FULL CONGLOMERABILITY

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ABSTRACT. We do a thorough mathematical study of the notion of full conglomerability, that is, conglomerability with respect to all the partitions of an infinite possibility space, in the sense considered by Peter Walley [1991]. We consider both the cases of precise and imprecise probability (sets of probabilities). We establish relations between conglomerability and countable additivity, continuity, super-additivity and marginal extension. Moreover, we discuss the special case where a model is conglomerable with respect to a subset of all the partitions, and try to sort out the different notions of conglomerability present in the literature. We conclude that countable additivity, which is routinely used to impose full conglomerability in the precise case, appears to be the most well-behaved way to do so in the imprecise case as well by taking envelopes of countably additive probabilities. Moreover, we characterise these envelopes by means of a number of necessary and sufficient conditions.

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1. INTRODUCTION

If you decide to work with probabilistic (or statistical) models in infinite spaces of possibilities, you should also decide whether or not requiring these models to be conglomerable.

Conglomerability of a probability P was first discussed by de Finetti [1930]. If we consider a partition $\mathcal{B}$ of the possibility space Ω such that $P(B) > 0$ for every $B \in \mathcal{B}$, conglomerability means that

$$(\forall A \subseteq \Omega) \inf_{B \in \mathcal{B}} P(A|B) \leq P(A) \leq \sup_{B \in \mathcal{B}} P(A|B). \tag{1.1}$$

This notion, in a slightly stronger form, was later studied by Dubins, with the name *disintegrability* [Dubins, 1974], and also by Schervisch et al. [1984, 2014], Seidenfeld et al. [1998] and by Armstrong and Prikry [1982], Armstrong [1990], amongst many others.

Conglomerability holds trivially when Ω is finite, as a consequence of the common axioms of probability. In the infinite case it does not, and therefore one has to decide

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whether or not to impose it. In fact, one could say that conglomerability is the essential difference between probability in the finite and infinite cases.

If you wonder why you have never heard of conglomerability or felt the need to take a stance about it, in spite of its peculiar role, that may be because there is a consolidated habit to work with countably additive probabilities: in fact, countable additivity with respect to Ω implies Eq. (1.1) *for all* countable partitions $\mathcal{B}$ of Ω; this is what we shall call *full conglomerability* later on. On the other hand, there seems to be little motivation to require countable additivity other than mathematical convenience or as a means to impose full conglomerability. Moreover, countable additivity with respect to a partition $\mathcal{B}$ implies the measurability with respect to this partition, and countable additivity with respect to all partitions means that our model is a discrete probability measure.[1] So it makes sense to look behind countable additivity and rather directly target the core notion of conglomerability.

Unlike in the case of countable additivity, imposing as well as checking conglomerability can be particularly difficult. Partly for this reason, there are different schools of thought about the previous question: those who reject that conglomerability should be a rationality requirement—among them looms the figure of de Finetti himself; and those who think it should be imposed, often in the light of the paradoxical situations that the lack of conglomerability may lead to. Among the latter stands Peter Walley, who has proposed a behavioural theory of *imprecise* probabilities, where the core modelling unit is a closed convex set of finitely additive probabilities [Walley, 1991]. This theory is essentially Peter Williams' earlier theory of imprecise probability [Williams, 1975] with an additional axiom of conglomerability for sets of probabilities, which coincides with Eq. (1.1) in the special case of precise probability.[2]

In [Zaffalon and Miranda, 2013], we have shown that conglomerability follows as a theorem whenever conditioning a probabilistic model is understood as a way to compute—or, more reasonably, constrain—future beliefs (which is somewhat improperly called *updating* beliefs). This is exactly the case for precise probability. With imprecise probability, the theorem needs an additional assumption; yet, this can be formulated in such a way that the theorem remains broadly applicable in the imprecise case as well (we shall detail this reformulation elsewhere).

If we take this result seriously, then we should try to embed conglomerability in probability in some kind of "practical" way. This is already the case in Walley's theory, but unfortunately only because conglomerability is treated approximately [Miranda et al., 2012].

[1] Under the assumptions of Ulam's theorem [Ulam, 1930].

[2] Walley's notion becomes equivalent to disintegrability when we also require that the conditional model is precise; see Section 4 for a detailed discussion.

Actual conglomerability has instead eluded simple treatments, rather pointing to some inherent complexity of such a notion; this becomes manifest in some recent work [Miranda and Zaffalon, 2013, 2015].

In this paper we analyse whether at least the notion of full conglomerability admits a simple enough treatment. The underlying idea is that full conglomerability requires a model to be much more regular than conglomerability, and for this reason it could make it easier to handle. After all, the same idea is what makes countable additivity, which is even stronger than full conglomerability [Walley, 1991, Section 6.9], mathematically convenient in precise probability.

To this end, we make a thorough mathematical study of the properties of full conglomerability and its relations to other notions.

One important remark that we must stress throughout is that the notion of conglomerability is not univocally defined in the literature, particularly in what concerns the precise case. In that context, there are quite a few works (see for instance Schervisch et al. [1984, 2014], Seidenfeld et al. [1998]) based on full conditional measures, where a compatibility requirement is imposed upon the conditional and the unconditional models; on the other hand, Walley's approach, which is the one we follow in this paper, regards conglomerability as a property of the unconditional model only, and this ultimately helps simplifying the treatment somewhat (for instance it allows us to be concerned with countable partitions only). In Section 3 we try to sort out the situation by examining and comparing the different proposals, after recalling some preliminary notions in Section 2. Next, we investigate the connections of full conglomerability with countable additivity, continuity and super-additivity both in the precise (Section 4) and the imprecise (Section 5) cases. Then Section 6 we show that full conglomerability can be characterised in terms of the supremum of a family of imprecise models defined by a generalised law of total probability (marginal extension); we study the properties of this functional and in particular also when we require conglomerability with respect to an arbitrary family of partitions, not necessarily all the possible ones. The paper concludes in Section 7 with our summary thoughts a posteriori and some discussion.

2. Preliminary notions

Let us introduce the elements of the theory of coherent lower previsions that we shall use in the remainder of the paper. We refer to Walley [1991] for more details.

2.1. **Coherent lower previsions.**

Consider a possibility space Ω. A *gamble* is a bounded map $f : \Omega \to \mathbb{R}$. Its *support* is given by $\operatorname{supp} f = \{\omega \in \Omega : f(\omega) \neq 0\}$. A gamble represents an uncertain reward that depends on the outcome of an experiment, so that we get the (possibly negative) reward $f(\omega)$ when the outcome of the experiment is the element $\omega \in \Omega$. One instance of gambles are the *indicator* gambles of sets $B \subseteq \Omega$, that take the value 1 on the elements of B, and 0 elsewhere; we shall denote these by I_B or B. We denote by $\mathcal{L}(\Omega)$ the space of all gambles on Ω, and by $\mathcal{L}^+(\Omega)$ the space of all non-negative non-zero ones (we call them *positive*).

A *coherent lower prevision* on $\mathcal{L}(\Omega)$ is a mapping $\underline{P} : \mathcal{L}(\Omega) \to \mathbb{R}$ satisfying the following conditions:

- $\underline{P}(f) \geq \inf f$;
- $\underline{P}(\lambda f) = \lambda \underline{P}(f)$;
- $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g)$

for every $f, g \in \mathcal{L}(\Omega)$ and every $\lambda > 0$. It follows from these conditions that a coherent lower prevision is always *constantly additive*, meaning that $\underline{P}(f + \mu) = \underline{P}(f) + \mu$ for any gamble f and any constant μ.

The lower prevision of a gamble f represents a subject's supremum acceptable buying price for this gamble, in the sense that for every $\mu < \underline{P}(f)$ he is disposed to accept the transaction $f - \mu$, which is equivalent to pay μ utiles in exchange for the uncertain reward given by f. The word *coherent* means that these supremum acceptable buying prices are consistent, in the sense that:

- A finite combination of acceptable transactions should not yield to a sure loss, irrespective to the outcome of the experiment: i.e., it should be impossible to make a Dutch book based on these prices.
- The lower prevision of a gamble f should be tight, in the sense that we should not be able to derive greater acceptable buying prices for f taking into account other acceptable transactions.

One instance of coherent lower prevision is the *vacuous* one, given by $\underline{P}(f) := \inf f$ for every $f \in \mathcal{L}(\Omega)$. It models the case that our subject has no information about the outcome of the experiment.

The conjugate function $\overline{P}$ of the coherent lower prevision $\underline{P}$, given by $\overline{P}(f) := -\underline{P}(-f)$ for every gamble f, is called a *coherent upper prevision*. It may be interpreted as the subject's infimum acceptable selling price for f, i.e., the infimum value of μ such that $\mu - f$ is an acceptable transaction to him. A coherent lower prevision on $\mathcal{L}(\Omega)$ satisfying $\underline{P}(f + g) = \underline{P}(f) + \underline{P}(g)$ for every pair of gambles f, g is called a *linear prevision*. In that case, for a gamble f the supremum acceptable buying price coincides with the infimum acceptable selling price, and this common value is called the *fair price* for f. Linear previsions were studied by de Finetti [1974–1975].

There is a one-to-one correspondence between coherent lower previsions and closed and convex sets of linear previsions: $\underline{P}$ is a coherent lower prevision on $\mathcal{L}(\Omega)$ if and only if

$$(\forall f \in \mathcal{L}(\Omega))\ \underline{P}(f) = \min\{P(f) : P \in \mathcal{M}(\underline{P})\},$$

where $\mathcal{M}(\underline{P}) \coloneqq \{P \text{ linear prevision} : P(f) \geq \underline{P}(f)\ \forall f\}$ is called the *credal set* associated with $\underline{P}$. This correspondence gives coherent lower previsions a sensitivity analysis interpretation.

We also say that a map $\underline{P} : \mathcal{L}(\Omega) \to \mathbb{R}$ *avoids sure loss* when it is dominated by some coherent lower prevision. The smallest such prevision is called its *natural extension*, and it coincides with the lower envelope of the non-empty set $\mathcal{M}(\underline{P})$. The same procedure allows to extend a lower prevision $\underline{P}_1$ from a domain $\mathcal{K} \subsetneq \mathcal{L}(\Omega)$ to $\mathcal{L}(\Omega)$: we only need to consider the lower prevision $\underline{P}'_1$ given by

$$\underline{P}'_1(f) \coloneqq \inf\{P(f) : (\forall g \in \mathcal{K})\ P(g) \geq \underline{P}_1(g)\}$$

for any gamble $f \in \mathcal{L}(\Omega)$. The natural extension determines the buying prices whose acceptability may be derived from those in $\underline{P}$ and the notion of coherence.

A linear prevision corresponds to the expectation operator with respect to its restriction to events, which is a finitely additive probability. When this restriction is moreover countably additive, meaning that $P(\cup_n B_n) = \sum_n P(B_n)$ for any countable family $(B_n)_n$ of pairwise disjoint events, we say that P is a *countably additive linear prevision*. On the other hand, a coherent lower prevision is not uniquely determined by the *coherent lower probability* that is its restriction to events: many different coherent lower previsions on $\mathcal{L}(\Omega)$ may have the same restriction to the power set $\mathcal{P}(\Omega)$.

Let us consider two examples that we shall use later on. The first are linear previsions whose restrictions to events are $\{0, 1\}$-valued: they are thus determined by the class of events $\mathcal{F} \coloneqq \{A \subseteq \Omega : P(A) = 1\}$. This class of events satisfies the following properties:

- $\emptyset \notin \mathcal{F}$,
- $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,
- $A \in \mathcal{F}, A \subseteq B \Rightarrow B \in \mathcal{F}$,
- $\forall A \subseteq \Omega$, either A or $A^c \in \mathcal{F}$,

and is therefore an *ultrafilter*. There are two types of ultrafilters: the *fixed* ones are those for which the intersection $\cap\{A : A \in \mathcal{F}\}$ is equal to some $\omega \in \Omega$. Their associated probability measure satisfies $P(\omega) = 1$, i.e., it corresponds to the degenerate probability measure on ω. On the other hand, the *free* ultrafilters are those where $\cap\{A : A \in \mathcal{F}\} = \emptyset$. Its associated probability measure on $\mathcal{P}(\Omega)$ satisfies then $P(\omega) = 0\ \forall\omega \in \Omega$, and as a consequence it will not be countably additive in general. See [Walley, 1991, Section 3.6].[3]

The second example is an instance of coherent *upper* previsions: the 2-*alternating* ones, which are those satisfying

$$(\forall f, g \in \mathcal{L}(\Omega))\ \overline{P}(f \wedge g) + \overline{P}(f \vee g) \leq \overline{P}(f) + \overline{P}(g),$$

where $\wedge$ denotes the point-wise minimum and $\vee$, the point-wise maximum. They correspond to the Choquet integral with respect to their restriction to events, which is a 2-alternating upper probability.

A coherent lower prevision is in a one-to-one correspondence with its associated set of *strictly desirable gambles* $\underline{\mathcal{R}} \coloneqq \{f : \underline{P}(f) > 0\} \cup \mathcal{L}^+(\Omega)$, in the sense that

$$(\forall f \in \mathcal{L}(\Omega))\ \underline{P}(f) = \sup\{\mu : f - \mu \in \underline{\mathcal{R}}\}; \tag{2.1}$$

the closure $\overline{\mathcal{R}}$ of the set of strictly desirable gambles in the topology of uniform convergence is called the set of *almost-desirable gambles*, and it satisfies $\overline{\mathcal{R}} = \{f : \underline{P}(f) \geq 0\}$.

2.2. **Conditional lower previsions.** The notion of coherence can also be extended to the conditional case. Let $\mathcal{B}$ be a partition of Ω. A *separately coherent* conditional lower prevision is a map $\underline{P}(\cdot|\mathcal{B})$ such that $\underline{P}(f|\mathcal{B})$ is defined as the gamble $\sum_{B\in\mathcal{B}} I_B \underline{P}(f|B)$, and where for every $B \in \mathcal{B}$ the functional $\underline{P}(\cdot|B) : \mathcal{L}(\Omega) \to \mathbb{R}$ satisfies the following conditions:

- $\underline{P}(f|B) \geq \inf_B f$;
- $\underline{P}(\lambda f|B) = \lambda \underline{P}(f|B)$;
- $\underline{P}(f + g|B) \geq \underline{P}(f|B) + \underline{P}(g|B)$

for every $f, g \in \mathcal{L}(\Omega)$ and every $\lambda > 0$. In the particular case where $\mathcal{B} = \{\Omega\}$ we recover the notion of (unconditional) coherence introduced at the beginning of Section 2.1. The interpretation of the conditional lower prevision on B is that the transaction is called off unless the outcome of the experiment belongs to B. In this paper, we are focusing on Walley's formalism, and as a consequence we shall assume that the set of conditioning events may be structured in a partition; this is not imposed in the approach by Williams [1975], which we shall also discuss later on in relation to conglomerability.

The behavioural interpretation of separately coherent conditional lower previsions is similar to the unconditional ones: they model the consistency of the buying prices that have been considered acceptable, now for transactions that are called off unless the conditioning event occurs.

