Conformity and independence with coherent lower previsions

Enrique Miranda
University of Oviedo
Oviedo (Spain)
mirandaenrique@uniovi.es

Marco Zaffalon
IDSIA
Lugano (Switzerland)
zaffalon@idsia.ch

Abstract

We study the conformity of marginal unconditional and conditional models with a joint model under assumptions of epistemic irrelevance and independence, within Walley's theory of coherent lower previsions. By doing so, we make a link with a number of prominent models within this theory: the marginal extension, the irrelevant natural extension, the independent natural extension and the strong product.

Keywords. Coherent lower previsions, sets of desirable gambles, epistemic irrelevance, epistemic independence, marginal extension, strong product.

1 Introduction

The theory of coherent lower previsions was developed by Peter Walley [22], with some influence from earlier work by Peter Williams [23], as a generalisation to the imprecise case of the behavioural approach to probability championed by de Finetti [10]. One of its advantages is that it includes as particular cases most of the models of non-additive measures existing in the literature, such as Choquet capacities [3], belief functions [21] or possibility measures [12].

Coherent lower previsions can be used to express both unconditional and conditional information, and several coherent lower previsions can be used to build a joint model that puts together the assessments present in each of the underlying sources. This is usually done by means of the notion of *natural extension*, which in some cases can be combined with other structural assessments, such as independence or exchangeability [1, Chapter 3].

The conformity of some marginal lower previsions with a joint model is easy to understand (it means simply that the joint model produces this marginals when restricted to gambles that depend on one of the variables); however, the relationship with the conditional models is more problematic. This is due to two reasons: on the one hand, there are several ways in which we can consider that a number of conditional models are consistent with a joint model, as the different notions of coherence by Williams and Walley testify. In this paper, we are going to use Walley's theory of coherent lower previsions, which makes use of the notion of *conglomerability*. This is an assumption that is not considered in de Finetti and Williams' approaches, and that has been subject to some controversy.

On the other hand, even if we stick to Walley's approach (but also in the finite case, where conglomerability is not an issue), there are several ways in which we can derive conditional models from an unconditional ones, so it is not immediate how to tell which conditional assessments are the ones derived from the unconditional model.

Our choice in this paper is to consider the notion of conditional natural extension, which, according to Walley, provides the most conservative behavioural implications of the assessments present in the unconditional model.

Under this setting, we are going to define a notion of conformity of marginal and conditional assessments with the meaning of the existence of a joint that induces them with the procedures of marginalization and natural extension mentioned above. We shall consider three different scenarios: that where we start from two marginal models and make an assessment of epistemic irrelevance, that where we make an assessment of epistemic independence, and that where our starting point is a marginal and a conditional lower prevision. In each of these cases we shall show that the notion of conformity does not always hold, we shall give necessary and sufficient conditions for its existence, and determine the least conservative model satisfying this notion.

Interestingly, we shall prove that this so-called conforming natural extension coincides under some conditions with some well-known models within the theory of coherent lower previsions: the marginal extension, the irrelevant natural extension, and the independent natural extension. This has led us to deepen our study of independent models, by completing some recent work in [19, 25]. In particular, we study in detail two properties that we have recently linked with independent products, and more specifically with the strong product. We investigate to which extent they are satisfied by other independent products and also by the marginal extension. The properties are in some cases formulated in terms of sets of desirable gambles, which provide the behavioural interpretation underlying coherent lower previsions.

The paper is organized as follows: in Section 2, we introduce the basics of the theory of coherent lower previsions that we shall use in the rest of the paper. Our study begins in Section 3 with the definition of conformity for a marginal and a conditional model, when the latter is defined by means of an assessment of epistemic irrelevance. This is completed in Section 4 with a study of the relationship between conformity and independent products. Then in Section 5 we consider the general case of conformity of a marginal and a conditional lower prevision, where the latter need not satisfy the property of epistemic irrelevance. The paper ends in Section 6 with some additional comments and remarks.

2 Preliminaries

2.1 Coherent lower previsions

Let us give the basics of the theory of coherent lower previsions necessary to follow the remainder of this paper. An in-depth study with details on the behavioural interpretation of the following notions may be found in [22].

Consider a possibility space Ω. A *gamble* on Ω is a bounded real-valued function $f:\Omega\to\mathbb{R}$. The set of all gambles on Ω is denoted $\mathcal{L}(\Omega)$. In particular, we shall let $\mathcal{L}^+(\Omega) \coloneqq \{f\in\mathcal{L}(\Omega): f\geq 0, f\neq 0\}$. For any subset B of Ω, we use I_B to denote its indicator gamble, that takes the value 1 on the elements of B and 0 otherwise.

Definition 1. A coherent lower prevision on $\mathcal{L}(\Omega)$ is a function $\underline{P}:\mathcal{L}(\Omega)\to\mathbb{R}$ satisfying the following properties:

(C1) $\underline{P}(f)\geq \inf f$;

(C2) $\underline{P}(\lambda f)=\lambda\underline{P}(f)$;

(C3) $\underline{P}(f+g)\geq\underline{P}(f)+\underline{P}(g)$

for every $f,g\in\mathcal{L}(\Omega)$ and every $\lambda>0$.

One example of a coherent lower prevision is the vacuous prevision with respect to a subset B of Ω, given by $\underline{P}(f)\coloneqq \inf_{\omega\in B} f(\omega)$.

A coherent lower prevision satisfying (C3) with equality for every $f,g\in\mathcal{L}(\Omega)$ is called a *linear prevision*. Linear previsions can be used to characterise coherence: a lower prevision $\underline{P}$ is coherent if and only if it is the lower envelope of its associated *credal set* $\mathcal{M}(\underline{P})\coloneqq\{P \text{ linear prevision}: P(f)\geq\underline{P}(f)\ \forall f\}$, meaning that $\underline{P}(f)=\min\{P(f):P\in\mathcal{M}(\underline{P})\}$ for every gamble f.

Given a partition $\mathcal{B}$ of Ω, a gamble $f\in\mathcal{L}(\Omega)$ is called $\mathcal{B}$*-measurable* when it is constant on the elements of $\mathcal{B}$. A *separately coherent conditional lower prevision* is a map $\underline{P}(\cdot|\mathcal{B})$ such that for every $B\in\mathcal{B}$, $\underline{P}(\cdot|B)$ is a coherent lower prevision satisfying $\underline{P}(B|B)=1$, and where $\underline{P}(f|\mathcal{B})$ is the $\mathcal{B}$-measurable gamble given by $\underline{P}(f|\mathcal{B})=\sum_{B\in\mathcal{B}} I_B\underline{P}(f|B)$.

Definition 2. Given a coherent lower prevision $\underline{P}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, they are called (jointly) coherent when

(GBR) $\underline{P}(I_B(f-\underline{P}(f|B)))=0$;

(CNG) $\underline{P}(f-\underline{P}(f|\mathcal{B}))\geq 0$

for every gamble f and every $B\in\mathcal{B}$.

The first of these conditions is called the *Generalised Bayes rule*, and determines $\underline{P}(f|B)$ uniquely when $\underline{P}(B)>0$. The second is usually referred to as a *conglomerability* condition, and follows from the first when the partition $\mathcal{B}$ is finite.

