1+a}.$$

This implies that $Q_\alpha(g'')$ is given by

$$\underbrace{\sum_n g''(n)\frac{1}{2^{n+1}}\cdot\frac{\alpha-a}{1-a}}_{\geq 0}+\underbrace{P(g''^-)\cdot\frac{1}{4}}_{\leq 0}\cdot\frac{3\alpha+2a-1}{1+a},$$

so that $3\alpha+2a-1\leq 0\Rightarrow Q_\alpha(g'')\geq 0$. On the other hand, if $3\alpha+2a-1>0$, we can reason as in the case of $Q_\alpha(g')$ to conclude that we can always find some g'' such that $Q_\alpha(g'')<0$.

Let us consider the case where $3\alpha+2b-1\geq 0$ and $3\alpha+2a-1\leq 0$ (note that we can attain this case given that $3b+2b-1\geq 0$ and $3a+2a-1\leq 0$ if and only if $a\leq 0.2\leq b$). Then $Q_\alpha(g')\geq 0, Q_\alpha(g'')\geq 0$ and therefore $Q_\alpha(g)\geq 0$; using Proposition 4 we obtain that $Q_\alpha\in\mathcal{M}(\underline{E}_1)$. On the other hand, in the case where $3\alpha+2b-1<0$ or $3\alpha+2a-1\leq 0$, we know that there is g' s.t. $Q_\alpha(g')<0$, and g'' s.t. $Q_\alpha(g'')=0$ (it is enough to use an f, in the definition of g'', s.t. $f(n)=f(-n)$ for all $n\notin A$); applying again Proposition 4, we obtain that $Q_\alpha\notin\mathcal{M}(\underline{E}_1)$. Analogous considerations hold for the remaining cases.

Thus, recalling that $\mathcal{M}(\underline{P})=\{Q_\alpha:\alpha\in[a,b]\}$, with $0<a<0.2<b<1$, it follows that $\mathcal{M}(\underline{E}_1)$ is given by the linear previsions Q_α where $\alpha\in\left[\max\left\{a,\frac{1-2b}{3}\right\},\min\left\{\frac{1-2a}{3},b\right\}\right]$. Note that if $a<b$ then it must be the case that $[\max\{a,\frac{1-2b}{3}\},\min\{\frac{1-2a}{3},b\}]\subsetneq[a,b]$, because it is not possible that both $a\geq\frac{1-2b}{3}$ and $b\leq\frac{1-2a}{3}$ hold. This means that at least one of the two extreme points of $[a,b]$ must change. Moreover, note that the new interval will have still to contain the value 0.2 properly, in the sense that 0.2 will have to be an interior point of the new interval, because

$$a<0.2<b\Rightarrow\max\left\{a,\frac{1-2b}{3}\right\}<0.2\text{ and}$$
$$a<0.2<b\Rightarrow\min\left\{b,\frac{1-2a}{3}\right\}>0.2.$$

Thus, the infinite sequence $\underline{P},\underline{E}_1,\underline{E}_2,\ldots$ is in correspondence with an infinite sequence of intervals of strictly decreasing length, each one containing 0.2 properly.

Let us show now that 0.2 is actually the limit of this sequence. We must consider a number of cases:

- If in the passage from $\mathcal{M}(\underline{P})$ to $\mathcal{M}(\underline{E}_1)$ both extreme points of the interval change, then we go from $[a,b]$ to $[\frac{1-2b}{3},\frac{1-2a}{3}]$, and the length of the new interval is two thirds of the length of the previous one.

- Assume otherwise that that in the passage from $\mathcal{M}(\underline{P})$ to $\mathcal{M}(\underline{E}_1)$ only the left extreme of the interval $[a,b]$ changes (if it were the right extreme, we would eventually obtain analogous conclusions). We can then rewrite the interval as $[\max\{a,\frac{1-2b}{3}\},\min\{\frac{1-2a}{3},b\}]=[\frac{1-2b}{3},\min\{\frac{1-2a}{3},b\}]$. If we now do one more step, to get to $\mathcal{M}(\underline{E}_2)$, we see that the left extreme cannot change and hence the new interval will be $[\frac{1-2b}{3},\frac{1+4b}{9}]$. Hence, in two steps we go from $[a,b]$ to $[\frac{1-2b}{3},\frac{1+4b}{9}]$, and the length of the latter interval is $\frac{10b-2}{9}$. Now, since $a\leq\frac{1-2b}{3}$, we deduce that $3a+2b\leq 1$, and as a consequence $\frac{3}{2}\cdot\frac{10b-2}{9}=\frac{5b-1}{3}\leq b-a$. This means that the length of $[\frac{1-2b}{3},\frac{1+4b}{9}]$ is at most two thirds of the length of $[a,b]$.