[3] Fixed and free ultrafilters are sometimes called *principal* and *non-principal*, respectively.

The problem arises when we want to verify the consistency of several separately coherent conditional lower previsions that are conditional on different partitions (and in particular if we consider an unconditional and a conditional lower prevision, for instance). To this end, for any separately coherent lower prevision $\underline{P}(\cdot|\mathcal{B})$ and a gamble f, let $G(f|B) \coloneqq B(f - \underline{P}(f|B))$ and $G(f|\mathcal{B}) \coloneqq \sum_{B\in\mathcal{B}} G(f|B) = f - \underline{P}(f|\mathcal{B})$.

Now, consider separately coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_n)$ with domain $\mathcal{L}(\Omega)$. They are said to:

- *avoid partial loss* when $(\forall f_1, \dots, f_n \in \mathcal{L}(\Omega))(\exists B \in \cup S_i(f_i))$:
$$\sup_B \sum_{i=1}^n G(f_i|\mathcal{B}_i) \geq 0;$$
- be *coherent* when $(\forall f_0, f_1, \dots, f_n \in \mathcal{L}(\Omega), j \in \{1, \dots, n\}, B_j \in \mathcal{B}_j)(\exists B \in \cup S_i(f_i) \cup \{B_j\})$:
$$\sup_B \sum_{i=1}^n G(f_i|\mathcal{B}_i) - G(f_0|B_j) \geq 0;$$
- be *Williams coherent* when $(\forall f_0, f_1, \dots, f_n \in \mathcal{L}(\Omega), j \in \{1, \dots, n\}, B_j \in \mathcal{B}_j : (\forall i)\ S_i(f_i) \text{ finite })(\exists B \in \cup S_i(f_i) \cup \{B_j\})$:
$$\sup_B \sum_{i=1}^n G(f_i|\mathcal{B}_i) - G(f_0|B_j) \geq 0,$$

where $S_i(f_i) \coloneqq \{B_i \in \mathcal{B}_i : B_i f_i \neq 0\}$.

The two most important conditions here are that of coherence and of Williams coherence, which can both be given a behavioural interpretation similar to the one in the unconditional case. In particular, given a coherent lower prevision $\underline{P}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, they are (jointly) *coherent* when the following two conditions are satisfied:

JC1. $(\forall f \in \mathcal{L}(\Omega), B \in \mathcal{B})\ \underline{P}(G(f|B)) = 0$,

JC2. $(\forall f \in \mathcal{L}(\Omega))\ \underline{P}(G(f|\mathcal{B})) \geq 0$.

The coherence of the conditional lower previsions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_n)$ implies, but is not equivalent to, the existence of an unconditional lower prevision $\underline{P}$ that is pairwise coherent with each of them [Miranda and Zaffalon, 2013, Lemma 1].

The notion of natural extension can also be considered in the conditional case. Given a coherent lower prevision $\underline{P}$ and a partition $\mathcal{B}$ of Ω, its *conditional natural extension* $\underline{P}(\cdot|\mathcal{B})$ is given by

$$\underline{P}(f|B) \coloneqq \begin{cases} \inf_B f & \text{if } \underline{P}(B) = 0, \\ \sup\{\mu : \underline{P}(B(f-\mu)) \geq 0\} & \text{otherwise} \end{cases} \tag{2.2}$$

for any $f \in \mathcal{L}(\Omega)$. This conditional lower prevision always satisfies JC1 with $\underline{P}$, so $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent if and only if $\underline{P}(G(f|\mathcal{B})) \geq 0\ \forall f \in \mathcal{L}(\Omega)$. Moreover, if P is a linear prevision and $P(B) > 0$ for every $B \in \mathcal{B}$ (so that its conditional natural extension $P(\cdot|\mathcal{B})$) is linear), then P is coherent with $P(\cdot|\mathcal{B})$ if and only if P coincides with the concatenation $P(P(\cdot|\mathcal{B}))$. In general, condition JC2 follows from JC1 and the super-additivity of the coherent lower prevision $\underline{P}$ when the partition $\mathcal{B}$ is finite, and in particular when Ω is finite.

On the other hand, given $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_n)$ with domain $\mathcal{L}(\Omega)$ and avoiding partial loss, their *unconditional natural extension* is given by

$$\underline{E}(f) := \sup \left\{ \mu : (\exists g_1, \dots, g_n \in \mathcal{L}(\Omega)) \; f - \mu \geq \sum_{i=1}^{n} G(g_i|\mathcal{B}_i) \right\} \tag{2.3}$$

for any $f \in \mathcal{L}(\Omega)$.

2.3. **Shorthand notations.** In the following we shall often make use of the set of *positive* natural numbers $\mathbb{N}$. Yet, we shall hide the symbol $\mathbb{N}$ when its use is obvious, so as to avoid repeating the symbol over and over. For instance, sequences of gambles indexed by natural numbers will be denoted by $(f_n)_n$ rather than $(f_n)_{n\in\mathbb{N}}$. The notation will be analogous for sequences of lower previsions or of other objects. Similarly we shall write $\sum_n$ (or $\cup_n$, $\inf_n$, ...) to represent $\sum_{n\in\mathbb{N}}$ (or $\cup_{n\in\mathbb{N}}$, $\inf_{n\in\mathbb{N}}$, ...). Also in the case of limits of gambles (or of lower previsions), we shall use the shorthand $\lim_n f_n$ in the place of $\lim_{n\to\infty} f_n$.

3. DIFFERENT NOTIONS OF CONGLOMERABILITY IN THE LITERATURE

Although this paper places in the framework of Walley's theory of coherent lower prevision and we shall investigate the notion of conglomerability as considered in Walley [1991], the term has also appeared in the literature with a somewhat different meaning. In order to avoid confusions, we discuss here the different approaches and clarify the connections between them. A somewhat more detailed summary of the precise case may be found in [Petturiti and Vantaggi, 2017, Section 3].

3.1. **Traditional approaches to conglomerability for the precise case.** As we mentioned in the Introduction, conglomerability was first introduced by de Finetti [1930] in terms of Eq. (1.1). This property was also discussed by Lévy, as reported by Cantelli [1935]. The conditional probability $P(A|B)$ in that equation is derived from the unconditional one by Bayes' rule, so that $P(A|B) = P(A \cap B)/P(B)$, whenever $P(B) \neq 0$. However, as argued by de Finetti [1972, Chapter 5], it also makes sense to consider the conditional probability $P(A|B)$ when the event B has probability 0 but is not deemed impossible. In that case, he suggested to define what Dubins called a *full conditional probability* [Dubins, 1975, Section 3] as a functional $P : \mathcal{A} \times (\mathcal{A} \setminus \emptyset) \to [0, 1]$, where $\mathcal{A}$ is a field of subsets of Ω, satisfying the following conditions:

FC1. $(\forall B \in \mathcal{A} \setminus \emptyset)\ P(\cdot|B)$ is a probability measure on $\mathcal{A}$.

FC2. $(\forall B \in \mathcal{A} \setminus \emptyset) P(B|B) = 1$.

FC3. $(\forall A, B, C \in \mathcal{A} : B \cap C \in \mathcal{A} \setminus \emptyset)\ P(A \cap B|C) = P(B|C)P(A|B \cap C)$. [Conditional coherence]

Note that in particular we recover the product (Bayes') rule by considering $C = \Omega$ in the last condition.

There exists a connection between full conditional measures and the theory of coherent previsions: if we represent a full conditional measure on $\mathcal{P}(\Omega) \times (\mathcal{P}(\Omega) \setminus \emptyset)$ as a family of conditional and unconditional assessments $\{P(\cdot|B) : B \subseteq \Omega\}$, then the axioms FC1–FC3 above are equivalent to Williams coherence of any finite subset of conditional linear previsions [Williams, 2007, Proposition 6].

As a consequence, Dubins [1975] result guaranteeing that any linear prevision can be extended into a full conditional measure can also be understood as the possibility of deriving conditional linear previsions satisfying Williams coherence. However, these conditional previsions may violate the notion of conglomerability: as established in [Schervisch et al., 1984, Seidenfeld et al., 1998], if the linear prevision that results from restricting a full conditional measure to $\mathcal{P}(\Omega)$ is not countably additive, then there is some partition $\mathcal{B}$ of Ω where Eq. (1.1) is violated. In other words, the only full conditional measures that may satisfy the notion of full conglomerability are the countably additive ones.[4]

3.2. **Walley's approach to conglomerability.** Walley [1991, Section 6.8.1] calls a coherent lower prevision $\underline{P}$ *$\mathcal{B}$-conglomerable* when it satisfies the following condition:

$$(\forall B \in \mathcal{B} \text{ with } \underline{P}(B) > 0)\ \underline{P}(Bf) \geq 0 \Rightarrow \underline{P}\left(\sum_{\underline{P}(B)>0} Bf\right) \geq 0 \tag{3.1}$$

for any $f \in \mathcal{L}(\Omega)$.

In this equation, only those events from the partition with positive lower probability are taken into account. Since there are at most a countable number of such events, this means that only countable partitions matter when studying full conglomerability, taking also into account that in the case of finite partitions we can use super-additivity to verify Eq. (3.1). See also [Walley, 1991, Section 6.8.3].

Interestingly, a coherent lower prevision is $\mathcal{B}$-conglomerable if and only if there exists a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ such that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are jointly coherent, and if and only if $\underline{P}$ is coherent with its conditional natural extension, given by Eq. (2.2). Thus,

[4]But note that there can be countably additive ones that are not conglomerable; this may happen with uncountable partitions, as shown by Kadane et al. [1986, Appendix] and more recently by Seidenfeld et al. [2013, Example 2].

conglomerability means that the coherent lower prevision $\underline{P}$ can be updated in a coherent way to a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$. The notion can be applied in particular to linear previsions. However, in that case we may also require that the linear prevision can be updated into a *linear* model; this gives rise to a stronger notion, called *$\mathcal{B}$-disintegrability*. From [Walley, 1991, Theorem 6.5.7], the $\mathcal{B}$-disintegrability of a linear prevision is equivalent to the existence of a conditional linear prevision $P(\cdot|\mathcal{B})$ such that $P = P(P(\cdot|\mathcal{B}))$.

Similarly, given a family $\mathbb{B}$ of countable partitions of Ω, we say that $\underline{P}$ is $\mathbb{B}$-conglomerable when it is $\mathcal{B}$-conglomerable for every $\mathcal{B} \in \mathbb{B}$, and we say that $\underline{P}$ is *fully conglomerable* when it is $\mathcal{B}$-conglomerable for every countable partition $\mathcal{B}$ of Ω. In a similar manner, we say that a linear prevision P is *fully $\mathcal{B}$-disintegrable* when for every countable partition $\mathcal{B}$ there is some conditional linear prevision $P(\cdot|\mathcal{B})$ such that $P = P(P(\cdot|\mathcal{B}))$.

Now, if a lower prevision $\underline{P}$ is fully conglomerable, then we can define a family of conditional lower previsions $\mathcal{H} := \{\underline{P}(\cdot|\mathcal{B}) : \mathcal{B} \text{ partition of } \Omega\}$ with the property that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent for every countable partition $\mathcal{B}$. However, this does not guarantee that these conditional lower previsions are also coherent with each other: we only have what Walley called *weak coherence*, that does not prevent some inconsistencies from arising [Walley, 1991, Section 7.3.5]. Next we are going to show that the conditional lower previsions we can derive from a fully conglomerable lower prevision are indeed coherent with each other.

To see how this comes about, we are going to consider the stronger notion of *conglomerable coherence* studied in much detail in Miranda and Zaffalon [2013]. We say that a coherent lower prevision $\underline{P}$ is $\mathcal{B}$-conglomerably coherent when there is a $\mathcal{B}$-conglomerable set of gambles $\mathcal{R}$ inducing it by means of Eq. (2.1), where $\mathcal{R}$ is called $\mathcal{B}$-conglomerable when

$$(\forall B \in \mathcal{B})\ Bf \in \mathcal{R} \Rightarrow f \in \mathcal{R}. \tag{3.2}$$

Eq. (3.2) is called the *conglomerative principle* by Walley. It lies at the core of the disagreement between Walley and de Finetti: although coherence only guarantees that a finite sum of desirable transactions should again be a desirable transaction, Walley considers that an infinite sum of desirable gambles is again desirable when the gambles only act on different elements of the same partition. For this reason, the gamble $f - \underline{P}(f|\mathcal{B})$ is almost desirable for Walley, while it need not be so for de Finetti. If we do not accept the conglomerative principle then the notion of conditional coherence we should employ is Williams coherence.

We refer to [Miranda et al., 2012] for a comparison of the notions of conglomerability for lower previsions and for sets of desirable gambles. Although $\mathcal{B}$-conglomerable coherence is not equivalent in general to $\mathcal{B}$-conglomerability [Miranda and Zaffalon, 2013, Example 1], the differences between the two notions vanish when we consider conglomerability with respect to all partitions:

Proposition 1. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$. It is fully conglomerable if and only if there exists a fully conglomerable coherent set of desirable gambles $\mathcal{R}$ inducing it. In that case, for any finite set of partitions $\mathcal{B}_1, \dots, \mathcal{B}_n$ the conditional natural extensions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_n)$ of $\underline{P}$ are coherent.*

Proof. Assume first of all that $\underline{P}$ is fully conglomerable and let $\underline{\mathcal{R}}$ be its associated set of strictly desirable gambles. It follows from [Miranda et al., 2012, Theorem 3] that $\underline{\mathcal{R}}$ is $\mathcal{B}$-conglomerable for every partition $\mathcal{B}$, and as a consequence it is a fully conglomerable set of gambles that induces $\underline{P}$.

Conversely, if $\mathcal{R}$ is a fully conglomerable set of gambles that induces $\underline{P}$, then for every partition $\mathcal{B}$ of Ω it holds that $\mathcal{R}$ is $\mathcal{B}$-conglomerable and again by [Miranda et al., 2012, Theorem 3] we conclude that $\underline{P}$ is $\mathcal{B}$-conglomerable.

Finally, given partitions $\mathcal{B}_1, \dots, \mathcal{B}_n$ of Ω, it follows from [Miranda et al., 2012, Theorem 25(i)] that the set $\underline{\mathcal{R}}$ of strictly desirable gambles associated with $\underline{P}$ induces the conditional natural extensions $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ of $\underline{P}$. Since $\underline{\mathcal{R}}$ is $\mathcal{B}_i$-conglomerable for $i = 1, \dots, m$, [Miranda and Zaffalon, 2013, Theorem 8] implies that $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_m)$ are coherent. □

In the same manner that the natural extension of a lower prevision is the smallest dominating coherent lower prevision, given a partition $\mathcal{B}$ of Ω the *$\mathcal{B}$-conglomerable natural extension* of $\underline{P}$ is the smallest dominating coherent lower prevision that is $\mathcal{B}$-conglomerable and the *fully conglomerable natural extension* is the smallest dominating fully conglomerable coherent lower prevision. Note that in both cases the extension may not exist.