In this paper, we will focus on the case where $\Omega\coloneqq\mathcal{X}_1\times\mathcal{X}_2$. By an abuse of notation, we shall use $\underline{P}(\cdot|\mathcal{X}_1)$ to refer to a lower prevision conditional on the partition $\{\{x_1\}\times\mathcal{X}_2\}$ of $\mathcal{X}_1\times\mathcal{X}_2$, and we shall say that a gamble is $\mathcal{X}_1$-measurable when it is measurable with respect to this partition. There is a one-to-one correspondence between $\mathcal{L}(\mathcal{X}_1)$ and the class of $\mathcal{X}_1$-measurable gambles (and also between $\mathcal{L}(\mathcal{X}_2)$ and the class of $\mathcal{X}_2$-measurable gambles); we shall use it throughout the paper to alleviate the notation.

In particular, we shall mention the notion of coherence of a joint lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}_1\times\mathcal{X}_2)$ with conditional lower previsions $\underline{P}(\cdot|\mathcal{X}_1),\underline{P}(\cdot|\mathcal{X}_2)$; for the purposes of this paper, we only need that it implies the coherence of $\underline{P}$ with each of $\underline{P}(\cdot|\mathcal{X}_1),\underline{P}(\cdot|\mathcal{X}_2)$. A more detailed account can be found in [22, Section 7.1].

2.2 Sets of desirable gambles

A more general model than coherent lower previsions are *coherent sets of desirable gambles*:

Definition 3. A subset $\mathcal{R}\subseteq\mathcal{L}(\Omega)$ is *coherent* when it is a convex cone that includes $\mathcal{L}^+(\Omega)$ and does not include 0.

If $\mathcal{R}$ is a coherent set of desirable gambles, then the lower prevision given by

$$\underline{P}(f)\coloneqq\sup\{\mu: f-\mu\in\mathcal{R}\} \tag{1}$$

is coherent. On the other hand, there are several coherent sets of desirable gambles that induce the same coherent lower prevision. The smallest such set is called the set of *strictly desirable gambles*, and it is given by $\underline{\mathcal{R}}\coloneqq\mathcal{L}^+(\Omega)\cup\{f:\underline{P}(f)>0\}$. On the other hand, the closure of any of these sets in the topology of uniform convergence is given by $\overline{\mathcal{R}} := \{f : \underline{P}(f) \geq 0\}$, and this is called the set of *almost-desirable gambles* associated with $\underline{P}$.

Similarly, a coherent set of desirable gambles can also be used to define a separately coherent conditional lower prevision, by means of the formula

$$\underline{P}(f|B) := \sup\{\mu : I_B(f-\mu) \in \mathcal{R}\} \tag{2}$$

for every gamble f and every conditioning event B.

3 Irrelevant products

Consider two possibility spaces $\mathcal{X}_1, \mathcal{X}_2$ and let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$. Its marginal lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are defined on $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$ as the restriction of $\underline{P}$ to $\mathcal{X}_1$- and $\mathcal{X}_2$-measurable gambles, respectively.

The conditional information encompassed by $\underline{P}$ can be determined by many different updating rules (see for instance [22, Chapter 6] or [16]). In this paper we are using the updating rule determined by the natural extension:

Definition 4. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$. For any $x_1 \in \mathcal{X}_1$, the *conditional natural extension* $\underline{E}(\cdot|x_1)$ is defined as

$$\underline{E}(f|x_1) := \begin{cases} \sup\{\mu : \underline{P}(I_{x_1}(f-\mu)) \geq 0\} & \text{if } \underline{P}(x_1) > 0 \\ \inf_{x \in \mathcal{X}_2} f(x_1, x) & \text{otherwise.} \end{cases} \tag{3}$$

Recall that when $\underline{P}(x_1) > 0$ the conditional lower prevision $\underline{E}(\cdot|x_1)$ is uniquely determined by (GBR). In general, $\underline{P}, \underline{E}(\cdot|\mathcal{X}_1)$ need not be coherent; when they are, $\underline{E}(\cdot|\mathcal{X}_1)$ is the smallest, or least committal, conditional lower prevision that is jointly coherent with $\underline{P}$. This is equivalent to the *conglomerability* of $\underline{P}$, a notion discussed in much detail in [22, Section 6.8].

Definition 5. $\underline{P}$ is said to *model $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance* when its conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ satisfies epistemic irrelevance, meaning that $\underline{E}(f|x_1) = \underline{E}(f|x_1')$ for every $\mathcal{X}_2$-measurable f and every $x_1, x_1' \in \mathcal{X}_1$.

Note that given marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ on $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$, we can always make an assessment of irrelevance and obtain a conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$ on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ by means of the formula

$$\underline{P}(f|x_1) := \underline{P}_{\mathcal{X}_2}(f(x_1,\cdot)) \ \forall f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), \forall x_1 \in \mathcal{X}_1. \tag{4}$$

However, there may be no joint $\underline{P}$ modelling $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance and inducing some given $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$: the reason is that as soon as $\underline{P}_{\mathcal{X}_1}(x_1) = 0$ for some x_1 it follows from Eq. (3) that $\underline{P}(\cdot|x_1)$ should be vacuous, and then by irrelevance $\underline{P}_{\mathcal{X}_2}$ should be vacuous too.

When instead we can find such a $\underline{P}$, we say that $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are conforming with an $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevant model, or, more briefly, that they are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance:

Definition 6. We say that $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are *conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance* when there is a coherent lower prevision $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ and whose conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ satisfies Eq. (4).

We shall denote $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ the set of coherent lower previsions satisfying the conditions of the definition above for given $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$.

It is interesting to remark that $\underline{P}$ may be an $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevant model with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ while its conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ does not satisfy Eq. (4):

Example 1. Consider $\mathcal{X}_1 := \mathcal{X}_2 := \{0,1\}$, and let $\underline{P}$ be the lower envelope of the linear previsions $\{P_1, P_2\}$ associated with the mass functions $\mathcal{X}_1 \times \mathcal{X}_2 = \{(0,0,0.5,0.5),(0.5,0.5,0,0)\}$ on $\{(0,0),(0,1),(1,0),(1,1)\}$. Then $\underline{P}_{\mathcal{X}_1}(0) = \underline{P}_{\mathcal{X}_1}(1) = 0$, so $\underline{E}(\cdot|\mathcal{X}_1)$ is vacuous and as a consequence it satisfies $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance. However, the $\mathcal{X}_2$-marginal of $\underline{P}$ is the linear prevision given by $P_{\mathcal{X}_2}(f) = (f(0)+f(1))/2$ for every $f \in \mathcal{L}(\mathcal{X}_2)$. Thus, Eq. (4) is not satisfied. ◊

In other words, the conditional natural extension may satisfy an irrelevance condition with respect to an unconditional coherent lower prevision different from the marginal of $\underline{P}$. This is why we are explicitly requiring this to hold in Definition 6.