By iterating the argument, we conclude that every two steps the length of the intervals decreases at least exponentially fast by $\frac{2}{3}$. As a consequence, given that 0.2 is always included in the intervals, the sequence $(\underline{E}_n)_n$ will converge towards $Q_{0.2}$, which, being conglomerable, is the conglomerable natural extension of $\underline{P}$. ♦

6 Conglomerability and Marginal Extension

The previous example shows that the sequence $(\underline{E}_n)_n$ may not stabilise in a finite number of steps. When $\underline{Q}$ does not coincide with $\underline{E}_n$ for any n, it is an open problem whether $\underline{Q}$ always coincides with the conglomerable natural extension or not. Here, we shall give a number of sufficient conditions for the equality $\underline{Q}=\underline{F}$. We shall show that one particular case of interest is that where $\underline{Q}$ is a marginal extension model and we are going to explore in more detail the connection between conglomerably coherent lower previsions and marginal extensions. We begin by proving an elementary and yet interesting result:

Proposition 6. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, $\mathcal{B}$ a partition of Ω and $\underline{P}(\cdot|\mathcal{B})$ the conditional natural extension of $\underline{P}$. Define $\underline{M}:=\underline{P}(\underline{P}(\cdot|\mathcal{B}))$. Then $\underline{M}\leq\underline{P}\Leftrightarrow\underline{P}$ conglomerable.*

It is possible to find examples that show that not every conglomerably coherent lower prevision is a marginal extension, or, in other words, that we do not necessarily have the equality $\underline{P}=\underline{M}$.

Next, we investigate the properties of the sequence of marginal extensions $(\underline{M}_n)_n$ associated to $(\underline{E}_n)_n$, where $\underline{M}_n:=\underline{E}_{n-1}(\underline{E}_{n-1}(\cdot|\mathcal{B}))$ for every $n>1$ and $\underline{M}_1:=\underline{P}(\underline{P}(\cdot|\mathcal{B}))$. It follows from Proposition 4 that $\mathcal{M}(\underline{E}_n)=\mathcal{M}(\underline{E}_{n-1})\cap\mathcal{M}(\underline{M}_n)$, so $\underline{M}_n\leq\underline{E}_n$ for all n. Since the sequence $(\underline{E}_n(\cdot|\mathcal{B}))_n$ is also increasing, we deduce that so is the sequence $(\underline{M}_n)_n$. Thus, $(\underline{M}_n)_n$ is an increasing sequence of conglomerable and coherent lower previsions that is dominated by $\underline{F}$, the conglomerable natural extension of $\underline{P}$. Moreover, if $\underline{E}_n$ is not conglomerable, then it cannot be $\underline{M}_n\geq\underline{P}$, because then it would be $\underline{M}_n=\underline{F}$, and therefore also $\underline{E}_n=\underline{F}$ would be conglomerable.

However, it may be that the conglomerable natural extension is not a marginal extension model, and therefore that the increasing sequence of marginal extensions stabilises on a model that is not the conglomerable natural extension, as the following example shows.

Example 5. Consider $\Omega := \mathbb{N} \cup -\mathbb{N}$, $B_n := \{n, -n\}$ and $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$. Let P be a finitely additive probability on $\mathcal{P}(\mathbb{N})$ s.t. $P(\{n\}) = 0$ for all n, and P_1 a σ-additive probability on $\mathcal{P}(\Omega)$ s.t. $P_1(\{n\}) = P_1(\{-n\}) = \frac{1}{2^{n+1}}$ for all n. Consider also the linear previsions

$$\begin{aligned} P_2(h) &:= \frac{1}{2}\sum_n h(n)\frac{1}{2^n} + \frac{1}{2}P(h^-) \\ P_3(h) &:= \frac{3}{4}P(h^+) + \frac{1}{4}P(h^-) \\ P_4(h) &:= \frac{1}{2}P_1(h) + \frac{1}{2}P_3(h), \end{aligned}$$

where $h \in \mathcal{L}$ and h^+, h^- are derived by Eq. (5). Finally, let $\underline{P} := \min\{P_1, P_2, P_4\}$. Given $f := \mathbb{I}_{-\mathbb{N}}$, it holds that: $\underline{P}(f) = \min\left\{\frac{1}{2}, \frac{1}{2}, \frac{3}{8}\right\} = \frac{3}{8}$. In [5, Example 5] it is showed that the unconditional natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ is given by

$$\underline{E}_1 = \min\left\{P_1, P_4, \frac{1}{3}P_2 + \frac{2}{3}P_4\right\},$$

that the conditional natural extension of $\underline{E}$ is given by

$$\underline{E}_1(h|B_n) = \min\left\{\frac{h(n) + h(-n)}{2}, \frac{2h(n) + h(-n)}{3}\right\},$$

and that $P_4(G_{\underline{E}}(h|\mathcal{B})) < 0$ for some h, so $\underline{E}_1$ is not conglomerable.