3.3. **Comparison between the two approaches.** One immediate difference between the traditional approaches to conglomerability and Walley's is that the former are only established for the precise case, while the latter is also valid for coherent lower previsions.[5] This means that a linear prevision that is conglomerable in the traditional sense should satisfy what we have called full disintegrability in this paper.

However, even when we stick to the precise case, the two approaches are not equivalent, due to the different treatment they give to the problem of conditioning on events of probability zero.

In fact, if we look at the particular case of partitions $\mathcal{B}$ of Ω such that $P(B) > 0$ for every $B \in \mathcal{B}$, then the conditional models $\{P(\cdot|B) : B \in \mathcal{B}\}$ follow from the unconditional

[5]But see Doria [2011, 2015] for an interesting approach in the imprecise case based on disintegrability.

one by means of Bayes' rule. If we consider the analogue of Eq. (1.1) in terms of gambles, which would be

$$(\forall f \in \mathcal{L}(\Omega)) \inf_{B \in \mathcal{B}} P(f|B) \leq P(f) \leq \sup_{B \in \mathcal{B}} P(f|B), \tag{3.3}$$

then Eq. (3.3) is equivalent to the following condition:

$$(\forall B \in \mathcal{B})\ P(Bf) \geq 0 \Rightarrow P(f) \geq 0,$$

which is precisely Walley's notion of conglomerability in Eq. (3.1). In other words, in the precise case, and for partitions with positive probability on all its elements, the traditional notion of conglomerability is equivalent to Walley's. This case was called *positive conglomerability* by Armstrong [1990].

In spite of this, any time the possibility space Ω is uncountable, we are bound to find partitions $\mathcal{B}$ where some of the conditioning events have zero probability, and in those cases the two approaches yield different results: in Walley's case, the notion of conglomerability for coherent lower previsions is established in terms of the unconditional model only; in particular, the elements of the partition with zero lower probability are not taken into account. This means for instance that a linear prevision whose restriction to events is $\{0, 1\}$-valued is always fully conglomerable in the sense of Walley. On the other hand, in the traditional approaches the conditional previsions should at least satisfy condition FC3, which eventually implies that full conglomerability will only hold when the unconditional prevision is countably additive.

The problem of conditioning on events of probability zero is one of the most important, both in the precise and in the imprecise case. Within the behavioural theory of imprecise probabilities, which is based on the notion of desirability, it can be overcome by means of conditioning on sets of desirable gambles [Walley, 1991, Section 3.7, Appendix F]. See Augustin et al. [2014, Chapter 2], Miranda and Zaffalon [2010], Moral [2005] for more information. Conditioning on sets of desirable gambles produces in general more informative models that those derived from lower previsions by the notion of coherence, and in general the definition of conglomerability is different (see Miranda et al. [2012] for a comparison). The approach based on desirable gambles is also relevant for full conditional measures, because these are related to Williams coherence and by [Miranda and Zaffalon, 2013, Williams, 1975] any finite family of conditional lower previsions that is Williams coherent can be obtained by conditioning a coherent set of desirable gambles.

Then, the key issue here in our view is that, both in the traditional approaches and in Walley's, a principle of conditional coherence is established. The principles are different, because Walley's makes use of the conglomerative principle and, e.g., de Finetti's does not (and this is what gives rise to the notion of Williams coherence). In Walley's case, it is shown that full conglomerability of a linear prevision is equivalent to the existence of a conditional model that is coherent with the unconditional one, but this conditional model may not be linear; in fact, there are cases of linear previsions that are coherent with some conditional lower prevision but not with any conditional linear prevision [Walley, 1991, Example 6.6.10]. And although this kind of models would be considered acceptable in Walley's sense, they would not be so in the traditional precise frameworks, for once imprecision enters the picture they cannot be represented as full conditional measures. In other words, one of the differences between the two approaches is that full conglomerability and full disintegrability are not equivalent for linear previsions.

In summary, Walley's approach to conglomerability can be seen as weaker/more general, than the traditional approaches, in two respects: the neglect of the events of the partition with zero (lower) probability, for which the traditional approaches adopt the axiom of conditional coherence FC3; and the focus on coherent *lower* and *upper* previsions, not necessarily linear.

Although conglomerability for full conditional measures has been quite extensively studied by Seidenfeld and colleagues, we think that full conglomerability under Walley's approach has not been investigated in all detail, other than the results from Walley [1991, Sections 6.8, 6.9]. For this reason, in the coming sections we shall investigate the connection of some families of fully conglomerable lower previsions with continuity, super-additivity and with concatenation models, both in the precise and the imprecise case.

4. Full conglomerability in the precise case

We begin by studying the notion of conglomerability for precise previsions. Let us stress, once again, that we are dealing with Walley's approach, where a linear prevision on $\mathcal{L}(\Omega)$ is fully conglomerable if and only if for every partition $\mathcal{B}$ of Ω there exists a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ such that $P, \underline{P}(\cdot|\mathcal{B})$ are (Walley-) coherent. Because of the features of Walley's notion of coherence discussed above, full conglomerability (in the sense of Walley) holds if and only if P is coherent with some conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, and this for any countable partition $\mathcal{B}$ of Ω.

To see the implications of this notion, we consider three properties:

M1. P is countably additive.
M2. P is fully disintegrable.
M3. P is fully conglomerable.

A countably additive linear prevision is always fully disintegrable, as shown by Walley [1991, Theorem 6.9.1]; on the other hand, it follows from its definition that a fully disintegrable linear prevision is in particular fully conglomerable. With respect to the converse implication, we shall consider two cases: linear previsions whose restrictions to events have a finite range (called *molecular* by Armstrong and Prikry [1982]) and those whose restrictions to events have infinite range (called *non-molecular* in Armstrong and Prikry [1982]).

All molecular linear previsions are fully conglomerable:

Proposition 2. *Let P be a molecular linear prevision on $\mathcal{L}(\Omega)$. Then for every partition $\mathcal{B}$ of Ω, $|\{B \in \mathcal{B} : P(B) > 0\}| < +\infty$, and as a consequence, P is fully conglomerable.*

Proof. Consider a partition $\mathcal{B}$ of Ω, and let $\alpha := \min\{P(A) : A \subseteq \Omega, P(A) > 0\}$. Then $\alpha > 0$ because the range of P is finite. Then $\{B \in \mathcal{B} : P(B) > 0\} = \{B \in \mathcal{B} : P(B) \geq \alpha\}$, and since the elements of $\mathcal{B}$ are pairwise disjoint we deduce that this class is finite. Using now the finite additivity of P, we conclude that it satisfies Eq. (3.1). □

On the other hand, there are molecular linear previsions that are fully conglomerable but not fully disintegrable [Walley, 1991, Example 6.6.10]. This shows that the implication M2 $\Rightarrow$ M3 is not an equivalence.

It was shown by Armstrong and Prikry [1982] that any molecular probability measure is a convex combination of $\{0, 1\}$-valued measures. This representation was used by Schervisch et al. [1984, Theorem 3.3] to show that any full conditional measure whose associated unconditional probability is molecular and not countably additive is not fully disintegrable. In other words, countable additivity and full disintegrability are equivalent in the molecular case provided we enter the framework of full conditional measures. But it follows from [Berti et al., 1991, Theorem 1.6] and [Berti and Rigo, 1992, Corollary 2.6] that a fully disintegrable probability on $\mathcal{P}(\Omega)$ can be represented as a full conditional measure with respect to the family of conditional previsions $P(\cdot|\mathcal{B})$ it satisfies disintegrability with. Then, applying Schervisch et al. [1984, Theorem 3.3] we deduce that P must be countably additive. See also [Petturiti and Vantaggi, 2017, Section 3.1].

Thus, in the case of molecular linear previsions full disintegrability in the sense of Walley is equivalent to countable additivity. With respect to non-molecular linear previsions, we have the following:

Proposition 3. *A non-molecular linear prevision is countably additive if and only if it is fully conglomerable. Moreover, if P is countably additive and the cardinality of $|\Omega|$ is smaller or equal than $\aleph_1$, then $P(\{\omega \in \Omega : P(\omega) > 0\}) = 1$.*

Proof. The equivalence has been established by Walley [1991, Theorem 6.9.2].

With respect to the second statement, it holds trivially when Ω is countable. Assume then that it has cardinality $\aleph_1$, and denote $A := \{\omega \in \Omega : P(\omega) > 0\}$. Take the partition $\mathcal{B} := \{A^c, \{\omega\} : \omega \in A\}$. If $P(A) < 1$, then we can define the functional $P' : \mathcal{L}(A^c) \to \mathbb{R}$ by $P'(f) := P(f'|A^c)$, where $f' \in \mathcal{L}(\Omega)$ denotes the gamble on Ω given by $f'(\omega) := f(\omega)$ if $\omega \in A^c$, and $f(\omega) = 0$ otherwise.

It follows from [Walley, 1991, Section 6.4.1] that P' is a linear prevision on $\mathcal{L}(A^c)$; moreover, by construction $P'(\omega) = 0$ for every $\omega \in A^c$. Since P is countably additive by assumption we deduce that P' is also countably additive. This means that the restriction of P' to events is a countably additive probability that gives zero probability to all the singletons. Since moreover, $|A^c| = \aleph_1$, we obtain a contradiction with Ulam's theorem Ulam [1930]. As a consequence, it must be $P(A) = 1$. □

The extension of the proof above to arbitrary cardinalities is not immediate, due to issues of *measurable cardinals*; see for instance [Bogachev, 2007, Section 1.12(x)].

Next we study the connection between full conglomerability and continuity. We consider the following continuity conditions:

C1. $(f_n)_n \to f \Rightarrow (\underline{P}(f_n))_n \to \underline{P}(f)$.
C2. $(f_n)_n \downarrow f \Rightarrow (\underline{P}(f_n))_n \downarrow \underline{P}(f)$.
C3. $(f_n)_n \downarrow 0 \Rightarrow (\underline{P}(f_n))_n \downarrow 0$.
C4. $(f_n)_n \uparrow f \Rightarrow (\underline{P}(f_n))_n \uparrow \underline{P}(f)$.

It is easy to prove that

$$\begin{array}{ccccc} \text{C1} & \Rightarrow & \text{C2} & \Rightarrow & \text{C3} \\ \Downarrow & & & & \\ \text{C4.} & & & & \end{array}$$

Countably additive linear previsions can be characterised by most of (but not all) the continuity conditions above:

Proposition 4. *Let P be a linear prevision on $\mathcal{L}(\Omega)$. The following are equivalent:*

- *P satisfies M1.*
- *P satisfies C2.*
- *P satisfies C3.*
- *P satisfies C4.*

Proof. The restriction to events of any linear prevision is a finitely additive probability, and it is well known that countable additivity is equivalent to the continuity for sequences of events $(A_n)_n \downarrow \emptyset$ and also for sequences $(A_n)_n \uparrow \Omega$. As a consequence, any of the conditions C2, C3, C4 implies M1. That the converse implication also holds follows from the monotone convergence theorem, taking into account that gambles are bounded and that they are trivially measurable with respect to the σ-field $\mathcal{P}(\Omega)$ where P is defined. $\square$

We deduce from this that a linear prevision satisfying condition C1 is always countably additive. To prove that the converse is not true, consider the following example:

Example 1. Consider $\Omega := \mathbb{N}$ and let P be the linear prevision whose restriction to events is the countably additive probability with mass function $P(\{n\}) := \frac{1}{2^n}$. The sequence of gambles $(f_n)_n$ given by $f_n := 2^n I_{\{n\}}$ converges point-wise to $f = 0$ but $P(f_n) = 1$ for every n. $\lozenge$

Next we focus on condition M3. It follows from Proposition 4 that any of the continuity conditions is sufficient for P to be fully conglomerable. Let us show that none of them (not even the weakest ones, C2 and C4) is necessary:

Example 2. First of all, note that if P is a linear prevision then conditions C2 and C4 are equivalent, because P is self-conjugate. Consider now $\Omega := \mathbb{N}$ and let $\mathcal{A}$ be a free ultrafilter of subsets of Ω. Let P be the linear prevision it induces, so that its restriction to events satisfies $P(A) = 1$ if $A \in \mathcal{A}$, and $P(A) = 0$ otherwise.

P is fully conglomerable, since for any partition $\mathcal{B}$ of Ω there is at most one $B \in \mathcal{B}$ with $P(B) > 0$. On the other hand, since $\mathcal{A}$ is free, $\{n\} \notin \mathcal{A}$ for any $n \in \Omega$. As a consequence, $A_n := \{m \in \Omega : m \geq n\} \in \mathcal{A}$ for every $n \in \mathbb{N}$, and $(A_n)_n \downarrow \emptyset \notin \mathcal{A}$. Thus, P does not satisfy C2. $\lozenge$

We conclude that when P is a linear prevision,

$$\text{C1} \Rightarrow \text{M1} \Leftrightarrow \text{C2} \Leftrightarrow \text{C3} \Leftrightarrow \text{C4} \Rightarrow \text{M2} \Rightarrow \text{M3},$$

and that moreover M1 is equivalent to M2 when the cardinality of Ω is $\aleph_1$.

5. FULL CONGLOMERABILITY IN THE IMPRECISE CASE

Next we consider fully conglomerable coherent lower previsions. We shall consider the following properties for a coherent lower prevision $\underline{P}$:

M4. $\underline{P}$ is the lower envelope of a family of countably additive linear previsions.

M5. $\underline{P}$ is the lower envelope of a family of fully conglomerable linear previsions.

M6. $\underline{P}$ is a fully conglomerable coherent lower prevision.

Note that conditions analogous to M4, M5 can be established for coherent upper previsions $\overline{P}$, considering upper envelopes.[6]

[6]Note also that condition M4 does not imply that any linear prevision in the credal set $\mathcal{M}(\underline{P})$ is countably additive, as this credal set shall also include in general linear previsions associated with finitely additive

It is immediate to prove that the following relations hold:

$$\begin{array}{ccccc} & & \text{M1} & & \\ & \swarrow & & \searrow & \\ \text{M3} & & & & \text{M4} \\ & \searrow & & \swarrow & \\ & & \text{M5} & & \\ & & \Downarrow & & \\ & & \text{M6.} & & \end{array}$$

However, the remaining implications do not hold: on the other hand, a linear prevision may be fully conglomerable without being countably additive, as we can see from Example 2; moreover, there are fully conglomerable coherent lower previsions that are not dominated by any fully conglomerable (and as consequence by any countably additive) linear prevision [Walley, 1991, Example 6.9.6].