Note also that, given the marginal coherent lower prevision $\underline{P}_{\mathcal{X}_1}$ on $\mathcal{L}(\mathcal{X}_1)$ and the conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$ derived from $\underline{P}_{\mathcal{X}_2}$ by Eq. (4), Walley models their behavioural implications by means of the smallest lower prevision $\underline{E}$ that is coherent with them, and which in this case is given by the concatenation $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$ [22, Theorem 8.1.7]. However, the conditional natural extension of $\underline{E}$ may not agree with $\underline{P}(\cdot|\mathcal{X}_1)$, whence it is arguable that with $\underline{E}$ we encompass assessments different from the ones we started with. Indeed, the notion of conformity differs from that of coherence of conditional and unconditional lower previsions considered by Walley (although the latter follows from conformity in the finite case). Conformity can be characterised as follows:

Proposition 1. *Let $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ be two coherent lower previsions with respective domains $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$. Then $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})} \neq \emptyset$ if and only if either $\underline{P}_{\mathcal{X}_1}(x_1) > 0$ for every x_1 or $\underline{P}_{\mathcal{X}_2}$ is vacuous.*

Proof. Let us start with the direct implication. Assume that $\underline{P}$ is a coherent lower prevision with marginal $\underline{P}_{\mathcal{X}_1}$ and whose conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ coincides with the conditional lower prevision that the marginal $\underline{P}_{\mathcal{X}_2}$ induces by means of (4). If there is some $x_1 \in \mathcal{X}_1$ such that $\underline{P}_{\mathcal{X}_1}(x_1) = 0$, it follows from Eq. (3) that $\underline{E}(\cdot|x_1)$ must be vacuous, and from Eq. (4) that $\underline{P}_{\mathcal{X}_2}$ must then be the vacuous lower prevision.

Conversely, consider the coherent lower prevision

$\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$, where $\underline{P}(\cdot|\mathcal{X}_1)$ is induced from $\underline{P}_{\mathcal{X}_2}$ by Eq. (4). It follows by definition that its marginals are $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$. Thus, to see that it belongs to $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ under any of the conditions of the proposition statement, it suffices to show that in those cases $\underline{P}(\cdot|\mathcal{X}_1)$ coincides with the conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ of $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$.

By [22, Theorem 6.7.2], $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$ is coherent with $\underline{P}(\cdot|\mathcal{X}_1)$. In particular, this means that $\underline{E}(\cdot|\mathcal{X}_1)$ is dominated by $\underline{P}(\cdot|\mathcal{X}_1)$. If $\underline{P}_{\mathcal{X}_2}$ is vacuous, then so is $\underline{P}(\cdot|\mathcal{X}_1)$, and as a consequence it is equal to $\underline{E}(\cdot|\mathcal{X}_1)$. On the other hand, if $\underline{P}_{\mathcal{X}_1}(x_1) > 0$ for every $x_1 \in \mathcal{X}_1$, it follows from the coherence of $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$ and $\underline{P}(\cdot|\mathcal{X}_1)$ that the latter is equal to $\underline{E}(\cdot|\mathcal{X}_1)$, because this is the only conditional lower prevision that satisfies (GBR) with $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$. □

This shows that conformity is quite a stringent notion, because the assumption $\underline{P}_{\mathcal{X}_1}(x_1) > 0$ for every $x_1 \in \mathcal{X}_1$ can only hold when the space $\mathcal{X}_1$ is countable.

In other words, for most pairs of marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ there is no joint $\underline{P}$ whose conditional natural extension is the one that $\underline{P}_2$ induces by epistemic irrelevance. There are two reasons for this: one is the use of the natural extension as an updating rule; the other is that, as we have showed in Example 1, a $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevant model may induce a different conditional prevision than the one determined by its marginal and the notion of irrelevance.

Next, we are going to compare our definition above with another notion that has been considered in the literature [9]: it may be considered that $\underline{P}$ models irrelevance with respect to $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ when it is coherent (in the sense considered in Definition 2) with the conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$ given by Eq. (4) and has marginal $\underline{P}_{\mathcal{X}_1}$.

In order to make this comparison, we prove first of all that the set $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ is closed under lower envelopes:

Proposition 2. *The lower envelope $\underline{P}$ of a family $\{\underline{P}^{\lambda} : \lambda \in \Lambda\}$ of elements of $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ also belongs to $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$.*

Proof. It follows trivially that the marginals of $\underline{P}$ are $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$. Moreover, the conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ of $\underline{P}$ must be dominated by that of each $\underline{P}^{\lambda}$, that is, $\underline{P}(\cdot|\mathcal{X}_1)$. Now we apply Proposition 1.

If $\underline{P}_{\mathcal{X}_1}(x_1) > 0$ then $\underline{E}(f|x_1) = \sup\{\mu : \underline{P}(I_{x_1}(f-\mu)) \geq 0\} = \sup\{\mu : \underline{P}^{\lambda}(I_{x_1}(f-\mu)) \geq 0\ \forall \lambda \in \Lambda\} = \underline{P}(f|x_1)$: it suffices to note that for every $\mu < \underline{P}(f|x_1)$ we obtain that $\underline{P}(I_{x_1}(f-\mu)) = \inf_{\lambda\in\Lambda} \underline{P}^{\lambda}(I_{x_1}(f-\mu)) \geq 0$, whence $\underline{E}(f|x_1) \geq \mu$ and as a consequence $\underline{E}(f|x_1) \geq \underline{P}(f|x_1)$.

On the other hand, if $\underline{P}_{\mathcal{X}_2}$ is vacuous then so is $\underline{P}(\cdot|\mathcal{X}_1)$, and as a consequence it coincides with $\underline{E}(\cdot|\mathcal{X}_1)$. Applying Proposition 1, we conclude that $\underline{P}$ is an $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevant model. □

In this manner, we may define a *conforming natural extension*, and regard conformity as a structural assessment that can be made together with irrelevance. It models the implications of the assessments present in the marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ and our notion of conformity. In the finite case, it is easy to establish the following:

Proposition 3. *Consider marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ on $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$, and assume that $\mathcal{X}_1$ is finite. If $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})} \neq \emptyset$, then the smallest model in this set is $\underline{P}_{\mathcal{X}_1}(\underline{P}_{\mathcal{X}_2}) := \underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$, where $\underline{P}(\cdot|\mathcal{X}_1)$ is derived from $\underline{P}_{\mathcal{X}_2}$ by* (4).

Proof. When $\mathcal{X}_1$ is finite, any coherent lower prevision $\underline{P}$ is coherent with its conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$. Applying [22, Section 6.7.2], we deduce that any element of $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ must dominate the marginal extension $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$.

Moreover, from the proof of Proposition 1 we see that when there is an $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevant model that is conforming with $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$, then $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$ is one such irrelevant model. These two facts imply that $\underline{P}_{\mathcal{X}_1}(\underline{P}_{\mathcal{X}_2})$ is the smallest irrelevant model. □

We shall refer to $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$ as the *irrelevant natural extension* of $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$. What Proposition 3 shows is that this coincides with the conforming natural extension in the finite case, *whenever the latter exists*.

Note that when $\mathcal{X}_1$ is infinite the above result may not hold, as we see from the next example. The key is that in our definition of conformity with $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance, we are not requiring the joint model to be coherent with the conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$ that $\underline{P}_{\mathcal{X}_2}$ induces by means of Eq. (4):

Example 2. Let $\mathcal{X}_1 := \mathbb{N}, \mathcal{X}_2 := \{0,1\}$ and let P_1 be the linear prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ whose restriction to events is the σ-additive probability given by $P_1(n,0) = P_1(n,1) = \frac{1}{2^{n+1}}\ \forall n$. Let P_2 be a linear prevision whose restriction to events satisfies $P_2(\{2n : n \in \mathbb{N}\} \times \{0\}) = P_2(\{2n+1 : n \in \mathbb{N}\} \times \{1\}) = 0.5$ and $P_2(n,0) = P_2(n,1) = 0$ for every n. Let $P = 0.5(P_1 + P_2)$. Then $P(n) > 0$ for every $n \in \mathbb{N}$, and the conditional natural extension of P is given by $P(f|n) = 0.5f(n,0) + 0.5f(n,1)\ \ \forall f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$. This shows that P is an $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevant model that is conforming with its marginals $P_{\mathcal{X}_1}, P_{\mathcal{X}_2}$.