On the other hand, it can be showed that both $P_1(G_{\underline{E}_1}(\cdot|\mathcal{B})) \geq 0$ and $P_5(G_{\underline{E}_1}(\cdot|\mathcal{B})) \geq 0$. It follows from Proposition 4 that the unconditional natural extension $\underline{E}_2$ of $\underline{E}_1, \underline{E}_1(\cdot|\mathcal{B})$ is dominated by the lower envelope of $\{P_1, P_5\}$, from which we obtain that $\underline{E}_2(\cdot|B_n) \leq \min\{P_1(\cdot|B_n), P_5(\cdot|B_n)\}$ and in particular that $\underline{E}_2(h|B_n)$ is dominated by

$$\min\left\{\frac{h(n) + h(-n)}{2}, \frac{2h(n) + h(-n)}{3}\right\} = \underline{E}_1(h|B_n)$$

for every $h \in \mathcal{L}$ and every $n \in \mathbb{N}$, which implies that $\underline{E}_2(h|B_n) = \underline{E}_1(h|B_n)$ for every gamble h. Applying [5, Proposition 16], we deduce that $\underline{E}_2$ is conglomerable and therefore it is the conglomerable natural extension of $\underline{P}$.

Now, if we reconsider $f := \mathbb{I}_{-\mathbb{N}}$, then $\underline{E}_2(f|B_n) = \frac{1}{3}$ for all n, so if $\underline{E}_2$ was a marginal extension model, we would have $\underline{E}_2(f) = \underline{E}_2(\underline{E}_2(f|\mathcal{B})) = \underline{E}_2(\frac{1}{3}) = \frac{1}{3}$. But we know that $\underline{E}_2(f) \geq \underline{P}(f) = \frac{3}{8} > \frac{1}{3}$. This shows that the sequence of marginal extensions may not stabilise on the conglomerable natural extension. ♦

Let us study in more detail the sequence $(\underline{M}_n)_n$ of marginal extensions. We begin by characterising their relationship with $\underline{Q}$ in terms of credal sets.

Proposition 7. *Let* $\underline{Q} := \lim_n \underline{E}_n$ *and let* $\underline{Q}' := \lim_n \underline{Q}(\underline{E}_n(\cdot|\mathcal{B}))$. *Then* $\mathcal{M}(\underline{Q}') = \cap_n \mathcal{M}(\underline{M}_n)$, *whence:*

1. $\mathcal{M}(\underline{Q}) = \mathcal{M}(\underline{P}) \cap (\cap_n \mathcal{M}(\underline{M}_n)) = \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{Q}')$.
2. $\underline{Q}'$ *conglomerable* $\Leftrightarrow \underline{Q}' = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$.

Thus, the limit of the increasing sequence $(\underline{M}_n)_n$ is the coherent lower prevision $\underline{Q}' = \lim_n \underline{Q}(\underline{E}_n(\cdot|\mathcal{B}))$. Taking this into account, we can establish a sufficient condition for the conglomerable natural extension to be the limit of the sequence of marginal extensions:

Proposition 8. *Let* $\underline{Q}, \underline{Q}'$ *be given as in Proposition 7, and consider the following possibilities:*

(a) $\underline{Q}(\cdot|\mathcal{B})$ *is the uniform limit of* $(\underline{E}_n(\cdot|\mathcal{B}))_n$.

(b) $\underline{Q} = \underline{Q}' = \underline{F}$.

(c) $\underline{Q}'$ *is conglomerable.*

(d) $\underline{Q}$ *is conglomerable.*

(e) $\underline{Q} = \underline{F}$.

Then (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e) *and* (b) $\Rightarrow$ (c). *If in particular* $\underline{Q}' \geq \underline{P}$, *then:*

1. (b) $\Leftrightarrow$ (c) $\Leftrightarrow \underline{Q}' = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$.
2. (d) $\Leftrightarrow$ (e) $\Leftrightarrow \underline{Q} = \underline{Q}(\underline{Q}(\cdot|\mathcal{B}))$.
3. (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e).