With respect to envelopes of countably additive linear previsions, the following result was established by Krätschmer [2003]:

Proposition 5. *[Krätschmer, 2003, Section 5] Let $\overline{P}$ be a 2-alternating upper probability on $\mathcal{P}(\Omega)$. Then $\overline{P}$ is the upper envelope of a family of countably additive probabilities if and only if*

$$(\forall A \subseteq \Omega)\ \overline{P}(A) = \sup\{\overline{P}(B) : B \subseteq A, B \textit{ finite}\}. \tag{5.1}$$

More in general, we have proven the following characterisation of M4 when $\Omega = \mathbb{N}$:

Proposition 6. *Let $\overline{P}$ be a coherent upper prevision on $\mathcal{L}(\mathbb{N})$. The following are equivalent:*

(1) $\overline{P}$ *is an upper envelope of a family of countably additive linear previsions.*
(2) $(\forall n \in \mathbb{N})\ \overline{P} = \sup \mathcal{M}_n$, *where* $\mathcal{M}_n := \{P \leq \overline{P} : \lim_m P(\{1,\dots,m\}) \geq 1 - \frac{1}{n}\}$.
(3) $(\forall f)\ \overline{P}(f) = \lim_n \overline{P}(fI_{\{1,\dots,n\}}) \geq 0$.
(4) $(\forall f \geq 0)\ \overline{P}(f) = \sup\{\overline{P}(g) : g \leq f, \operatorname{supp}(g) \textit{ finite}\}$.

Proof. Since the class $\mathcal{M} = \{P \leq \overline{P} : P \text{ countably additive}\}$ is included in $\mathcal{M}_n$ for every natural number n, we deduce that the first statement implies the second. Conversely, consider a gamble $f \in \mathcal{L}(\mathbb{N})$. By constant additivity, we can assume without loss of generality that $f \geq 0$.

For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is some $P^n_\varepsilon \in \mathcal{M}_n$ such that $\overline{P}(f) - P^n_\varepsilon(f) < \varepsilon$. The mass function p^n_ε of P^n_ε can be regarded as an element of $[0,1]^{\mathbb{N}}$ such that $\sum_m p^n_\varepsilon(m) \geq 1 - \frac{1}{n}$. Since $[0,1]$ is a compact metric space, it is sequentially compact; and since a countable product of sequentially compact spaces is again sequentially compact [Joshi,

probabilities that are not countably additive. Moreover, the condition means that $\underline{P}(f) = \inf\{P(f) : P \geq \underline{P}, P \text{ countably additive}\}$ for every f, but this infimum need not be a minimum.

1983], we deduce that the space $[0,1]^{\mathbb{N}}$ is sequentially compact. Thus, the sequence $(p_\varepsilon^n)_n$ has a convergent subsequence $(p_\varepsilon^{n'})_{n'}$ to some $p \in [0,1]^{\mathbb{N}}$, and by construction $\sum_m p(m) \geq 1 - \frac{1}{n}$ for all n, whence $\sum_m p(m) = 1$. This means that p determines a countably additive linear prevision P.

For any fixed n, we can consider the linear functional P_ε^{n*} given by $P_\varepsilon^{n*}(g) := \sum_m g(m) p_\varepsilon^n(m)$ for all $g \in \mathcal{L}(\Omega)$. Then for any gamble $g \in \mathcal{L}(\Omega)$, it holds that

$$P_\varepsilon^{n*}(g) = \lim_m P_\varepsilon^{n*}(gI_{\{1,\dots,m\}}) \text{ and } P_\varepsilon^{n*}(gI_{\{1,\dots,m\}}) = P_\varepsilon^n(gI_{\{1,\dots,m\}}).$$

As a consequence, given $h \in \mathcal{L}^+(\Omega)$,

$$(\forall m \in \mathbb{N})\ P_\varepsilon^n(h) \geq P_\varepsilon^n(hI_{\{1,\dots,m\}}) = P_\varepsilon^{n*}(hI_{\{1,\dots,m\}}) \Rightarrow P_\varepsilon^n(h) \geq P_\varepsilon^{n*}(h).$$

Now there are two possibilities:

(a) If $P_\varepsilon^{n*}(\Omega) = 1$, then P_ε^{n*} is a linear prevision. Since $P_\varepsilon^{n*} \leq P_\varepsilon^n$ on non-negative gambles, it follows from constant additivity that $P_\varepsilon^{n*}(g) \leq P_\varepsilon^n(g)$ for every gamble g. But this is only possible if $P_\varepsilon^{n*} = P_\varepsilon^n$.

(b) If $P_\varepsilon^{n*}(\Omega) < 1$, then we can define the functional $Q_\varepsilon^n := \frac{P_\varepsilon^n - P_\varepsilon^{n*}}{(P_\varepsilon^n - P_\varepsilon^{n*})(\Omega)}$. This is a linear functional and it satisfies moreover $Q_\varepsilon^n(1) = 1, Q_\varepsilon^n(h) \geq 0$ for every $h \in \mathcal{L}^+(\Omega)$, whence by Walley [1991, Corollary 2.8.5] it is a linear prevision. Thus, for any gamble g on Ω, $Q_\varepsilon^n(g) \leq \sup g$, and therefore $P_\varepsilon^n(g) - P_\varepsilon^{n*}(g) \leq \frac{1}{n} \sup g$ for all $g \in \mathcal{L}(\Omega)$.

Now, for any gamble $h \in \mathcal{L}^+(\Omega)$,

$$\begin{aligned} P(h) = \sum_m h(m)p(m) &= \lim_{n'} \sum_m h(m) p_\varepsilon^{n'}(m) \\ &= \lim_{n'} P_\varepsilon^{n'*}(h) \leq \lim_{n'} P_\varepsilon^{n'}(h) \leq \lim_{n'} \overline{P}(h) = \overline{P}(h). \end{aligned}$$

Using constant additivity, we deduce that also $P(g) \leq \overline{P}(g)$ for any $g \in \mathcal{L}(\Omega)$. Thus, the countably additive linear prevision P is dominated by $\overline{P}$.

If we now consider the non-negative gamble f fixed at the beginning, we obtain that

$$\begin{aligned} \overline{P}(f) - P(f) = \lim_{n'} \overline{P}(f) - P_\varepsilon^{n'*}(f) &\leq \lim_{n'} \overline{P}(f) - P_\varepsilon^{n'}(f) + \frac{1}{n'} \sup f \\ &\leq \lim_{n'} \left(\varepsilon + \frac{1}{n'} \sup f \right) = \varepsilon, \end{aligned}$$

where the first inequality follows from points (a) and (b) above.

Thus, given $f \geq 0$ and $\varepsilon > 0$ there exists a countably additive linear prevision P dominated by $\overline{P}$ such that $\overline{P}(f) - P(f) \leq \varepsilon$. Since this can be done for any $\varepsilon > 0$, we conclude that $\overline{P}$ is the upper envelope of a family of countably additive linear previsions.

To prove that the second statement implies the third, consider a gamble $f \geq 0$ and fix $\varepsilon > 0$. Since the equality holds trivially when $f = 0$, let us assume that $\sup f > 0$. Then we can find some $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} < \frac{\varepsilon}{2\sup f}$. By assumption, for f, ε fixed there is some $n_2 \in \mathbb{N}$ and $P \in \mathcal{M}_{n_2}$ such that

$$\overline{P}(f) - P(f) < \frac{\varepsilon}{2}. \tag{5.2}$$

Moreover, since the sequence $(\mathcal{M}_n)_n$ is increasing, we may assume without loss of generality that $n_2 \geq n_1$.

On the other hand, $P \in \mathcal{M}_{n_2}$ implies that there is some finite set $\{1, \dots, m\}$ such that $P(\{1, \dots, m\}) \geq 1 - \frac{1}{n_2}$, so that $P(f) - P(fI_{\{1,\dots,m\}}) = P(fI_{\{1,\dots,m\}^c}) \leq P(\{1, \dots, m\}^c) \sup f \leq \frac{\sup f}{n_1} < \frac{\varepsilon}{2}$, whence

$$P(f) - \overline{P}(fI_{\{1,\dots,m\}}) < \frac{\varepsilon}{2}. \tag{5.3}$$

Adding up (5.2) and (5.3), we get $\overline{P}(f) - \overline{P}(fI_{\{1,\dots,m\}}) < \varepsilon$ and since we can do this for every $\varepsilon > 0$ we deduce that $\overline{P}(f) = \lim_n \overline{P}(fI_{\{1,\dots,n\}})$.

To prove the converse, assume ex-absurdo that there is a gamble f, a natural number n and some $\varepsilon > 0$, such that $\overline{P}(f) - \varepsilon > \sup_{P \in \mathcal{M}_n} P(f)$. Then we can assume without loss of generality that $\inf f = 1$, because of constant additivity. For every $P \in \mathcal{M}(\underline{P}) \setminus \mathcal{M}_n$, it holds that, $\forall m \in \mathbb{N}$,

$$P(f) - P(fI_{\{1,\dots,m\}}) = P(fI_{\{1,\dots,m\}^c}) \geq P(\inf fI_{\{1,\dots,m\}^c}) = P(\{1, \dots, m\}^c) \geq \frac{1}{n},$$

taking into account that $P \notin \mathcal{M}_n$. As a consequence,

$$\begin{aligned}(\forall m \in \mathbb{N})\ \overline{P}(fI_{\{1,\dots,m\}}) &= \max\{\sup_{P \in \mathcal{M}_n} P(fI_{\{1,\dots,m\}}), \sup_{P \in \mathcal{M}(\underline{P}) \setminus \mathcal{M}_n} P(fI_{\{1,\dots,m\}})\} \\ &\leq \max\left\{\sup_{P \in \mathcal{M}_n} P(f), \overline{P}(f) - \frac{1}{n}\right\} \leq \max\left\{\overline{P}(f) - \varepsilon, \overline{P}(f) - \frac{1}{n}\right\},\end{aligned}$$

whence $\lim_m \overline{P}(fI_{\{1,\dots,m\}}) \leq \max\{\overline{P}(f) - \varepsilon, \overline{P}(f) - \frac{1}{n}\} < \overline{P}(f)$. This contradicts the third statement.

Finally, to prove that the third and fourth statements are equivalent, note that given $f \geq 0$, any $g \leq f$ with finite support will be bounded by $fI_{\{1,\dots,n\}}$ for some n. As a consequence, $\sup\{\overline{P}(g) : g \leq f, \operatorname{supp}(g) \text{ finite}\} = \sup_n \overline{P}(fI_{\{1,\dots,n\}}) = \lim_n \overline{P}(fI_{\{1,\dots,n\}})$. □

This result provides a number of necessary and sufficient conditions for a coherent upper prevision to be the supremum of a family of countably additive linear previsions. Nevertheless, this does not imply that it is the maximum, or, in other words, the set of dominated countably additive linear previsions is not closed: if we let $\overline{P}$ be the vacuous upper prevision on $\mathbb{N}$, given by $\overline{P}(f) = \sup f$ for every f, it follows immediately that $\overline{P}$ is the upper envelope of a family of countably additive linear previsions: those associated with the degenerate probability measures. However, the gamble f given by $f(n) = 1 - \frac{1}{n}$ satisfies $\overline{P}(f) = 1$ while for any countably additive linear prevision P it holds that $P(f) < 1$.

Proposition 6 allows us to extend Krätschmer's Proposition 5 to the case of gambles:

Corollary 7. *Let $\overline{P}$ be a 2-alternating upper prevision on $\mathcal{L}(\mathbb{N})$. Then $\overline{P}$ is the upper envelope of a family of countably additive linear previsions if and only if its restriction to events satisfies Eq.* (5.1).

Proof. The direct implication follows applying Proposition 6(4) to indicators of events. To see the converse, consider a gamble $f \geq 0$. Since any gamble may be approximated as a uniform limit of simple gambles, given $\varepsilon > 0$ there is some simple gamble $0 \leq g \leq f$ such that $\overline{P}(f) - \overline{P}(g) < \frac{\varepsilon}{2}$. We can denote $g = \sum_{i=1}^{n} x_i I_{A_i}$ for $x_1 > x_2 > \cdots > x_n = 0, \{A_1, \ldots, A_n\}$ being a partition of $\mathbb{N}$. Since $\overline{P}$ is 2-alternating, it follows from Walley [1981] that it can be computed as the Choquet integral of its restriction to events, meaning that Denneberg [1994]

$$\overline{P}(g) = \sum_{i=1}^{n-1} (x_i - x_{i+1}) \overline{P}(A_1 \cup \cdots \cup A_i).$$

By Eq. (5.1), for every $i = 1, \ldots, n-1$ there exists a finite set $B_i \subseteq A_1 \cup \cdots \cup A_i$ such that $\overline{P}(A_1 \cup \cdots \cup A_i) - \overline{P}(B_i) < \frac{\varepsilon}{2nx_1}$, and we may assume without loss of generality that $B_i \subseteq B_{i+1}$ for $i = 1, \ldots, n-1$. Thus, the gamble $h := x_1 I_{B_1} + \sum_{i=2}^{n} x_i I_{B_i \setminus B_{i-1}}$ satisfies

$$\overline{P}(g) - \overline{P}(h) = \sum_{i=1}^{n-1} (x_i - x_{i+1})(\overline{P}(A_1 \cup \cdots \cup A_i) - \overline{P}(B_i)) < \frac{\varepsilon}{2},$$

whence $\overline{P}(f) - \overline{P}(h) < \varepsilon$. Since h has finite support and we can do this for every $\varepsilon > 0$, we deduce that condition (4) in Proposition 6 holds and as a consequence $\overline{P}$ is the upper envelope of a family of countably additive linear previsions. □

5.1. **Full conglomerability and continuity.** Next, we study the connections between the continuity properties C1–C4 considered in Section 4 and fully conglomerable coherent lower previsions. We begin by studying which of these continuity conditions is necessary for full conglomerability. In this respect, we deduce from Example 2 that none of them is necessary for $\underline{P}$ to satisfy M5, M6. With respect to condition M4, we can establish the following:

Proposition 8. *If $\underline{P}$ satisfies M4, then it also satisfies C2 (and as a consequence also C3) but not necessarily C4.*

Proof. Let $\mathcal{M}$ be a family of countably additive linear previsions such that $\underline{P} = \inf \mathcal{M}$, and consider a decreasing sequence of gambles $(f_n)_n$ that converges towards f. Then for any $\varepsilon > 0$ there is some $P \in \mathcal{M}$ such that $P(f) - \underline{P}(f) < \frac{\varepsilon}{2}$. Since by Proposition 4 $P(f) =$

$\lim_n P(f_n)$, there is some $n \in \mathbb{N}$ such that $P(f_n) - P(f) < \frac{\varepsilon}{2}$, whence $\underline{P}(f_n) - \underline{P}(f) < \varepsilon$. Therefore, $\underline{P}(f) = \lim_n \underline{P}(f_n)$ and condition C2 holds.