To see that it does not coincide with the marginal extension $P_{\mathcal{X}_1}(P_{\mathcal{X}_2})$, take $A = \{2n : n \in \mathbb{N}\} \times \{1\}$. Then $P(A) = \frac{1}{8} \neq P_{\mathcal{X}_1}(P(A|\mathcal{X}_1)) = 0.5P_{\mathcal{X}_1}(\{2n : n \in \mathbb{N}\}) = 0.25$. This means that P is not coherent with the conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$, because it does not satisfy the notion of conglomerability. ◊

We can immediately characterise conformity in the precise case when $\mathcal{X}_1$ is finite:

Proposition 4. *When $\mathcal{X}_1$ is finite, two linear previsions $P_{\mathcal{X}_1}, P_{\mathcal{X}_2}$ are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance if and only if $P_{\mathcal{X}_1}(x_1) > 0$ for every x_1. In that case, the only conforming model is given by $P_{\mathcal{X}_1}(P_{\mathcal{X}_2})$.*

4 Independent products

Similarly to the previous section, we say that $\underline{P}$ models $\mathcal{X}_1$-$\mathcal{X}_2$ independence when each of its conditional natural extensions satisfies epistemic irrelevance:

Definition 7. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ and let $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ denote its marginals on $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$. We say that $\underline{P}$ models $\mathcal{X}_1$-$\mathcal{X}_2$ *independence* when $\underline{E}(f|x_1) = \underline{E}(f|x_1')$ for every $\mathcal{X}_2$-measurable f and $x_1, x_1' \in \mathcal{X}_1$, and $\underline{E}(g|x_2) = \underline{E}(g|x_2')$ for every $\mathcal{X}_1$-measurable g and $x_2, x_2' \in \mathcal{X}_2$.

Similarly to what we did in the previous section, we may also study which pairs of marginals are conforming with a $\mathcal{X}_1$-$\mathcal{X}_2$ independent model, in the sense that its conditional natural extensions coincide with the conditionals $\underline{P}(\cdot|\mathcal{X}_1), \underline{P}(\cdot|\mathcal{X}_2)$ determined by $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ and the assumption of irrelevance. This produces the following definition:

Definition 8. Given two marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ on $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$ we say that they are *conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ independence* when there is a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ satisfying

$$\underline{E}(f|x_1) = \underline{P}_{\mathcal{X}_2}(f(x_1,\cdot)) \ \forall \mathcal{X}_2\text{-measurable } f$$
$$\underline{E}(f|x_2) = \underline{P}_{\mathcal{X}_1}(f(\cdot,x_2)) \ \forall \mathcal{X}_1\text{-measurable } f.$$

We shall denote $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ the set of coherent lower previsions satisfying the conditions of the definition above for given $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$.

From Proposition 1, we immediately derive the following:

Proposition 5. *Let $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ be two coherent lower previsions with respective domains $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$. Then $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})} \neq \emptyset$ if and only if (a) either $\underline{P}_{\mathcal{X}_1}(x_1) > 0$ for every x_1 or $\underline{P}_{\mathcal{X}_2}$ is vacuous; and (b) either $\underline{P}_{\mathcal{X}_2}(x_2) > 0$ for every x_2 or $\underline{P}_{\mathcal{X}_1}$ is vacuous.*

In particular, it follows that if $\mathcal{X}_1, \mathcal{X}_2$ are uncountable, then the only marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ that are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ independence are the vacuous ones.

When two marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ independence, there are in general many different coherent lower previsions in $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$. One example is given by the following family:

Definition 9. A coherent lower prevision $\underline{P}$ on $\mathcal{X}_1 \times \mathcal{X}_2$ is called an *independent product* of the marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ if and only if $\underline{P}, \underline{P}(\cdot|\mathcal{X}_1), \underline{P}(\cdot|\mathcal{X}_2)$ are coherent, where $\underline{P}(f|x_2) := \underline{P}_{\mathcal{X}_1}(f(\cdot,x_2))$ and $\underline{P}(f|x_1) := \underline{P}_{\mathcal{X}_2}(f(x_1,\cdot))$ for every $f \in \mathcal{X}_1 \times \mathcal{X}_2$.

Note, however, that independent products of given marginals always exist when $\mathcal{X}_1, \mathcal{X}_2$ are finite, whereas this is not the case for $\mathcal{X}_1$-$\mathcal{X}_2$ independent models, as shown by Proposition 5.

Proposition 6. *Assume that $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance (resp., independence). Then any independent product of $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ belongs to $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$.*

Proof. Let us establish the result for $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance; the proof for $\mathcal{X}_1$-$\mathcal{X}_2$ independence is analogous.

One the one hand, any independent product $\underline{P}$ is coherent with $\underline{P}(\cdot|\mathcal{X}_1)$ and has marginal $\underline{P}_{\mathcal{X}_1}$, so it suffices to show that it must induce $\underline{P}(\cdot|\mathcal{X}_1)$ by means of Eq. (3). If $\underline{P}_{\mathcal{X}_1}(x_1) > 0$ for every x_1, it follows from coherence that $\underline{P}(\cdot|x_1)$ coincides with $\underline{E}(\cdot|x_1)$; if $\underline{P}_{\mathcal{X}_1}(x_1) = 0$ for some x_1, then if $\underline{P}'$ is an $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevant conforming joint it must be $\underline{P}'(\cdot|x_1)$ vacuous, whence $\underline{P}_{\mathcal{X}_2}$ is the vacuous lower prevision. But then since $\underline{P}$ is coherent with the vacuous conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$, it follows that it must be $\underline{E}(\cdot|x_1)$ vacuous even if $\underline{P}(x_1) > 0$. Thus, $\underline{P}(\cdot|\mathcal{X}_1)$ agrees with the conditional natural extension of $\underline{P}$. □

Independent products were studied in [9] in the case when $\mathcal{X}_1, \mathcal{X}_2$ are finite and in [19] when they are infinite. In particular, in [19, Theorem 3] it is proved that, if two marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ have an independent product, the smallest one corresponds to the smallest coherent lower prevision that dominates the two concatenations $\underline{P}_{\mathcal{X}_1}(\underline{P}_{\mathcal{X}_2}), \underline{P}_{\mathcal{X}_2}(\underline{P}_{\mathcal{X}_1})$. This coherent lower prevision is called the *independent natural extension*, and it is denoted by $\underline{P}_{\mathcal{X}_1} \otimes \underline{P}_{\mathcal{X}_2}$. Two interesting surveys of the notion of independence within imprecise probabilities are [5, 7].

The result above, together with Proposition 3, allows us to establish the following:

Proposition 7. *Assume that $\mathcal{X}_1, \mathcal{X}_2$ are finite, and let $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ be two coherent lower previsions with respective domains $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$. If $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})} \neq \emptyset$ then the smallest element of this set is the independent natural extension $\underline{P}_{\mathcal{X}_1} \otimes \underline{P}_{\mathcal{X}_2}$.*

When both $\mathcal{X}_1, \mathcal{X}_2$ are infinite, independent products may not exist [19, Example 1]. Taking into account that this case is also problematic with respect to conformity, that reduces in most cases just to the combination of the vacuous marginal coherent lower previsions, in the remainder of this section we shall assume that at least one of $\mathcal{X}_1, \mathcal{X}_2$ is finite.