7 The Finitary Case: Sufficient Conditions

As we have showed in Example 4, the sequence $(\underline{E}_n)_n$ of coherent lower previsions that provides a lower bound on the conglomerable natural extension may not stabilise in a finite number of steps. On the other hand, in Proposition 8 we have showed that a sufficient condition for $(\underline{E}_n)_n$ to converge towards the conglomerable natural extension is the uniform convergence of the sequence of conditional lower previsions. In this section, we give two sufficient conditions for this uniform convergence.

We focus on the case of an initial lower prevision $\underline{P}$ characterised by an associated credal set $\mathcal{M}(\underline{P})$ that contains *finitely many* extreme points. We call this a *finitary* model, or a finitary lower prevision.

In other words, we consider finitely many linear previsions $P_1, \ldots, P_k$ on $\mathcal{L}$ and let $\underline{P} := \min\{P_1, \ldots, P_k\}$. Then $\mathcal{M}(\underline{P}) = \{P_{\bar{\alpha}} : \bar{\alpha} \in \Delta\}$, where $\Delta := \{(\alpha_1, \ldots, \alpha_k) : \alpha_i \geq 0\ \forall i, \sum_{i=1}^k \alpha_i = 1\}$ is the $(k-1)$-dimensional simplex, and simplifying the notation by letting $P_{\bar{\alpha}} := \alpha_1 P_1 + \cdots + \alpha_k P_k$, with $\bar{\alpha} := (\alpha_1, \ldots, \alpha_k)$. We consider as usual a partition $\mathcal{B}$ of Ω and the sequence $(\underline{E}_n)_n$ of coherent lower previsions that we use to approximate the conglomerable natural extension $\underline{F}$ of $\underline{P}$ (provided that it exists), and $\underline{Q} = \lim_n \underline{E}_n$. We aim at giving sufficient conditions for $\underline{Q}$ to coincide with $\underline{F}$.

If there is $m \in \mathbb{N}$ such that $\underline{E}_m = \underline{E}_{m-1}$, then $\lim_n \underline{E}_n = \underline{E}_m = \underline{F}$ and in particular $\underline{Q} = \underline{F}$. Otherwise, if the sequence never stabilises, then $\underline{E}_n \lneq \underline{E}_{n+1}$ for all n, whence $\mathcal{M}(\underline{E}_n) \supsetneq \mathcal{M}(\underline{E}_{n+1})$. For each natural number n, we have that $\mathcal{M}(\underline{E}_n) = \{P_{\bar{\alpha}} : \bar{\alpha} \in \Delta_n\}$, where Δ_n is a closed and convex subset of Δ.

Hence $(\Delta_n)_n$ is a strictly decreasing sequence of closed and convex subsets of Δ; since Δ is a compact subset of $\mathbb{R}^k$, we deduce that $\lim_n \Delta_n =: \Delta'$ is a compact subset of Δ, that determines moreover $\underline{Q} = \lim_n \underline{E}_n$.

Next, we are going to use these sets to give a sufficient condition for the uniform convergence of the sequence of conditional natural extensions. One important issue here is that of the positivity of the lower probabilities of the conditioning events: as we have showed in (2), $\underline{Q}(f|B)$ can only be non-vacuous when $\underline{Q}(B) > 0$, and similarly for $\underline{E}_n$. Then it may be that $\underline{Q}(B) > 0$ for all B in $\mathcal{B}$ while for every n there is an infinity of B for which $\underline{E}_n(B) = 0$, thus preventing the uniform convergence. Our next result shows that for finitary models this is not an issue:

Lemma 9. *If $\underline{P} = \min\{P_1, \dots, P_k\}$, then there is some natural number n such that, for every $B \in \mathcal{B}$, $\underline{Q}(B) > 0 \Rightarrow \underline{E}_n(B) > 0$.*

Since the conglomerable natural extension of $\underline{P}$ coincides with that of $\underline{E}_n$ for every $n \in \mathbb{N}$, we are going to assume that $\underline{P}(B) > 0$ whenever $\underline{Q}(B) > 0$; otherwise, it suffices to start the sequence at the n for which the condition in Lemma 9 holds.

Let us give now two sufficient conditions for the uniform convergence of the sequence $(\underline{E}_n(\cdot|\mathcal{B}))_n$.

Theorem 10. *Under any of the following conditions:*

1. $\exists N > 0$ *s.t.* $\frac{\overline{P}(B)}{\underline{P}(B)} < N \; \forall B \in \mathcal{B}$,

2. $\exists \nu > 0$ *s.t.* $\min_{i=1}^k \alpha_i \geq \nu > 0 \; \forall \bar{\alpha} \in \Delta'$,

$\underline{Q}(f|\mathcal{B})$ is the uniform limit of $(\underline{E}_n(f|\mathcal{B}))_n \; \forall f \in \mathcal{L}$ and therefore $\underline{Q}$ is the conglomerable natural extension of $\underline{P}$.