To prove that condition C4 may not hold, take $\Omega := \mathbb{N}$ and let $\underline{P}$ be the vacuous coherent lower prevision. For any $n \in \mathbb{N}$, let δ_n be the countably additive linear prevision associated with the degenerate probability measure on n. It is given by $\delta_n(f) := f(n)$ for any gamble f. Then $\underline{P}(f) = \inf f = \inf_{n\in\mathbb{N}} \delta_n(f)$, whence $\underline{P}$ satisfies M4. On the other hand, the sequence $(A_n)_n$ given by $A_n := \{1, \dots, n\}$ converges to Ω but $\lim_n \underline{P}(A_n) = 0$. Thus, $\underline{P}$ does not satisfy C4. □

Next we study which of the continuity conditions is sufficient for full conglomerability. With respect to M4, we have the following:

Proposition 9. *Let $\underline{P}$ be a coherent lower prevision. If it satisfies C4, then it is the lower envelope of a family of countably additive linear previsions.*

Proof. Consider a linear prevision $P \in \mathcal{M}(\underline{P})$. Then for any sequence $(A_n)_n \uparrow \Omega$ it holds that $\lim_n P(A_n) \geq \lim_n \underline{P}(A_n) = 1$. Thus, P satisfies C4, and applying Proposition 4 we deduce that it is countably additive. Therefore, $\underline{P}$ satisfies M4. □

As a consequence, condition C1 is also sufficient for $\underline{P}$ to satisfy M4 (and therefore also M5, M6). Note that a similar result has been established for coherent lower probabilities by Krätschmer [2003, Proposition 2.3].

Our next example shows that condition C3 is not sufficient for $\underline{P}$ to be fully conglomerable (and as a consequence it does not imply that $\underline{P}$ satisfies M4, M5 either). The key here is that a coherent lower prevision satisfies C3 as soon as one of the dominating linear previsions does:

Example 3. Consider $\Omega := \mathbb{N}$, and let P_1 be the countably additive linear prevision whose restriction to events satisfies $P_1(\{n\}) = \frac{1}{2^n}$ for every n. Consider now a linear prevision P_2 satisfying $P_2(\{2n\}) := P_2(\{2n-1\}) := \frac{1}{2^{n+2}}$, $P_2(\{2n : n \in \mathbb{N}\}) > \frac{1}{2}$. Let $\underline{P} := \min\{P_1, P_2\}$.

If we take $\mathcal{B} := \{\{2n, 2n-1\} : n \in \mathbb{N}\}$, and $f := I_{\{2n-1:n\in\mathbb{N}\}} + \varepsilon - I_{\{2n:n\in\mathbb{N}\}}$ for $0 < \varepsilon < P_2(\{2n : n \in \mathbb{N}\}) - P_2(\{2n-1 : n \in \mathbb{N}\})$, we obtain that $\underline{P}(B_n) > 0$ for all $n \in \mathbb{N}$, $\underline{P}(B_n f) \geq 0$ for all f because $P_1(B_n f), P_2(B_n f) \geq 0$ for all f. However, $\underline{P}(f) \leq P_2(f) < 0$, so $\underline{P}$ is not fully conglomerable.

On the other hand, for any decreasing sequence $(f_n)_n \downarrow 0$, it holds that $\lim_n \underline{P}(f_n) \leq \lim_n P_1(f_n) = 0$, whence $\underline{P}$ satisfies C3. ♢

With respect to condition C2, we can establish the following:

Proposition 10. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$. If it satisfies C2, then it satisfies M6.*

Proof. We begin by showing that for any sequence $(f_n)_n$ such that $\underline{P}(f_n) \geq 0 \; \forall n$, it holds that $\underline{P}(\limsup f_n) \geq 0$, when the gamble $\limsup f_n$ is well defined. For this, note that $\limsup f_n = \inf_k \sup_{n \geq k} f_n = \lim_k g_k$, where $g_k := \sup_{n \geq k} f_n$. Since the sequence $(g_k)_k$ is decreasing, we deduce that $\underline{P}(\lim_k g_k) = \lim_k \underline{P}(g_k) \geq \lim_k \underline{P}(f_k) \geq 0$.

Now, consider a partition $\mathcal{B}$ and a gamble f such that $\underline{P}(Bf) \geq 0$ whenever $\underline{P}(B) > 0$. Denote $\{B_n : n \in \mathbb{N}\}$ the family of elements of $\mathcal{B}$ with positive lower probability. Then the sequence $f_n := f I_{\cup_{i=1}^n B_i}$ satisfies $\underline{P}(f_n) \geq 0$ by super-additivity, and $\limsup f_n = f I_{\cup_n B_n}$. Applying the first part, we deduce that $\underline{P}(f I_{\cup_n B_n}) \geq 0$, whence $\underline{P}$ is $\mathcal{B}$-conglomerable. Since this holds for any partition $\mathcal{B}$, we deduce that $\underline{P}$ is fully conglomerable. $\square$

To prove that the converse does not hold, we refer to Example 2. On the other hand, condition C2 does not imply M5 (and therefore it does not imply M4 either):

Example 4. Let $\underline{P}$ be the natural extension of Lebesgue measure P on [0,1], which is given by

$$\underline{P}(f) := \sup\{P(g) : f \geq g, \; g \text{ is } \beta_{[0,1]}\text{-measurable}\}. \tag{5.4}$$

It has been established by Walley [1991, Section 6.9.6] that $\underline{P}$ is fully conglomerable but has no dominating fully conglomerable linear prevision (whence $\underline{P}$ does not satisfy M5).

To prove that $\underline{P}$ satisfies condition C2, consider a decreasing sequence of gambles $(f_n)_n \downarrow f$. By Eq. (5.4), for any $\varepsilon > 0$, there is some $\beta_{[0,1]}$-measurable gamble g_n such that $g_n \leq f_n$ and $\underline{P}(f_n) \leq P(g_n) + \varepsilon$. If we also consider a $\beta_{[0,1]}$-measurable gamble $g \leq f$ such that $\underline{P}(f) \leq P(g) + \varepsilon$, then we can assume without loss of generality that $g_n \geq g$ for every n: otherwise, it suffices to consider $g'_n := \max\{g_n, g\}$, which satisfies $g'_n \leq \max\{f_n, f\} = f_n$ and also $\underline{P}(g'_n) \geq \underline{P}(g_n) \geq \underline{P}(f_n) - \varepsilon$. As a consequence,

$$\underline{P}(f) \leq \lim_n \underline{P}(f_n) = \limsup \underline{P}(f_n) \leq \limsup P(g_n) + \varepsilon \leq P(\limsup g_n) + \varepsilon \leq \underline{P}(f) + \varepsilon,$$

where the third inequality follows from Fatou's lemma and the fourth from monotonicity, because $\limsup g_n \leq \limsup f_n = \lim f_n = f$. Since this holds for any $\varepsilon > 0$, then $\underline{P}(f) = \lim_n \underline{P}(f_n)$. Thus, $\underline{P}$ satisfies C2. $\Diamond$

We conclude this section by summarising the implications between full conglomerability and continuity in the imprecise case:

$$
\begin{array}{ccccccc}
 & & \text{C1} \Rightarrow \text{C4} & \Rightarrow & \text{M4} & & \\
 & & & \swarrow & & \searrow & \\
\text{C3} & \Leftarrow & \text{C2} & & & & \text{M5} \\
 & & & \searrow & & \swarrow & \\
 & & & & \text{M6}. & &
\end{array}
$$

No additional implication other than the ones that derive from this diagram holds.

5.2. **Full conglomerability and super-additivity.** Next we investigate the connection between full conglomerability and countable super-additivity of the coherent lower prevision. Consider the following condition:

M7. $(\forall (f_n)_n \subseteq \mathcal{L}(\Omega) : \sum_n f_n \in \mathcal{L}(\Omega))\ \underline{P}\left(\sum_n f_n\right) \geq \sum_n \underline{P}(f_n)$.

The reason for our investigation is that both countable super-additivity and conglomerability are quite related to the closedness of the set of desirable gambles under countable sums. Specifically, we have proven the following:

Proposition 11. *Let $\underline{P}$ be a coherent lower prevision and let $\underline{\mathcal{R}}, \overline{\mathcal{R}}$ denote its associated sets of strictly desirable and almost desirable gambles, respectively. Then each of the following statements implies the next:*

- *$\underline{P}$ satisfies M7.*
- $(\forall (f_n)_n \subseteq \underline{\mathcal{R}} : \sum_n f_n \in \mathcal{L}(\Omega))\ \sum_n f_n \in \underline{\mathcal{R}}$.
- $(\forall (f_n)_n \subseteq \underline{\mathcal{R}} : \sum_n f_n \in \mathcal{L}(\Omega))\ \sum_n f_n \in \overline{\mathcal{R}}$.
- *$\underline{P}$ satisfies C3.*

Proof. Let us show that the first statement implies the second. Consider a sequence $(f_n)_n \subseteq \underline{\mathcal{R}}$ such that $\sum_n f_n \in \mathcal{L}(\Omega)$. If $f_n \gneq 0$ for every n, we deduce that $\sum_n f_n \gneq 0$, and therefore it belongs to $\underline{\mathcal{R}}$. If there is some natural number m such that $\underline{P}(f_m) > 0$, then since for any other n it holds that $\underline{P}(f_n) \geq 0$, we deduce that $\sum_n \underline{P}(f_n) > 0$, whence, applying condition M7, we deduce that $\underline{P}(\sum_n f_n) > 0$. Hence, $\sum_n f_n \in \underline{\mathcal{R}}$.

That the second statement implies the third is trivial.

Finally, to prove that the third statement implies the fourth, consider a sequence $(f_n)_n \downarrow 0$, and assume ex-absurdo that $\lim_n \underline{P}(f_n) = \inf_n \underline{P}(f_n) = \varepsilon > 0$.

For every $\omega \in \Omega$, $\lim_n f_n(\omega) = 0$, whence there is some $n_\omega \in \mathbb{N}$ such that $f_n(\omega) < \frac{\varepsilon}{4}$ for every $n \geq n_\omega$. As a consequence, $\sum_n (f_n(\omega) - \frac{\varepsilon}{2}) = -\infty$ for every ω, and therefore there is some minimum n^*_ω such that $\sum_{i=1}^{n^*_\omega} (f_i(\omega) - \frac{\varepsilon}{2}) < -\frac{\varepsilon}{4}$.

Consider the gamble g_n given by

$$
g_n(\omega) := \begin{cases} f_n(\omega) - \frac{\varepsilon}{2} & \text{if } n \leq n^*_\omega \\ 0 & \text{otherwise.} \end{cases}
$$

By construction $g_n \geq f_n - \frac{\varepsilon}{2}$, whence $\underline{P}(g_n) > 0$ for every n. On the other hand, $\sum_n g_n(\omega) < -\frac{\varepsilon}{4}$ for every ω, and therefore $\underline{P}(\sum_n g_n) < -\frac{\varepsilon}{4}$, meaning that $\sum_n g_n \notin \overline{\mathcal{R}}$. □

In fact, the third condition in the above proposition suffices to guarantee full conglomerability:

Proposition 12. *Let $\underline{P}$ be a coherent lower prevision, and assume that any countable sum of strictly desirable gambles is almost desirable (whenever the countable sum belongs to $\mathcal{L}(\Omega)$). Then $\underline{P}$ is fully conglomerable.*

Proof. Ex-absurdo, assume there is a partition $\mathcal{B}$ of Ω and a gamble f such that $\underline{P}(Bf) \geq 0$ for every B with $\underline{P}(B) > 0$ and $\underline{P}(\sum_{\underline{P}(B)>0} Bf) < 0$. Then there is some $\varepsilon > 0$ such that $\underline{P}(\sum_{\underline{P}(B)>0} B(f+\varepsilon)) < 0$. On the other hand, for every B with $\underline{P}(B) > 0$,

$$\underline{P}(B(f+\varepsilon)) \geq \underline{P}(Bf) + \underline{P}(B\varepsilon) \geq \varepsilon \underline{P}(B) > 0.$$

Thus, the gamble $B(f+\varepsilon)$ is strictly desirable for every B in the countable set $\{B \in \mathcal{B} : \underline{P}(B) > 0\}$, but the sum $\sum_{\underline{P}(B)>0} B(f+\varepsilon)$ is not almost desirable. This is a contradiction with our hypotheses. Thus, $\underline{P}$ is fully conglomerable. □

However, the converse does not hold, as we can deduce from Proposition 11 and Example 2.

Next, we study the connection between this condition and the continuity conditions we studied in the previous section. With respect to conditions C1–C4, we have established the following:

Proposition 13. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$. Then*

$$C2 \Rightarrow M7 \Rightarrow \ C3.$$

Proof. Let us begin with the first implication. Consider a sequence $(f_n)_n$ of gambles such that their sum $\sum_n f_n$ exists and belongs to $\mathcal{L}(\Omega)$. Then $\sum_n f_n = \lim_n S_n$, where $S_n := f_1 + \cdots + f_n$. Since the limit of the sequence $(S_n)_n$ exists, then we have $\lim_n S_n = \limsup_n S_n = \inf_k \sup_{n \geq k} S_n$, and if we consider the sequence of gambles $(g_k)_k$ given by $g_k := \sup_{n \geq k} S_n$, we obtain that $(g_k)_k$ is a decreasing sequence and moreover $\lim_k g_k = \inf_k g_k = \lim_n S_n$. Applying C2,

$$\lim_k \underline{P}(g_k) = \underline{P}(\lim_k g_k) = \underline{P}(\lim_n S_n) = \underline{P}\left(\sum_n f_n\right),$$

and since on the other hand

$$\underline{P}(g_k) = \underline{P}(\sup_{n \geq k} S_n) \geq \underline{P}(S_k) \geq \sum_{i=1}^{k} \underline{P}(f_i),$$

we deduce that

$$\lim_k \underline{P}(g_k) \geq \lim_k \sum_{i=1}^{k} \underline{P}(f_i) = \sum_n \underline{P}(f_n),$$

whence $\underline{P}$ satisfies $M7$.

The second implication follows from Proposition 11. $\square$

This, together with our previous results, implies that a coherent lower prevision satisfying conditions C1 or C4 also satisfies M7.

Next we investigate the connection between condition M7 and full conglomerability. It is not difficult to prove that any coherent lower prevision $\underline{P}$ satisfying M7 is a fully conglomerable coherent lower prevision, but it need not satisfy M5 (and therefore not M4 either):

Proposition 14. *M7 implies M6, but not necessarily M5.*

Proof. Consider a partition $\mathcal{B}$ on Ω and a gamble f such that $\underline{P}(Bf) \geq 0$ for every $B \in \mathcal{B}$ with positive lower probability. Since there are at most a countable number of such B, we deduce from M7 that $\underline{P}(\sum_{\underline{P}(B)>0} Bf) \geq \sum_{\underline{P}(B)>0} \underline{P}(Bf) \geq 0$. Thus, $\underline{P}$ is $\mathcal{B}$-conglomerable; since we can do this for any partition $\mathcal{B}$, we conclude that $\underline{P}$ satisfies M6.