One particular family of independent products are the lower envelopes of factorising linear previsions:

Definition 10. A coherent lower prevision $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ is called an *independent envelope* when there is some $\mathcal{M} \subseteq \text{ext}(\mathcal{M}(\underline{P}_{\mathcal{X}_1})) \times \text{ext}(\mathcal{M}(\underline{P}_{\mathcal{X}_2}))$ such that $\underline{P} = \min\{P : P \in \mathcal{M}\}$.

The smallest independent envelope corresponds to the case where $\mathcal{M} = \text{ext}(\mathcal{M}(\underline{P}_{\mathcal{X}_1})) \times \text{ext}(\mathcal{M}(\underline{P}_{\mathcal{X}_2}))$. It is called the *strong product* of $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$, and we shall denote it $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$. The strong product may not coincide with the independent natural extension [22, Section 9.3.4]; moreover, it is only guaranteed to exist when at least one of the possibility spaces is finite [19]: otherwise, the two products $P_1 \times P_2$ and $P_2 \times P_1$ of any marginal linear previsions may not coincide.

To see that the strong product is not the only independent envelope with given marginals, consider the following example:

Example 3. Consider $\mathcal{X}_1 := \{\omega_1, \omega_2\}, \mathcal{X}_2 := \{x_1, x_2\}$ and let $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ be determined by $\underline{P}_{\mathcal{X}_1}(\omega_1) := 0.4$, $\overline{P}_{\mathcal{X}_1}(\omega_1) := 0.5$, $\underline{P}_{\mathcal{X}_2}(x_1) := 0.4$, $\overline{P}_{\mathcal{X}_2}(x_1) := 0.5$. Let $\underline{P}$ be the coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ given by $\underline{P} := \min\{P_1, P_2\}$, where P_1, P_2 are associated with the mass functions $(0.25, 0.25, 0.25, 0.25)$ and $(0.16, 0.24, 0.24, 0.36)$ on $\{(\omega_1, x_1), (\omega_1, x_2), (\omega_2, x_1), (\omega_2, x_2)\}$, respectively.

It follows from [22, Section 9.3.4] that $\underline{P}$ dominates the strong product of $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$. Moreover, it is the lower envelope of two factorising previsions, and as a consequence [9, Section 4.3] it is an independent product of its marginals, that coincide with $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$. To see that it does not coincide with the strong product $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$, note that $\underline{P}((\omega_1, x_2)) = 0.24 > 0.2 = \underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}((\omega_1, x_2))$. ◊

A similar example (involving zero lower probabilities) can be found in [9, Example 3].

The independent natural extension and the strong product are the two most important independent products in the literature of imprecise probabilities: the first one, because it corresponds to the most conservative product under the notion of *epistemic independence*; and the second because it is the one that models adequately the notion of *strong independence* [5]. Indeed, if we want to give a sensitivity analysis interpretation, we see that if $\underline{P}$ is a lower envelope of precise models P that are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ independence then Proposition 4 implies that $\underline{P}$ must be an independent envelope; and the smallest such envelope is given by the strong product.

In [25, Theorem 28], we established that when $\mathcal{X}_2$ is finite a coherent lower prevision $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ is dominated by the strong product if and only if it satisfies the following condition:

$$\underline{P}(f) \leq \underline{P}(P_{\mathcal{X}_2}(f|\mathcal{X}_1)) \ \forall f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), P_{\mathcal{X}_2} \geq \underline{P}_{\mathcal{X}_2}, \quad (5)$$

where $P_{\mathcal{X}_2}(\cdot|\mathcal{X}_1)$ is derived from $P_{\mathcal{X}_2}$ using Eq. (4).

In particular, this property is satisfied by the independent natural extension. However, it is not a sufficient condition for independence, as we show next:

Example 4. Consider $\mathcal{X}_1 := \{0, 1\} =: \mathcal{X}_2$, and let $\underline{P}$ be the coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ given by $\underline{P}(f) := \min\left\{\frac{f(0,0)+f(0,1)}{2}, \frac{f(1,0)+f(1,1)}{2}, \frac{f(0,0)+f(1,1)}{2}\right\}$. Then the marginals of $\underline{P}$ are $\underline{P}_{\mathcal{X}_1}(f) = \min\{f(0), f(1)\}, P_{\mathcal{X}_2}(g) = \frac{g(0)+g(1)}{2}$ for every $f \in \mathcal{L}(\mathcal{X}_1), g \in \mathcal{L}(\mathcal{X}_2)$. The strong product of these marginals is given by $\underline{P}_{\mathcal{X}_1} \boxtimes P_{\mathcal{X}_2}(f) = \min\left\{\frac{f(0,0)+f(0,1)}{2}, \frac{f(1,0)+f(1,1)}{2}\right\}$ for every $f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$. Moreover, it follows from [9, Proposition 25] that $\underline{P}_{\mathcal{X}_1} \boxtimes P_{\mathcal{X}_2}$ is the only independent product of these marginals. Since it dominates $\underline{P}$, we deduce from [25, Theorem 28] that $\underline{P}$ satisfies (5). However, they do not coincide, since $\underline{P}(\{(0,1),(1,0)\}) = 0 < 0.5 = (\underline{P}_{\mathcal{X}_1} \boxtimes P_{\mathcal{X}_2})((0,1),(1,0))$, and as a consequence $\underline{P}$ is not an independent product. ◊

Note also that in general being an independent product is not sufficient for condition (5), since there exist independent products that dominate the strong product; one instance is given in Example 3.

Next we discuss another condition that has been linked with independent products, as an attempt to give a behavioral interpretation of the strong product. For every $\omega \in \mathcal{X}_1$ and $f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$, let us define

$$\begin{aligned} f^{\omega} : \mathcal{X}_1 \times \mathcal{X}_2 &\to \mathbb{R} \\ (\omega', x) &\mapsto f(\omega, x). \end{aligned}$$

Proposition 8. *[25, Proposition 30] Let $\mathcal{X}_1, \mathcal{X}_2$ be finite spaces, and let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$. Then $\underline{P}$ is an independent envelope if and only if*

$$\underline{P}(g - f) \geq \min_{\omega \in \mathcal{X}_1} \underline{P}(g - f^{\omega}) \ \forall g, f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2). \quad (6)$$

The condition above can be regarded as a kind of dissociation between the marginal and conditional beliefs, and in the case of linear previsions it can be written more simply as $P(f) \leq \max_{\omega \in \mathcal{X}_1} P(f^{\omega}) = \max_{\omega \in \mathcal{X}_1} P_{\mathcal{X}_2}(f(\omega, \cdot)) \ \forall f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$. From Proposition 8 it follows that we can regard the strong product $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$ as the smallest coherent lower prevision that satisfies (6).

Note however, not every coherent lower prevision that dominates the strong product satisfies (6), as we can see from the following example:

Example 5. Consider the spaces $\mathcal{X}_1, \mathcal{X}_2$ and the marginal coherent lower previsions $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ from Example 3. By [22, Example 9.3.4] the strong product $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$ of $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ is given by $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2} = \min\{P_1, P_2, P_3, P_4\}$, where these are associated with the mass functions $(0.16, 0.24, 0.24, 0.36), (0.25, 0.25, 0.25, 0.25)$, $(0.2, 0.3, 0.2, 0.3)$ and $(0.2, 0.2, 0.3, 0.3)$ on $\{(\omega_1, x_1), (\omega_1, x_2), (\omega_2, x_1), (\omega_2, x_2)\}$, respectively.