It can be checked that neither of these sufficient conditions is necessary for the limit to be conglomerable.

Remark 1. The second of these sufficient conditions is particularly revealing in the binary case, where we consider the lower envelope of two linear previsions, $\underline{P} := \min\{P_1, P_2\}$. If we denote $P_\alpha := \alpha P_1 + (1-\alpha) P_2$, then we can identify each Δ_n with a subset of $[0, 1]$:

$$\begin{aligned} \mathcal{M}(\underline{P}) &: = \{P_\alpha : \alpha \in [0,1]\}, \\ \mathcal{M}(\underline{E}_n) &: = \{P_\alpha : \alpha \in [a_n, b_n]\} \text{ and} \\ \mathcal{M}(\underline{Q}) &: = \{P_\alpha : \alpha \in [a, b]\}, \end{aligned}$$

where $0 \leq a_n \leq b_n \leq 1$ for all n, and $(a_n)_n \uparrow a$, $(b_n)_n \downarrow b$. There are a number of possibilities:

- If $a = b = 1$, then $\underline{Q} = P_1$, so the conglomerable natural extension exists if and only if it coincides with $\underline{Q} = P_1$.

- If $a = b = 0$, then $\underline{Q} = P_2$, so the conglomerable natural extension exists if and only if it coincides with $\underline{Q} = P_2$.

- If $a, b \in (0, 1)$, then Theorem 10 implies that $(\underline{E}_n(f|\mathcal{B}))_n$ converges uniformly to $\underline{Q}(f|\mathcal{B})$, and as a consequence $\underline{Q}$ is the conglomerable natural extension of $\underline{P}$.

- If $a = 0$ and $b \in (0, 1)$, then we can deduce from Theorem 10 that $(P_{b_n}(f|\mathcal{B}))_n$ converges uniformly to $P_b(f|\mathcal{B})$ for every gamble f, and from this we deduce that $\underline{Q}$ is the conglomerable natural extension of $\underline{P}$. A similar result applies when $a \in (0, 1)$ and $b = 1$.

This means that if we consider a binary model $\underline{P} = \min\{P_1, P_2\}$ and that the conglomerable natural extension of $\underline{P}$ exists, then it necessarily coincides with $\underline{Q}$. ♦

8 Conclusions

The importance of the conglomerable natural extension can be appreciated when one realises that it is the analog, for a theory of probability based on conglomerability, of the deductive closure in logic. Unfortunately, this paper shows that such a closure is not finitary, in the sense that to compute the conglomerable natural extension $\underline{F}$ of a coherent lower prevision $\underline{P}$, one might have to create an infinite sequence $(\underline{E}_n)_n$ of distinct approximating coherent lower previsions.

Moreover, at the moment it is still an open problem whether the point-wise limit $\underline{Q}$ of such a sequence actually attains $\underline{F}$ in general. However, in the special case where $\underline{P}$ is the envelope of finitely many linear previsions, this paper gives sufficient conditions for $\underline{Q} = \underline{F}$ that seem to have quite broad applicability. This gives reasons to believe that $\underline{Q}$ will equal $\underline{F}$ in many cases of practical interest.

Yet, solving the mentioned problem in general seems to us the most important question, and a very difficult one too, that should be addressed by future research.

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References

[1] B. de Finetti. Sulla proprietà conglomerativa delle probabilità subordinate. *Rendiconti del Reale Instituto Lombardo*, 63:414–418, 1930.

[2] B. de Finetti. *Teoria delle Probabilità*. Einaudi, Turin, 1970.

[3] B. de Finetti. *Theory of Probability: A Critical Introductory Treatment*. John Wiley & Sons, Chichester, 1974–1975. English translation of [2], two volumes.

[4] E. Miranda and M. Zaffalon. Conglomerable coherence. *International Journal of Approximate Reasoning*, 2013. Accepted for publication.

[5] E. Miranda, M. Zaffalon, and G. de Cooman. Conglomerable natural extension. *International Journal of Approximate Reasoning*, 53(8):1200–1227, 2012.

[6] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

[7] P. M. Williams. Notes on conditional previsions. Technical report, School of Mathematical and Physical Science, University of Sussex, UK, 1975. Reprinted in [8].

[8] P. M. Williams. Notes on conditional previsions. *International Journal of Approximate Reasoning*, 44:366–383, 2007. Revised journal version of [7].

[9] M. Zaffalon and E. Miranda. Probability and time. *Artificial Intelligence*, 198:1–51, 2013.