To prove that M7 does not imply that M5, it suffices to consider the coherent lower prevision $\underline{P}$ from Example 4 and apply Proposition 13. $\square$

Finally, to prove that condition M7 is not equivalent to C3, it suffices to consider the coherent lower prevision $\underline{P}$ from Example 3, which satisfies C3 but not M6 (whence, applying Proposition 14, it does not satisfy M7 either). As a consequence, the place of super σ-additivity in our diagram is given by the following:

$$\begin{array}{ccccccc}
\text{C1} & \Rightarrow & \text{C4} & \Rightarrow & \text{M4} & & \\
 & & \swarrow & & & \searrow & \\
 & & \text{C2} & & & & \text{M5} \\
 & & \Downarrow & & & \swarrow & \\
\text{C3} & \Leftarrow & \text{M7} & \Rightarrow & \text{M6}. & &
\end{array}$$

On the other hand, if P is a linear prevision, we deduce from our results in the previous section that:

$$\text{C1} \Rightarrow \text{M1} \Leftrightarrow \text{C2} \Leftrightarrow \text{M7} \Leftrightarrow \text{C3} \Leftrightarrow \text{C4} \Rightarrow \text{M2} \Rightarrow \text{M3}.$$

The only open problem at this stage is whether conditions M7 and C2 are equivalent. One particular case where both conditions hold is for *vacuous* lower previsions on some subset A of Ω, given by $\underline{P}_A(f) := \inf_{\omega \in A} \underline{P}(f)$ for every $f \in \mathcal{L}(\Omega)$, because these models satisfy trivially M4.

To prove that conditions C1, C4 do not necessarily hold for vacuous lower previsions, it suffices to consider the vacuous lower prevision $\underline{P}$ on $\mathbb{N}$, for which $\lim_n \underline{P}(I_{\{1,\dots,n\}}) = 0 \neq \underline{P}(\lim_n I_{\{1,\dots,n\}}) = \underline{P}(1) = 1$.

6. Conglomerability and marginal extension

From Walley [1991, Theorem 6.8.2], given a coherent lower prevision $\underline{P}$ and a partition $\mathcal{B}$ of Ω, it holds that $\underline{P}$ is $\mathcal{B}$-conglomerable if and only if $\underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, where $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{P}$, given by Eq. (2.2). As a consequence, we have the following:

$$\underline{P} \text{ is fully conglomerable } \Leftrightarrow \underline{P} \geq \sup_{\mathcal{B}} \underline{P}(\underline{P}(\cdot|\mathcal{B})). \tag{6.1}$$

Thus, given a coherent lower prevision $\underline{P}$, we can define

$$\underline{Q} := \sup\{\underline{P}(\underline{P}(\cdot|\mathcal{B})) : \mathcal{B} \text{ is a partition of } \Omega\}. \tag{6.2}$$

The concatenation $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ of a marginal and a conditional lower prevision is called a *marginal extension model* [Walley, 1991, Section 6.7], and is an extension of the product rule to the imprecise case. Thus, the functional $\underline{Q}$ given by Eq. (6.2) is the supremum of a family of marginal extension models. In this section, we are going to study the properties of this functional. It follows immediately that $\underline{Q}(f) \geq \inf f$ and $\underline{Q}(\lambda f) = \lambda \underline{Q}(f)$ for all $f \in \mathcal{L}(\Omega)$ and $\lambda > 0$. However, $\underline{Q}$ is not a coherent lower prevision in general:

Example 5. Let $\Omega := \mathbb{N}$, and consider a free ultrafilter $\mathcal{F}$ of $\mathbb{N}$. Let P_1 denote its associated linear prevision. Let P_2 be the countably additive linear prevision whose restriction to singletons satisfies $P_2(n) = \frac{1}{2^n}$ for every n, and define $P = \frac{1}{2}P_1 + \frac{1}{2}P_2$. Then since P_1 is not countably additive we deduce that neither is P. Since on the other hand P is non-molecular because so is P_2, we deduce from Proposition 3 that P is not fully conglomerable. Consider then a partition $\mathcal{B}$ such that P is not coherent with its conditional natural extension. Since by construction $P(n) > 0$ for every n, we deduce that $P(B) > 0$ for every $B \in \mathcal{B}$, and therefore this conditional natural extension is linear. Thus, $P \neq P(P(\cdot|\mathcal{B}))$, whence $\underline{Q} \geq \max\{P, P(P(\cdot|\mathcal{B}))\} \gneq P$, and since P is a linear prevision then it will be

$$\mathcal{M}(\underline{Q}) \subseteq \mathcal{M}(P) \cap \mathcal{M}(P(P(\cdot|\mathcal{B}))) = \{P\} \cap \{P(P(\cdot|\mathcal{B}))\} = \emptyset,$$

taking into account that for any linear prevision P' its associated credal set is $\mathcal{M}(P') = \{P'\}$. Thus, $\underline{Q}$ does not avoid sure loss, and as a consequence it is not coherent either. $\Diamond$

Next we investigate in more detail the relationship between a coherent lower prevision $\underline{P}$ and the functional $\underline{Q}$ given by Eq. (6.2). We can prove the following:

Proposition 15. $\underline{P} \leq \underline{Q}$. *As a consequence,* $\underline{P}$ *is fully conglomerable if and only if coincides with* $\underline{Q}$*, and the fully conglomerable natural extension of* $\underline{P}$ *dominates* $\underline{Q}$.

Proof. To prove that $\underline{Q}(f) \geq \underline{P}(f)$ for every gamble f, it suffices to consider the partition $\mathcal{B} := \{\{\omega\} : \omega \in \Omega\}$, for which we obtain $\underline{P}(f|\{\omega\}) = f(\omega)$ for every $\omega \in \Omega$, whence $\underline{P}(f|\mathcal{B}) = f$ and therefore $\underline{P}(\underline{P}(f|\mathcal{B})) = \underline{P}(f)$. Now, since by Eq. (6.1) $\underline{P}$ is fully conglomerable if and only if it dominates $\underline{Q}$, we deduce that $\underline{P} \geq \underline{Q}$ if and only if they are equal.

To prove the second statement, note that if $\underline{P}' \geq \underline{P}$ is fully conglomerable, then $\underline{P}' \geq \sup_{\mathcal{B}} \underline{P}'(\underline{P}'(\cdot|\mathcal{B})) \geq \sup_{\mathcal{B}} \underline{P}(\underline{P}(\cdot|\mathcal{B})) = \underline{Q}$. □

It is an immediate consequence of Proposition 15 that if $\underline{P}$ is fully conglomerable then $\underline{Q}$ avoids sure loss. The converse is not true: if $\underline{Q}$ avoids sure loss but is not coherent, then it will not coincide with $\underline{P}$, and therefore the latter will not be fully conglomerable.

Example 6. Consider [Miranda et al., 2012, Example 9], where $\Omega = \mathbb{N} \cup -\mathbb{N}$, $\underline{P} := \min\{P_1, P_2\}$, P_1 is fully conglomerable and P_2 is not, and where for a given $\mathcal{B}$ it holds that the $\mathcal{B}$-conglomerable natural extension of $\underline{P}$ is P_1, which is then also the fully conglomerable natural extension of $\underline{P}$. Then it follows from Proposition 15 that $\underline{P} \leq \underline{Q} \leq P_1$, whence $\mathcal{M}(\underline{Q}) \subseteq \mathcal{M}(\underline{P}) = \{\alpha P_1 + (1-\alpha)P_2 : \alpha \in [0,1]\}$. Thus, we can identify $\mathcal{M}(\underline{Q})$ with a closed and convex subset of $[0,1]$, namely $\{\alpha \in [0,1] : \alpha P_1 + (1-\alpha)P_2 \geq \underline{Q}\}$. This set is not empty as $\alpha = 1$ belongs to it. Also, it is shown in the example that for every $\alpha \neq 1$ there is a gamble h such that

$$(\alpha P_1 + (1-\alpha)P_2)(h) < \underline{P}(\underline{P}(h|\mathcal{B})) \Rightarrow (\alpha P_1 + (1-\alpha)P_2)(h) < \underline{Q}(h),$$

whence $\mathcal{M}(\underline{Q}) = P_1$.

On the other hand, if we take $f := I_{\{-n\}}$, there are the following options for an arbitrary partition $\mathcal{B}'$:

- If the set $B \in \mathcal{B}'$ that includes $\{-n\}$ strictly does so, then

$$\begin{cases} \underline{P}(f|B) \leq P_2(f|B) = 0 & \text{if } P_2(B) > 0, \\ \underline{P}(f|B) = \min_B f = 0 & \text{if } P_2(B) = 0, \end{cases}$$

  because the linear prevision P_2 in [Miranda et al., 2012, Example 9] satisfies $P_2(\{-n\}) = 0$; and hence $\underline{P}(\underline{P}(f|\mathcal{B}')) = 0$.
- The other possibility is that $B = \{-n\}$, whence $\underline{P}(f|B) = 1$; but then

$$\underline{P}(\underline{P}(f|\mathcal{B}')) = \min\{P_1(\{-n\}), P_2(\{-n\})\} = 0.$$

Thus $\underline{Q}(\{-n\}) = 0 < P_1(\{-n\})$, so $\underline{Q}$ avoids sure loss but is not coherent. ♢

This would mean that if $\underline{Q}$ avoids sure loss but is not coherent, then it cannot be $\underline{Q} = \underline{P}$ coherent so $\underline{P}$ would not be fully conglomerable. Thus, if $\underline{P}$ is fully conglomerable then $\underline{Q}$ is coherent. Although it is an open problem whether the converse holds in general, it is easy to prove that when P is linear, then $\underline{Q} \geq P$ is coherent if and only if $\underline{Q} = P$ (it cannot be that $\underline{Q}(f) > P(f)$ and still be that $\underline{Q}$ is coherent). Hence, in the precise case we have the equivalence.

Remark 1. The above example suggests a different procedure: for each partition $\mathcal{B}$, we define $\underline{P}_{\mathcal{B}}$ as the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$, and then let $\underline{Q}' := \sup_{\mathcal{B}} \underline{P}_{\mathcal{B}}$. Then

$$\underline{P}' \geq \underline{P} \text{ fully conglomerable} \Rightarrow \underline{P}' \geq \underline{Q}',$$

so we can also use $\underline{Q}'$ as a conservative approximation of the fully conglomerable natural extension. Moreover, $\underline{Q}' \geq \underline{Q} \geq \underline{P}$ by construction, so

$$\underline{P} \text{ fully conglomerable} \Leftrightarrow \underline{P} = \underline{Q}'.$$

A difference is that in Example 6 $\underline{Q}$ avoids sure loss but is not coherent, while the lower prevision $\underline{Q}'$ defined as above would be coherent and agree with the fully conglomerable natural extension. Moreover, we still have that if $\underline{P}$ fully conglomerable then $\underline{Q}'$ is coherent, and Example 6 shows that the converse is not true. $\Diamond$

On the other hand, even if the results above establish a connection between coherent lower previsions and marginal extension models, there are fully conglomerable coherent lower previsions that are not marginal extension models:

Example 7. Consider $\Omega := \{1, 2, 3, 4\}$ and $\mathcal{M} := \{P_{ij} : i, j \in \Omega, i \neq j\}$, where P_{ij} is the linear prevision given by

$$P_{ij}(f) := \frac{1}{2} f(i) + \frac{1}{2} f(j)$$

for all f. Let $\underline{P}$ be the coherent lower prevision that we obtain by taking the lower envelope of $\mathcal{M}$. It is trivially fully conglomerable because Ω is finite. Let us show that we do not have the equality $\underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ for any non-trivial partition $\mathcal{B}$. This will hold if for every $\mathcal{B}$ we can find a gamble f (possibly dependent on the partition) such that $\underline{P}(f) \neq \underline{P}(\underline{P}(f|\mathcal{B}))$. Since by construction $\underline{P}$ is invariant under permutations of the elements of Ω, it suffices to look at the following partitions:

- $\mathcal{B} = \{\{1, 2\}, \{3, 4\}\}$. If $\underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, then as $\underline{P}(\{1, 2\}) = 0 < \overline{P}(\{1, 2\}) = 1$, we should have $\underline{P}(f) = \min\{\underline{P}(f|\{1, 2\}), \underline{P}(f|\{3, 4\})\}$. Moreover, since $\underline{P}(\{1, 2\}) = 0 = \underline{P}(\{3, 4\}) = 0$, it follows from Eq. (2.2) that
$$\begin{cases} \underline{P}(f|\{1, 2\}) = \min(f(1), f(2)), \\ \underline{P}(f|\{3, 4\}) = \min(f(3), f(4)). \end{cases}$$

But then we obtain $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ vacuous, which will not coincide with $\underline{P}$ (use $f := I_{\{1,2,3\}}$, for instance).

- $\mathcal{B} = \{\{1,2,3\},\{4\}\}$. If $\underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, then by looking at the marginal distribution we get

$$\underline{P}(f) = \min\left\{\frac{1}{2}\underline{P}(f|\{1,2,3\}) + \frac{1}{2}f(4), \underline{P}(f|\{1,2,3\})\right\}.$$

On the other hand, using that

$$\begin{aligned}\underline{P}(f|\{1,2,3\}) &\leq \min\{P_{1,4}(f|\{1,2,3\}), P_{2,4}(f|\{1,2,3\}), P_{3,4}(f|\{1,2,3\})\}\\ &= \min(f(1), f(2), f(3))\end{aligned}$$

for all f, we obtain that

$$\underline{P}(f) = \min\left\{\frac{1}{2}\min(f(1),f(2),f(3)) + \frac{1}{2}f(4), \min(f(1),f(2),f(3))\right\}.$$

But this does not coincide with $\underline{P}$: take f equal to the identity function, then $\underline{P}(f) = \frac{3}{2}$ and the equation above gives $\underline{P}(f) = 1$.

- $\mathcal{B} = \{\{1,2\},\{3\},\{4\}\}$. Since $\underline{P}(\{1,2\}) = 0$, we get $\underline{P}(\cdot|\{1,2\})$ vacuous. Then

$$\begin{aligned}\underline{P}(\underline{P}(f|\mathcal{B})) = \min\{&\min(f(1),f(2)), \frac{1}{2}f(3) + \frac{1}{2}f(4),\\ &\frac{1}{2}f(3) + \frac{1}{2}\min(f(1),f(2)), \frac{1}{2}f(4) + \frac{1}{2}\min(f(1),f(2))\},\end{aligned}$$

and again with f equal to the identity function, we obtain $\underline{P}(\underline{P}(f|\mathcal{B})) = 1 < \underline{P}(f) = \frac{3}{2}$.

The other cases are just permutations of these. ♢

6.1. **Conglomerability for an arbitrary number of partitions.** Next, we are going to investigate a more general scenario: that where we study the conglomerability of a coherent lower prevision $\underline{P}$ with respect to a family $\mathbb{B}$ of partitions of Ω. This includes in particular the case where $\mathbb{B}$ consists of a single partition (which was discussed in Miranda and Zaffalon [2015], Miranda et al. [2012]) and the case where $\mathbb{B}$ is the set of all partitions, which would correspond to the case of full conglomerability.