Consider the coherent lower prevision $\underline{P} = \min\{P_1, P_2, 0.5P_3 + 0.5P_4\}$, that obviously dominates the strong product. To see that it does not satisfy (6), consider the gambles $g = (1,0,-9,-0.81), f = (0,0,1,-0.81)$, where the vector is made up of the images of $(\omega_1,x_1),(\omega_1,x_2),(\omega_2,x_1),(\omega_2,x_2)$, respectively. Then it holds that:

	(ω_1,x_1)	(ω_1,x_2)	(ω_2,x_1)	(ω_2,x_2)
$g-f$	1	0	-10	0
$g-f^{\omega_1}$	1	0	-9	-0.81
$g-f^{\omega_2}$	0	0.81	-10	0

and from this we derive that $\underline{P}(g-f) = -2.3, \underline{P}(g-f^{\omega_1}) = -2.293$ and $\underline{P}(g-f^{\omega_2}) = -2.2975$. Thus, we see that $\underline{P}(g-f) < \min_{\omega\in\mathcal{X}_1} \underline{P}(g-f^{\omega})$, so (6) does not hold. ◊

This is another way of showing that not every model dominating the strong product of its marginals is an independent envelope.

Condition (6) can be equivalently expressed in terms of the set of *almost-desirable* gambles associated with the coherent lower prevision $\underline{P}$. More generally, when $\underline{P}$ is induced by a set of desirable gambles $\mathcal{R}$, then the analogous condition would be given by

$$g - f^{\omega} \in \mathcal{R}\ \forall \omega \in \mathcal{X}_1 \Rightarrow g - f \in \mathcal{R}.$$

Note that this is not equivalent to Eq. (6), as we can tell from the fact that it is not always satisfied by the strong product:

Example 6. Let P be the uniform linear prevision on $\{\omega_1,\omega_2\}\times\{x_1,x_2\}$, which is trivially the strong (and only) product of its marginals. Consider the set of gambles $\mathcal{R} := \{f : P(f) > 0\} \cup \{f : P(f) = 0, f(\omega_1,x_2) + f(\omega_2,x_1) > 0\}$. It is easy to check that $\mathcal{R}$ is a coherent set of desirable gambles and that its associated coherent lower prevision coincides with P.

Consider now the gambles f, g given by:

	(ω_1,x_1)	(ω_1,x_2)	(ω_2,x_1)	(ω_2,x_2)
f	-1	3	3	-1
g	1	2	1	0.

Then the gambles $g-f^{\omega_1}, g-f^{\omega_2}$ and $g-f$ are given by:

	(ω_1,x_1)	(ω_1,x_2)	(ω_2,x_1)	(ω_2,x_2)
$g-f^{\omega_1}$	2	-1	2	-3
$g-f^{\omega_2}$	-2	3	-2	1
g-f	2	-1	-2	1

It follows that $g-f^{\omega_1}, g-f^{\omega_2} \in \mathcal{R}$ but $g-f$ does not. ◊

On the other hand, under some conditions the strong product satisfies an analogous condition for sets of strictly desirable gambles:

Proposition 9. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1\times\mathcal{X}_2)$, where $\mathcal{X}_1,\mathcal{X}_2$ are finite spaces, and let $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ denote the marginals of $\underline{P}$, respectively. If $\underline{P} = \underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$ and $\underline{P}_{\mathcal{X}_1}(\omega) > 0$ for every $\omega\in\mathcal{X}_1$, then it satisfies*

$$g - f^{\omega} \in \underline{\mathcal{R}}\ \forall\omega\in\mathcal{X}_1 \Rightarrow g - f \in \underline{\mathcal{R}}.$$

Proof. Consider gambles $f, g \in \mathcal{L}(\mathcal{X}_1\times\mathcal{X}_2)$ such that $g - f^{\omega} \in \underline{\mathcal{R}}$ for every $\omega\in\mathcal{X}_1$, and let us define $\mathcal{X}_1' := \{\omega\in\mathcal{X}_1 : g - f^{\omega} \ngeq 0\}$.

If $\mathcal{X}_1' = \emptyset$, then $g - f^{\omega} \geq 0$ for all ω, whence $I_{\omega}(g-f^{\omega}) \geq 0\ \forall\omega$ and therefore $g - f = \sum_{\omega} I_{\omega}(g-f^{\omega}) \geq 0$. But it cannot be $g = f$ because in that case the inequality $g - g^{\omega} \geq 0$ for every ω would imply that g is $\mathcal{X}$-measurable and then $g = g^{\omega}$ for all ω, a contradiction with the coherence of $\underline{\mathcal{R}}$.

Next, if $\mathcal{X}_1' \neq \emptyset$, then given $\omega\in\mathcal{X}_1'$ it must be $\underline{P}(g-f^{\omega}) > 0$, whence there is some $\delta_{\omega} > 0$ such that $\underline{P}(g-f^{\omega}) > \delta_{\omega} > 0$. This means that for every $P_{\mathcal{X}_1} \geq \underline{P}_{\mathcal{X}_1}$ and every $P_{\mathcal{X}_2} \geq \underline{P}_{\mathcal{X}_2}$ it holds that $(P_{\mathcal{X}_1}\times P_{\mathcal{X}_2})(g) \geq (P_{\mathcal{X}_1}\times P_{\mathcal{X}_2})(f^{\omega}) + \delta_{\omega} = P_{\mathcal{X}_2}(f(\omega,\cdot)) + \delta_{\omega}$. On the other hand, given $\omega\notin\mathcal{X}_1'$, $g - f^{\omega} \geq 0$, whence $(P_{\mathcal{X}_1}\times P_{\mathcal{X}_2})(g) \geq (P_{\mathcal{X}_1}\times P_{\mathcal{X}_2})(f^{\omega}) = P_{\mathcal{X}_2}(f(\omega,\cdot))$. We deduce that

$$\begin{aligned}(P_{\mathcal{X}_1}\times P_{\mathcal{X}_2})(f) &= \sum_{\omega\in\mathcal{X}_1, x\in\mathcal{X}_2} P_{\mathcal{X}_1}(\omega)P_{\mathcal{X}_2}(x)f(\omega,x)\\ &\leq \sum_{\omega\in\mathcal{X}_1'} P_{\mathcal{X}_1}(\omega)(P_{\mathcal{X}_1}\times P_{\mathcal{X}_2}(g) - \delta_{\omega})\\ &\quad + \sum_{\omega\notin\mathcal{X}_1'} P_{\mathcal{X}_1}(\omega)(P_{\mathcal{X}_1}\times P_{\mathcal{X}_2}(g))\\ &= (P_{\mathcal{X}_1}\times P_{\mathcal{X}_2})(g) - \sum_{\omega\in\mathcal{X}_1'} \delta_{\omega} P_{\mathcal{X}_1}(\omega),\end{aligned}$$

whence $(P_{\mathcal{X}_1}\times P_{\mathcal{X}_2})(g-f) \geq \sum_{\omega\in\mathcal{X}_1'} \delta_{\omega} P_{\mathcal{X}_1}(\omega) \geq \sum_{\omega\in\mathcal{X}_1'} \delta_{\omega} \underline{P}_{\mathcal{X}_1}(\omega)$; from this $\underline{P}(g-f) \geq \sum_{\omega\in\mathcal{X}_1'} \delta_{\omega}\underline{P}_{\mathcal{X}_1}(\omega) > 0$, and therefore $g - f \in \underline{\mathcal{R}}$. □