As we did in the previous section, we can define the lower prevision

$$\underline{Q} := \sup_{\mathcal{B}\in\mathbb{B}} \underline{P}(\underline{P}(\cdot|\mathcal{B})), \tag{6.3}$$

where $\underline{P}(\cdot|\mathcal{B})$ is derived from $\underline{P}$ by natural extension. Then $\underline{P} \leq \underline{Q}$ if we include in $\mathbb{B}$ either the finest $\{\{\omega\} : \omega \in \Omega\}$ or the coarsest $\{\Omega\}$ partition, and any dominating coherent lower prevision that is $\mathbb{B}$-conglomerable must dominate $\underline{Q}$; we shall always assume that this is the case in this section, and otherwise redefine $\underline{Q} := \max\{\underline{P}, \sup_{\mathcal{B}\in\mathbb{B}} \underline{P}(\underline{P}(\cdot|\mathcal{B}))\}$. This means that if the $\mathbb{B}$-conglomerable natural extension exists, then $\mathcal{M}(\underline{Q}) \neq \emptyset$. However, the converse is not true, as we see in the following example, based on [Miranda and Zaffalon, 2015, Example 3]:

Example 8. Consider $\Omega := \mathbb{N} \cup -\mathbb{N}$, let P_3, P_4 be the linear previsions from [Miranda and Zaffalon, 2015, Example 3] and take $\underline{P} := \min\{P_3, P_4\}$. Given the partition $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, for $B_n := \{-n, n\}$, it has been shown that $\{P \geq \underline{P} : P(G_{\underline{P}}(\cdot|\mathcal{B})) \geq 0\} = \{P_3\}$. Thus, $P_3 \geq P_3(\underline{P}(\cdot|\mathcal{B})) \geq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$.

Considered the partition $\mathcal{B}' := \{\{-2n, -2n+1\} : n \in \mathbb{N}\} \cup \{\mathbb{N}\}$, it holds that $\underline{P}(\{-2n, -2n+1\}) = P_4(\{-2n, -2n+1\}) = 0$, whence $\underline{P}(\cdot|\{-2n, -2n+1\})$ is vacuous for every n. From this it follows that, for every gamble f,

$$P_3(G_{\underline{P}}(f|\mathcal{B}')) \geq P_3(G_{\underline{P}}(f|\mathbb{N})) \geq P_3(G_{P_3}(f|\mathbb{N})) = 0,$$

whence $P_3 \geq P_3(\underline{P}(\cdot|\mathcal{B}')) \geq \underline{P}(\underline{P}(\cdot|\mathcal{B}'))$.

As a consequence, we deduce that $\mathcal{M}(\underline{Q}) = \{P_3\}$, so this is the only candidate to the $\{\mathcal{B}, \mathcal{B}'\}$-conglomerable natural extension of $\underline{P}$. However, P_3 is not $\mathcal{B}$-conglomerable, as shown in [Miranda and Zaffalon, 2015, Example 3]. $\lozenge$

When the family $\mathbb{B}$ is finite, we can extend some results from [Miranda et al., 2012, Section 4] and characterise the natural extension of $\underline{Q}$:

Proposition 16. *Let $\underline{P}$ be a coherent lower prevision, and let $\mathcal{B}_1, \dots, \mathcal{B}_n$ partitions of Ω. Define $\underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_n)$ by natural extension and let $\underline{E}$ be the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B}_1), \dots, \underline{P}(\cdot|\mathcal{B}_n)$, given by Eq.* (2.3)*.*

- $\mathcal{M}(\underline{E}) = \{P \in \mathcal{M}(\underline{P}) : (\forall i = 1, \dots, n)(\forall f)\ P(G_{\underline{P}}(f|\mathcal{B}_i)) \geq 0\}$.
- $\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}_1))) \cap \dots \cap \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}_n))) = \mathcal{M}(\underline{Q})$.

Proof.

- We begin with the direct inclusion. From Eq. (2.3), given that $\underline{P}$ can be regarded as a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with $\mathcal{B} = \{\Omega\}$, we deduce that

$$\underline{E}(f) = \sup\{\mu : (\exists g, h_1, \dots, h_n \in \mathcal{L}(\Omega))\ f - \mu \geq G_{\underline{P}}(g) + \sum_{i=1}^{n} G_{\underline{P}}(h_i|\mathcal{B}_i)\}, \quad (6.4)$$

from which $\underline{E} \geq \underline{P}$. If we take $h_i := f, h_j := g := 0$ for all $j \neq i$, we obtain that $\underline{E}(G_{\underline{P}}(f|\mathcal{B}_i)) \geq 0$ for every i.

Conversely, let P be a linear prevision satisfying $P \geq \underline{P}$ and $P(G_{\underline{P}}(f|\mathcal{B}_i)) \geq 0$ for every $i = 1, \dots, n$ and every $f \in \mathcal{L}(\Omega)$. If $P(f) < \underline{E}(f)$ for some f, then by Eq. (6.4) there are gambles $g, h_1, \dots, h_n$ such that

$$f - P(f) - \varepsilon \geq G_{\underline{P}}(g) + \sum_{i=1}^{n} G_{\underline{P}}(h_i|\mathcal{B}_i)$$

for some $\varepsilon > 0$, whence

$$-\varepsilon = P(f - P(f) - \varepsilon) \geq P(G_{\underline{P}}(g) + \sum_{i=1}^{n} G_{\underline{P}}(h_i|\mathcal{B}_i)) \geq 0,$$

a contradiction.

- We begin with the direct inclusion. From the first statement, $\mathcal{M}(\underline{E}) \subseteq \mathcal{M}(\underline{P})$ and given $P \in \mathcal{M}(\underline{E})$ it holds that

$$P(f) = P(G_{\underline{P}}(f|\mathcal{B}_i) + \underline{P}(f|\mathcal{B}_i)) \geq P(\underline{P}(f|\mathcal{B}_i)) \geq \underline{P}(\underline{P}(f|\mathcal{B}_i))$$

for every gamble f and every $i = 1, \ldots, n$.

Conversely, if $P \in \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}_1))) \cap \cdots \cap \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}_n)))$, then for every gamble f and every $i = 1, \ldots, n$ $P(G_{\underline{P}}(f|\mathcal{B}_i)) \geq \underline{P}(\underline{P}(G_{\underline{P}}(f|\mathcal{B}_i)|\mathcal{B}_i)) = 0$. Applying the first statement, we deduce that $P \in \mathcal{M}(\underline{E})$. Finally, the definition of $\underline{Q}$ implies that $\mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}_1))) \cap \cdots \cap \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}_n))) = \mathcal{M}(\underline{Q})$. □

We deduce from this result that $\underline{Q}$ avoids sure loss if and only if the lower previsions $\underline{P}, \underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_n)$ avoid partial loss, and it is coherent if and only if it agrees with $\underline{E}$, which corresponds to its natural extension. This equality can also be characterised in terms of the conditional natural extensions of $\underline{E}$, as we show next:

Proposition 17. *Under the previous conditions, $\underline{E} = \underline{Q}$ if and only if*

$$(\forall B_i \in \mathcal{B}_i, i = 0, \ldots, n)\ \underline{E}(f|B_i) = \max\{\underline{P}(f|B_i), \underline{M}_j(f|B_i), j = 1, \ldots, n\},$$

where we are denoting $\mathcal{B}_0 \coloneqq \{\Omega\}$ and $\underline{M}_j \coloneqq \underline{P}(\underline{P}(\cdot|\mathcal{B}_j))$ for $j = 1, \ldots, n$, and where $\underline{E}(\cdot|B_i), \underline{P}(\cdot|B_i), \underline{M}_j(\cdot|B_i)$ are defined by conditional natural extension.

Proof. Note that $\underline{E}(f|B_i) \geq \max\{\underline{P}(f|B_i), \underline{M}_j(f|B_i), j = 1, \ldots, n\}$ for all $B_i \in \mathcal{B}_i, i = 0, \ldots, n$ because $\underline{E} \geq \underline{Q}$.

We begin with the direct implication. The case $i = 0$ follows immediately from the equality $\underline{E} = \underline{Q}$. On the other hand, given $i = 1, \ldots, n$ and $B_i \in \mathcal{B}_i$, there are two possibilities: if $\underline{E}(B_i) = 0$, then $\underline{E}(\cdot|B_i)$ is vacuous, and as a consequence it coincides with $\underline{P}(f|B_i), \underline{M}_j(f|B_i), j = 1, \ldots, n$.

If $\underline{E}(B_i) > 0$, then, taking into account that $\underline{E} = \underline{Q}$, we get

$$\begin{aligned}
\underline{E}(f|B_i) &= \sup\{\mu : \underline{E}(B_i(f - \mu)) \geq 0\} \\
&= \max\{\underline{P}(f|B_i), \max_{j=1,\ldots,n} \sup\{\mu : \underline{M}_j(B_i(f - \mu)) \geq 0\}\} \\
&= \max\{\underline{P}(f|B_i), \underline{M}_j(f|B_i), j = 1, \ldots, n\}
\end{aligned}$$

for every $B_i \in \mathcal{B}_i, i = 0, \ldots, n$.

The converse implication follows by applying the equality with $i \coloneqq 0$ and $B_i \coloneqq \Omega$. □

We can also give a necessary condition for the coherence of $\underline{Q}$:

Proposition 18. *Let $\mathbb{B}$ be a family of partitions of Ω, and define the lower prevision $\underline{Q}$ by Eq.* (6.3)*. If $\underline{Q}$ is coherent, then for every $A \subseteq \Omega$, $\underline{Q}(A) > 0 \Leftrightarrow \underline{P}(A) > 0$.*

Proof. Since $\underline{Q} \geq \underline{P}$, it suffices to establish the direct implication. Assume ex-absurdo that there is some set A such that $\underline{P}(A) = 0 < \underline{Q}(A)$. Then there must be some partition $\mathcal{B}$ in $\mathbb{B}$ such that $\underline{P}(\underline{P}(A|\mathcal{B})) > 0$, and as a consequence there must be some $B \in \mathcal{B}$ such that $\underline{P}(A|B) > 0$.

If $B \nsubseteq A$, then we should have $\underline{P}(B) > 0$: otherwise, if $\underline{P}(B) = 0$ Eq. (2.2) implies that $\underline{P}(A|B) = \inf_B I_{A\cap B} = 0$, using that $A^c \cap B \neq \emptyset$. Now, $\underline{P}(B) > 0$ implies that for every $P \geq \underline{P}$ we have that $P(B) > 0, P(A|B) > 0$, and as a consequence

$$\begin{aligned}\underline{P}(A) \geq \underline{P}(A \cap B) &= \min\{P(A \cap B) : P \geq \underline{P}\} = \min\{P(A|B)P(B) : P \geq \underline{P}\} \\ &\geq \underline{P}(A|B)\underline{P}(B) > 0,\end{aligned}$$

a contradiction. Thus, for every $B \in \mathcal{B}$ such that $\underline{P}(A|B) > 0$ it must be $B \subseteq A$. Let $C := \cup\{B \in \mathcal{B} : B \subseteq A\}$. Then

$$\underline{P}(\underline{P}(A|\mathcal{B})) = \underline{P}(\underline{P}(C|\mathcal{B})) = \underline{P}(C) \leq \underline{P}(A) = 0,$$

where the second equality holds because the indicator of C is $\mathcal{B}$-measurable. But then we conclude that $\underline{P}(\underline{P}(A|\mathcal{B})) = 0$, a contradiction. □

6.1.1. *Two partitions.* Next we shall investigate the properties of $\underline{Q}$ in the scenario made of two partitions $\mathcal{B}_1, \mathcal{B}_2$:

Proposition 19. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega), \mathcal{B}_1, \mathcal{B}_2$ two partitions of Ω and $\underline{P}(\cdot|\mathcal{B}_1), \underline{P}(\cdot|\mathcal{B}_2)$ its conditional natural extensions. Let $\underline{E}_i$ be the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B}_i)$, $\underline{M}_i := \underline{P}(\underline{P}(\cdot|\mathcal{B}_i))$, and let $\underline{Q}$ be given by Eq.* (6.3)*.*

$$\underline{Q} \textit{ coherent } \Rightarrow \underline{E} = \max\{\underline{E}_1, \underline{E}_2\} \Leftrightarrow \mathcal{M}(\underline{E}_1) \cup \mathcal{M}(\underline{E}_2) \textit{ convex.}$$

Proof. On the one hand, $\underline{E}_1 \geq \max\{\underline{P}, \underline{M}_1\}, \underline{E}_2 \geq \max\{\underline{P}, \underline{M}_2\}$, so $\max\{\underline{E}_1, \underline{E}_2\} \geq \underline{Q}$. On the other hand, Proposition 16 implies that $\mathcal{M}(\underline{Q}) = \mathcal{M}(\underline{E}) = \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{M}_1) \cap \mathcal{M}(\underline{M}_2)$. Moreover, $\mathcal{M}(\underline{E}_i) = \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{M}_i)$ for $i = 1, 2$, whence $\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{E}_1) \cap \mathcal{M}(\underline{E}_2)$. Thus, $\underline{E} \geq \max\{\underline{E}_1, \underline{E}_2\}$.

Now, if $\underline{Q}$ is coherent then $\underline{Q} = \underline{E}$ and as a consequence $\underline{Q} = \underline{E} = \max\{\underline{E}_1, \underline{E}_2\}$.

The equivalence is a consequence of [Zaffalon and Miranda, 2013, Theorem 6]. □

Unfortunately, the converse of the first implication does not always hold: it may be that $\underline{E} = \max\{\underline{E}_1, \underline{E}_2\}$ while it does not coincide with $\underline{Q}$. In fact, the equality does not guarantee that $\underline{E}$ is the $\{\mathcal{B}_1, \mathcal{B}_2\}$-conglomerable natural extension of $\underline{P}$, as we show next:

Example 9. Consider $\Omega := \mathbb{N} \cup -\mathbb{N}$ and the partitions $\mathcal{B}_1 := \{\{-n, n\} : n \in \mathbb{N}\}, \mathcal{B}_2 := \{\{-2n, -2n+1\} : n \in \mathbb{N}\} \cup \{\mathbb{N}\}$.

Let P_1 be a linear prevision given by $P_1 := P_1(P_1(\cdot|\mathcal{B}_1))$ where $P_1(f|\{-n, n\}) = \frac{f(n)+f(-n)}{2}$ for all n, and whose $\mathcal{B}_1$-marginal satisfies $P_1(\{-2n, 2n\}) = P_1(\{-2n+1, 2n-1\}) = \frac{1}{2^{n+2}}$ for all n, $P_1(\{2n, -2n : n \in \mathbb{N}\}) = \frac{3}{4}$. Consider on the other hand P_2 given by

$$P_2(f) = \sum_n f(n)\frac{1}{2^{n+1}} + \frac{1}{2}P(f^-),$$

where P is a linear prevision on $\mathcal{L}(\mathbb{N})$ satisfying $P(\{n\}) = 0$ for every n and $f^- : \mathbb{N} \to \mathbb{R}$ is given by $f^-(n) := f(-n)\ \forall n$.

Let $\underline{P} := \min\{P_1, P_2\}$. Then $\underline{P}(f|\{n, -n\}) = \min\{f(n), \frac{f(n)+f(-n)}{2}\}$ for all $f \in \mathcal{L}(\Omega)$ and $n \in \mathbb{N}$. As a consequence, if we fix some natural number n and take $f := 2I_{n,n+1,\dots}$, we get $G_{\underline{P}}(f|\mathcal{B}_1) = I_{\{n,n+1,\dots\}} - I_{\{-n,-n-1,\dots\}}$, whence $P_2(G_{\underline{P}}(f|\mathcal{B}_1)) < 0$ and $P_1(G_{\underline{P}}(f|\mathcal{B}_1)) = 0$. Thus, $\mathcal{M}(\underline{E}) \subseteq \{P_1\}$. Since by construction P_1 is $\mathcal{B}_1$-conglomerable, we deduce that $\underline{E}_1 = P_1$.