To see that this does not always hold without the assumption of positive lower probabilities in $\mathcal{X}_1$, consider the following example:

Example 7. Let $\mathcal{X}_1 := \{\omega_1,\omega_2\}, \mathcal{X} := \{x_1,x_2\}$ and let $P_{\mathcal{X}_1}, P_{\mathcal{X}_2}$ be the linear previsions on $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_2)$ associated with the mass functions $P_{\mathcal{X}_1}(\omega_1) = 1 = P_{\mathcal{X}_2}(x_1)$. Consider the gambles f, g given by

	(ω_1,x_1)	(ω_1,x_2)	(ω_2,x_1)	(ω_2,x_2)
g	1	2	2	2
f	1	1	0	4

We obtain

	(ω_1,x_1)	(ω_1,x_2)	(ω_2,x_1)	(ω_2,x_2)
$g-f^{\omega_1}$	0	1	1	1
$g-f^{\omega_2}$	1	-2	2	-2
$g-f$	0	1	2	-2

Then $g - f^{\omega_1}$ is strictly desirable because it is non-negative; $g - f^{\omega_2}$ is strictly desirable because $(P_{\mathcal{X}_1} \times P_{\mathcal{X}_2})(g - f^{\omega_2}) = (g - f^{\omega_2})(\omega_1, x_1) = 1 > 0$; and $g - f$ is not strictly desirable because $(P_{\mathcal{X}_1} \times P_{\mathcal{X}_2})(g - f) = 0$ and it is not a non-negative gamble. Hence, the product $P_{\mathcal{X}_1} \times P_{\mathcal{X}_2}$ does not satisfy

$$g - f^{\omega} \in \mathcal{R} \ \forall \omega \in \mathcal{X}_1 \Rightarrow g - f \in \mathcal{R}. \lozenge$$

5 Conformity of marginal and conditional models

In the previous sections we have studied to which extent the notion of conformity can be imposed together with a structural assessment of epistemic irrelevance or independence. Next we study in more detail the properties of conformity, without making any structural assumptions.

Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$, and let $\underline{P}(\cdot|\mathcal{X}_1)$ be its conditional natural extension. It follows from [22, Theorem 6.8.2] that $\underline{P}$ is coherent with $\underline{P}(\cdot|\mathcal{X}_1)$ if and only if it is $\mathcal{X}_1$-conglomerable, and that this condition holds trivially when $\mathcal{X}_1$ is finite.

Similarly to what we discussed in Section 3, given a coherent lower prevision $\underline{P}_{\mathcal{X}_1}$ on $\mathcal{L}(\mathcal{X}_1)$ and a conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$, we say that they are *conforming* when there is a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ with marginal $\underline{P}_{\mathcal{X}_1}$ and conditional natural extension $\underline{P}(\cdot|\mathcal{X}_1)$.

It is easy to see that not every marginal and conditional models are conforming with a joint model $\underline{P}$ in the manner depicted above: this follows from the fact that if $\underline{P}(x_1) = 0$ then the conditional lower prevision $\underline{P}(\cdot|x_1)$ determined by natural extension must be vacuous. In fact, with a similar proof to that of Proposition 1, it is possible to show the following:

Proposition 10. *Let $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$ be a marginal and a conditional lower prevision with respective domains $\mathcal{L}(\mathcal{X}_1), \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$. Then $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$ are conforming if and only if $\underline{P}(\cdot|x_1)$ is vacuous whenever $\underline{P}_{\mathcal{X}_1}(x_1) = 0$.*

Moreover, when $\underline{P}_{\mathcal{X}_1}$ and $\underline{P}(\cdot|\mathcal{X}_1)$ are conforming with some joint model, they they may be conforming with more than one. In the finite case, the smallest of these is determined by the notion of marginal extension.

Proposition 11. *Assume $\mathcal{X}_1$ is finite, and let $\underline{P}_{\mathcal{X}_1}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1)$ and $\underline{P}(\cdot|\mathcal{X}_1)$ a conditional lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$. If there is some $\underline{P}$ conforming with $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$, then the smallest such model is given by $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$.*

Proof. It follows from [22, Section 6.7.2] that $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$ is coherent with $\underline{P}(\cdot|\mathcal{X}_1)$ and has marginal $\underline{P}_{\mathcal{X}_1}$, so it only remains to show that it induces $\underline{P}(\cdot|\mathcal{X}_1)$ by means of Eq. (3).

If $\underline{P}_{\mathcal{X}_1}(x_1) = 0$, then by Proposition 10 $\underline{P}(\cdot|x_1)$ is vacuous, whence by Eq. (3) it coincides with the conditional natural extension $\underline{E}(\cdot|x_1)$ of $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$. On the other hand, if $\underline{P}(x_1) > 0$ then it follows from the coherence of $\underline{P}(\underline{P}(\cdot|\mathcal{X}_1))$ and $\underline{P}(\cdot|\mathcal{X}_1)$ that $\underline{P}(\cdot|x_1)$ is uniquely determined by $\underline{P}(f|x_1) := \sup\{\mu : \underline{P}(I_{\{x_1\} \times \mathcal{X}_2}(f - \mu)) \geq 0\}$, and so it coincides with $\underline{E}(\cdot|x_1)$.

Finally, note that if $\mathcal{X}_1$ is finite then any joint model $\underline{P}$ that is conforming with $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$ is in particular coherent with $\underline{P}(\cdot|\mathcal{X}_1)$, and as a consequence it must dominate the marginal extension $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$. $\square$

To see that the result may not hold when $\mathcal{X}_1$ is infinite, we refer to Example 2.

In the precise case, the conditional natural extension $\underline{P}(\cdot|\mathcal{X}_1)$ of a linear prevision is precise if and only if $P(x_1) > 0$ for every $x_1 \in \mathcal{X}_1$, and then $P = P_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$ if and only if P is $\mathcal{X}_1$-disintegrable [11] (which holds trivially when $\mathcal{X}_1$ is finite).

With respect to the two conditions we discussed in the previous section, in general a marginal extension may not satisfy Eq. (5). To see this, it suffices to consider a linear prevision that is not the product of its marginals, as in the following example:

Example 8. Consider $\mathcal{X}_1 := \{\omega_1, \omega_2\}, \mathcal{X}_2 := \{x_1, x_2\}$ and let P be the linear prevision associated with the mass function $P(\{\omega_1, x_2\}) := P(\{\omega_2, x_1\}) := P(\{\omega_2, x_2\}) := \frac{1}{3}$. Since it is a linear prevision then it is a marginal extension model.

Consider $f := -I_{\{(\omega_1, x_1), (\omega_2, x_2)\}}$. Then $P(f) = -\frac{1}{3}$. Since the marginal of P is given by $P_{\mathcal{X}_1}(\omega_1) = \frac{1}{3}, P_{\mathcal{X}_1}(\omega_2) = \frac{2}{3}$, we obtain $P(P_{\mathcal{X}_2}(f)) = P(\omega_1) \cdot (-\frac{1}{3}) + P(\omega_2) \cdot (-\frac{2}{3}) = -\frac{5}{9} < P(f). \lozenge$

In fact, for a linear prevision P on a finite space $\mathcal{X}_1 \times \mathcal{X}_2$ it can be checked that conditions (5) and (6) are each of them equivalent to P being the product of its marginals [25].

When just one of the marginals is linear, condition (6) can be used to characterise the equality between the marginal extension and the strong product, as we show next:

Proposition 12. *Consider finite spaces $\mathcal{X}_1, \mathcal{X}_2$, and let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$, with respective marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$. If $\underline{P}_{\mathcal{X}_2}$ is linear, then $\underline{P}$ satisfies* (6) *if and only if $\underline{P} = \underline{P}_{\mathcal{X}_1}(P_{\mathcal{X}_2}(\cdot|\mathcal{X}_1))$.*

Proof. By [9, Proposition 25], when $P_{\mathcal{X}_2}$ is linear there is only one independent product of $\underline{P}_{\mathcal{X}_1}, P_{\mathcal{X}_2}$: the marginal extension $\underline{P}_{\mathcal{X}_1}(P_{\mathcal{X}_2}(\cdot|\mathcal{X}_1))$, which coincides thus with the strong product.

Now, by Proposition 8, if $\underline{P}$ satisfies (6) then it is an independent envelope, and as a consequence it is an independent product. This means that $\underline{P}$ must agree with $\underline{P}_{\mathcal{X}_1}(P_{\mathcal{X}_2}(\cdot|\mathcal{X}_1))$. Conversely, if $\underline{P}$ is equal to $\underline{P}_{\mathcal{X}_1}(P_{\mathcal{X}_2}(\cdot|\mathcal{X}_1))$ then it also coincides with the strong product, and by Proposition 8 it satisfies Eq. (6). $\square$

On the other hand, if $P_{\mathcal{X}_2}$ is linear then Eq. (5) holds if and only if $\underline{P} \leq \underline{P}_{\mathcal{X}_1}(P_{\mathcal{X}_2}(\cdot|\mathcal{X}_1))$. This would mean that the marginal extension satisfies Eq. (5), although it may not be the only one to do so: for a counterexample, consider again the coherent lower prevision $\underline{P}$ from Example 4.

We conclude this section by giving a property of the marginal extension in terms of sets of desirable gambles:

Proposition 13. *Assume $\mathcal{X}_1$ is finite, and let $\mathcal{R}$ be a coherent set of gambles on $\mathcal{X}_1 \times \mathcal{X}_2$, and let $\underline{P}, \underline{P}(\cdot|\mathcal{X}_1)$ be the lower previsions it induces by means of Eqs.* (1), (2). *Then $\underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{X}_1))$, and they coincide only if $\mathcal{R}$ is negatively additive, meaning that*

$$(\forall \omega \in \mathcal{X}_1) I_{\{\omega\}} g \notin \mathcal{R} \Rightarrow g \notin \mathcal{R}. \tag{7}$$

Proof. The inequality $\underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{X}_1))$ holds because $\underline{P}$ is coherent with the conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$ (use for instance [18, Thm. 8]), and applying then [22, Thm. 6.7.2].

Assume that $\mathcal{R}$ is not negatively additive, so that Eq. (7) is violated for some gamble f. Then it follows that $\underline{P}(f) > \max_{\omega \in \mathcal{X}_1} \underline{P}(f|\omega)$, whence for every $P \geq \underline{P}$ it holds that

$$P(\underline{P}(f|\mathcal{X}_1)) \leq \max_{\omega \in \mathcal{X}_1} \underline{P}(f|\omega) < \underline{P}(f),$$

whence $\underline{P}(\underline{P}(f|\mathcal{X}_1)) \leq \overline{P}(\underline{P}(f|\mathcal{X}_1)) < \underline{P}(f)$. □

To see that negative additivity is not sufficient for $\underline{P}$ to be a marginal extension, consider the following example:

Example 9. Let $\mathcal{X}_1 := \{\omega_1, \omega_2\}, \mathcal{X}_2 := \{x_1, x_2\}$ and let $\mathcal{H} = \{A \subseteq \mathcal{X}_1 \times \mathcal{X}_2 : |A| = 2\}$. For any $A \in \mathcal{H}$, define P_A as $P_A(f) := (\sum_{z \in A} f(z))/2$, and let $\underline{P}$ be the lower envelope of the family $\{P_A : A \in \mathcal{H}\}$. Consider $\mathcal{R}$ its associated set of strictly desirable gambles. To see that it is negatively additive, note that its associated conditional lower prevision $\underline{P}(\cdot|\mathcal{X}_1)$ is vacuous and that $\underline{P}, \underline{P}(\cdot|\mathcal{X}_1)$ satisfy the condition

$$\underline{P}(f) \leq \max\{\underline{P}(f|\omega_1), \underline{P}(f|\omega_2)\} \ \forall f; \tag{8}$$

It is not difficult to show that Eq. (8) is equivalent to the negative additivity of $\mathcal{R}$.

However, if we consider the gamble f given by $f(\omega_1, x_1) = 1, f(\omega_1, x_2) = 2, f(\omega_2, x_1) = 3, f(\omega_2, x_2) = 4$, we obtain that

$$\underline{P}(f) = 1.5 > 1 = \underline{P}(\underline{P}(f|\mathcal{X}_1))$$

and as a consequence $\underline{P}$ is not a marginal extension. ◊

A slightly related result can be found in [24, Proposition 13]: it is shown there that negative additivity implies the notion of *temporal consistency* between a coherent set of gambles $\mathcal{R}$ and its associated set of conditional beliefs, when the set $\mathcal{R}$ is defined by means of marginal extension.

6 Conclusions

The results in this paper show that the notion of conformity clashes (except in the vacuous case) with the existence of zero lower probabilities, which appear quite often within the theory of imprecise probabilities, and which have been the object or quite some discussion (see for instance [22, Section 6.10], [4]); note moreover that within our framework we obtain zero lower probabilities as soon as the conditioning space is uncountable.

One possible alternative that may help to deal better with this issue would be to use a different updating rule, such as regular extension, that produces more informative inferences and that in particular does not imply that the marginal lower previsions are vacuous as soon as one element has lower probability zero. The study of conformity under this scenario is left as an open problem.

If we restrict our attention to finite spaces, then it is easy to see that a structural assessment of conformity gives rise to the marginal extension, and in case it is combined with assessments of epistemic irrelevance and independence it produces the notions of irrelevant and independent natural extension. This has led us to study in more detail the properties of independent products, and more particularly those that are lower envelopes of factorising linear previsions: the independent envelopes. This allows us to give our notion a sensitivity analysis interpretation that usually gets lost when we move from the unconditional to the conditional case. We have considered two properties that imply that a lower prevision is dominated, and dominates, the strong product, respectively, and have shown that under some conditions they are satisfied by the marginal extension, too.

There are several lines of research that we can derive from the results in this paper: on the one hand, we should study the conformity of more than two (marginal or conditional) models. This has been studied from the point of view of coherence in [17, Section 8.2], where it was shown that Walley's *weak coherence* is quite related to the works in [2, 13] and also to the notion of *satisfiability* [14, 15].

Note also that in our treatment of epistemic irrelevance and independence we have only considered conditional information on the singletons; it would be interesting to consider a more general setting where we condition on arbitrary subsets of the possibility spaces $\mathcal{X}_1, \mathcal{X}_2$.

On the other hand, it would also be interesting to make a similar study using the more general language of *sets of desirable gambles*, that may help overcome some of the issues related to conditioning on sets of (lower) probability zero. We expect that links with the irrelevant and the independent natural extension for sets of gambles considered in [6, 8, 20] should arise in this context.

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