On the other hand, by construction, $\underline{P}(\{-2n, -2n+1\}) = 0$ for every n, whence $\underline{P}$ is $\mathcal{B}_2$-conglomerable and therefore $\underline{E}_2 = \underline{P}$. Thus, $\max\{\underline{E}_1, \underline{E}_2\} = P_1 = \underline{E}$, since $\mathcal{M}(\underline{E}) = \mathcal{M}(\underline{E}_1) \cap \mathcal{M}(\underline{E}_2) = \mathcal{M}(\underline{E}_1)$. However, P_1 is not $\mathcal{B}_2$-conglomerable: if we take $f := 2I_{\{-2n:n\in\mathbb{N}\}} - 2I_{\{-2n+1:n\in\mathbb{N}\}}$, we get $P_1(f|\mathcal{B}_2) = 0$, whence $P_1(P_1(f|\mathcal{B}_2)) = 0 \neq P_1(f) = P_1(P_1(f|\mathcal{B}_1)) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. $\Diamond$

Note that in the example above we do not have the equality $\underline{Q} = \underline{E} = P_1$, because

$$\underline{Q}(\{-n\}) = \max\{\underline{P}(\{-n\}), \underline{P}(\underline{P}(\{-n\}|\mathcal{B}_1)), \underline{P}(\underline{P}(\{-n\}|\mathcal{B}_2))\} = 0 < P_1(\{-n\}).$$

Taking into account Proposition 18, we immediately deduce that $\underline{Q}$ is not coherent.

6.1.2. *One partition.* Finally, we consider the case of a single partition $\mathcal{B}$. In that case, we can prove that $\underline{Q}$ is coherent if and only if it is the $\mathcal{B}$-conglomerable natural extension of $\underline{P}$.

Proposition 20. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$, $\mathcal{B}$ a partition of Ω and let $\underline{P}(\cdot|\mathcal{B})$ be the conditional natural extension of $\underline{P}$. Let $\underline{Q} := \max\{\underline{P}, \underline{P}(\underline{P}(\cdot|\mathcal{B}))\}$ and $\underline{E}$ be the natural extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$. Then*

$$\underline{Q} \text{ coherent} \Leftrightarrow \underline{Q} = \underline{E} \Leftrightarrow \underline{Q} \text{ is the } \mathcal{B}\text{-conglomerable natural extension of } \underline{P}.$$

If in addition $\underline{P}(\cdot|\mathcal{B})$ is linear then $\underline{Q}$ coherent $\Rightarrow \underline{Q} = \underline{P}$ or $\underline{Q} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$.

Proof. The first equivalence follows from Proposition 16, so let us establish the second. Assume that $\underline{E} = \underline{Q}$, and let us show that $\underline{E}(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$. We have that $\underline{E}(B) =$

$\underline{Q}(B) = \max\{\underline{P}(B), \underline{P}(\underline{P}(B|\mathcal{B}))\} = \underline{P}(B)$ for every $B \in \mathcal{B}$. Thus, if $\underline{P}(B) = 0$, then both $\underline{E}(\cdot|B)$ and $\underline{P}(\cdot|B)$ are vacuous.

On the other hand, if $\underline{E}(B) > 0$, then Proposition 17 implies that for any gamble f it holds that

$$\underline{E}(f|B) = \max\{\underline{P}(f|B), \underline{M}(f|B)\},$$

where $\underline{M}(\cdot|B)$ is the conditional natural extension of $\underline{M} := \underline{P}(\underline{P}(\cdot|\mathcal{B}))$. Since $\underline{M}$ is coherent with $\underline{P}(\cdot|\mathcal{B})$ and $\underline{M}(B) = \underline{P}(B) > 0$, it must be $\underline{M}(\cdot|B) = \underline{P}(\cdot|B)$.

Thus, $\underline{E}(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$ and this suffices to conclude that $\underline{E}$ (and therefore $\underline{Q}$) is the $\mathcal{B}$-conglomerable natural extension of $\underline{P}$, as established in [Miranda et al., 2012, Proposition 16].

The converse implication is trivial.

We move now towards the second part. Assume that $\underline{P}(\cdot|\mathcal{B})$ is linear and that the coherent lower prevision $\underline{Q}$ coincides neither with $\underline{P}$ nor with $\underline{P}(P(\cdot|\mathcal{B}))$. This means in particular that $\underline{P}$ does not dominate $\underline{P}(P(\cdot|\mathcal{B}))$ or viceversa, or, in other words, that both $\mathcal{M}(\underline{P}) \setminus \mathcal{M}(\underline{P}(P(\cdot|\mathcal{B})))$ and $\mathcal{M}(\underline{P}(P(\cdot|\mathcal{B}))) \setminus \mathcal{M}(\underline{P})$ are non-empty. Consider $P_1 \in \mathcal{M}(\underline{P}) \setminus \mathcal{M}(\underline{P}(P(\cdot|\mathcal{B})))$ and $P_2 \in \mathcal{M}(\underline{P}(P(\cdot|\mathcal{B}))) \setminus \mathcal{M}(\underline{P})$. If $P_1 \notin \mathcal{M}(\underline{P}(P(\cdot|\mathcal{B})))$, then it follows from Proposition 16 that P_1 does not dominate the natural extension of $\underline{P}, P(\cdot|\mathcal{B})$ and that there is some gamble f such that $P_1(G_P(f|\mathcal{B})) < 0$.

On the other hand, since $P_2 \in \mathcal{M}(\underline{P}(P(\cdot|\mathcal{B})))$, it follows from Walley [1991, Theorem 6.7.4] that it must be $P_2 = P_2(P(\cdot|\mathcal{B}))$, and as a consequence $P_2(G_P(f|\mathcal{B})) = 0$. But then it will be $(\alpha P_1 + (1-\alpha)P_2)(G_P(f|\mathcal{B})) < 0$ for any $\alpha \in (0,1)$, and applying Proposition 16 we deduce that $\alpha P_1 + (1-\alpha)P_2 \notin \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{P}(P(\cdot|\mathcal{B})))$.

We deduce that $\underline{Q} = \underline{E}$, which is the lower envelope of $\mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{P}(P(\cdot|\mathcal{B})))$, cannot coincide with the maximum of $\underline{P}, \underline{P}(P(\cdot|\mathcal{B}))$, as established in [Zaffalon and Miranda, 2013, Theorem 6, (b)⇔(c)]. This is a contradiction with the definition of $\underline{Q}$. □

The second part of this proposition gives an example where the coherence of $\underline{Q}$ is very stringent: it only holds when one of the coherent lower previsions $\underline{P}, \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ dominates the other. Our next result gives another scenario where this property holds: that where the coherent lower prevision $\underline{P}$ is the lower envelope of two linear previsions. This is a particular case of *finitary* models, which we investigated in [Miranda and Zaffalon, 2015, Section 6].

Proposition 21. *Let $\underline{P} := \min\{P_1, P_2\}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$, and consider a partition $\mathcal{B}$ of Ω. Define $\underline{Q} := \max\{\underline{P}, \underline{P}(\underline{P}(\cdot|\mathcal{B}))\}$. Then*

$$\underline{Q} \text{ coherent } \Leftrightarrow \underline{Q} = \underline{P} \text{ or } \underline{Q} = \underline{P}(\underline{P}(\cdot|\mathcal{B})).$$

Proof. Let us establish the direct implication; the converse is trivial.

First of all, if $\underline{Q}$ is coherent, then it coincides with the natural extension $\underline{E}$ of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$, and applying Proposition 16,

$$\mathcal{M}(\underline{Q}) = \{P \geq \underline{P} : (\forall f)\ P(G_{\underline{P}}(f|\mathcal{B})) \geq 0\}. \tag{6.5}$$

If $\underline{Q} \neq \underline{P}$, then $\mathcal{M}(\underline{Q})$ is a proper subset of $\mathcal{M}(\underline{P}) = \{\alpha P_1 + (1-\alpha)P_2 : \alpha \in [0,1]\}$, or, in other words, there is some $[a,b] \subsetneq [0,1]$ such that

$$\mathcal{M}(\underline{Q}) = \{\alpha P_1 + (1-\alpha)P_2 : \alpha \in [a,b]\}. \tag{6.6}$$

Let us assume without loss of generality that $b < 1$, so that $P_1 \in \mathcal{M}(\underline{P}) \setminus \mathcal{M}(\underline{Q})$.

Assume that $\underline{Q}$ does not coincide with $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ either, which means that the difference $\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) \setminus \mathcal{M}(\underline{P})$ is non-empty. Take $P_1' \in \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) \setminus \mathcal{M}(\underline{P})$.

If $\underline{Q} = \max\{\underline{P}, \underline{P}(\underline{P}(\cdot|\mathcal{B}))\}$ is coherent, then it follows from [Zaffalon and Miranda, 2013, Theorem 6] that there is some $\alpha \in (0,1)$ such that $\alpha P_1 + (1-\alpha)P_1' \in \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) = \mathcal{M}(\underline{Q})$, whence

$$\alpha P_1 + (1-\alpha)P_1' = \beta P_1 + (1-\beta)P_2 \text{ for some } \beta \in [a,b].$$

Note that it cannot be $\beta \geq \alpha$, or we would obtain that $P_1' \in \mathcal{M}(\underline{P})$; thus, $\beta < \alpha$, and then

$$P_1' = \frac{(\beta-\alpha)P_1 + (1-\beta)P_2}{1-\alpha}.$$

In other words, any linear prevision in the non-empty set $\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) \setminus \mathcal{M}(\underline{P})$ can be expressed as $\gamma P_1 + (1-\gamma)P_2$ for some $\gamma < 0$, and as a consequence

$$\begin{aligned}\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) &= (\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) \cap \mathcal{M}(\underline{P})) \cup (\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) \setminus \mathcal{M}(\underline{P}))\\ &= \mathcal{M}(\underline{Q}) \cup (\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) \setminus \mathcal{M}(\underline{P}))\\ &= \{\gamma P_1 + (1-\gamma)P_2 : \gamma \in [c,d]\} \end{aligned} \tag{6.7}$$

for some interval $[c,d]$, taking into account that the credal set $\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B})))$ is convex and that therefore it cannot be $\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) = \{\gamma P_1 + (1-\gamma)P_2 : \gamma \in [a',b'] \cup [a,b]\}$ with $b' < a$. From this we also deduce that $c < 0$ (because $\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) \setminus \mathcal{M}(\underline{P}) \neq \emptyset$) and that $d = b$ (because of Eq. (6.6)). In particular we deduce that $P_2 \in \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B})))$.

At this point we establish a couple of properties of $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$:

- On the one hand, for any gamble f that is constant on the elements of $\mathcal{B}$, we must have $P_1(f) = P_2(f)$; otherwise, we can find such a gamble g with $P_1(g) < P_2(g)$ and we would obtain $\underline{P}(g) = P_1(g) < bP_1(g) + (1-b)P_2(g) = \underline{Q}(g) = \max\{\underline{P}(g), \underline{P}(\underline{P}(g|\mathcal{B}))\} = \underline{P}(g)$, a contradiction.
- Thus, given $B \in \mathcal{B}$, if $P_1(B) = P_2(B) > 0$, then for any $\gamma \in \mathbb{R}$

$$(\gamma P_1 + (1-\gamma)P_2)(f|B) = \frac{(\gamma P_1 + (1-\gamma)P_2)(fB)}{P_1(B)} = \gamma P_1(f|B) + (1-\gamma)P_2(f|B),$$

taking into account that $P_1(B) = P_2(B)$ by the previous point.

From this it follows that it must be $P_1(f|B) = P_2(f|B)$ for every f: otherwise, if we can find a gamble f such that $P_1(f|B) < P_2(f|B)$, then we get

$$\begin{aligned}
&\min\{P(f|B) : P \geq \underline{P}(\underline{P}(\cdot|\mathcal{B}))\} \\
&= \min\{(cP_1 + (1-c)P_2)(f|B), (bP_1 + (1-b)P_2)(f|B)\} \\
&= \min\{cP_1(f|B) + (1-c)P_2(f|B), bP_1(f|B) + (1-b)P_2(f|B)\} \\
&= bP_1(f|B) + (1-b)P_2(f|B) > P_1(f|B) = \underline{P}(f|B),
\end{aligned}$$

because $b < 1$. And this means that $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ is not coherent with $\underline{P}(\cdot|\mathcal{B})$, a contradiction with [Walley, 1991, Section 6.7.2].

- The previous point means that $\underline{P}(\cdot|\mathcal{B})$ must be linear when $\underline{P}(B) > 0$, and vacuous otherwise. Let $C := \cup\{B : \underline{P}(B) = 0\}$; then $G_{\underline{P}}(f|\mathcal{B}) \geq G_{\underline{P}}(fI_{C^c}|\mathcal{B})$ for every gamble f. Since $P_2 \geq \underline{Q}$, we deduce from Eq. (6.5) that $P_2(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for every f, and in particular for fI_{C^c}. Since $-\underline{P}(\underline{P}(fI_{C^c}|\mathcal{B})) = \underline{P}(\underline{P}(-fI_{C^c}|\mathcal{B}))$ because $\underline{P}$ is linear on the gambles that are constant on $\mathcal{B}$ and $\underline{P}(\cdot|\mathcal{B})$ is linear on C^c, we deduce that $P_2(G_{\underline{P}}(fI_{C^c}|\mathcal{B})) = 0$ for every gamble f.

Since $P_1 \notin \mathcal{M}(\underline{Q})$, it follows from Eq. (6.5) that there exists a gamble f_1 such that $P_1(G_{\underline{P}}(f_1|\mathcal{B})) < 0$. We deduce from the third point above that $P_1(G_{\underline{P}}(f_1 I_{C^c}|\mathcal{B})) < 0$ and therefore $P_1(G_{\underline{P}}(-f_1 I_{C^c}|\mathcal{B})) > 0$.

The coherence of $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ with $\underline{P}(\cdot|\mathcal{B})$ implies that $\underline{P}(\underline{P}(G_{\underline{P}}(g|\mathcal{B}))) \geq 0$ for every gamble g, and therefore for any linear prevision $Q \in \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B})))$, it holds that $Q(G_{\underline{P}}(g|\mathcal{B})) \geq 0$. Taking the form of the credal set $\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B})))$ (Eq. (6.7)) into account, we observe that, if $\gamma > 0$,

$$(\gamma P_1 + (1-\gamma)P_2)G_{\underline{P}}(f_1 I_{C^c}|\mathcal{B}) < 0,$$

and if $\gamma < 0$,

$$(\gamma P_1 + (1-\gamma)P_2)G_{\underline{P}}(-f_1 I_{C^c}|\mathcal{B}) < 0.$$

Thus, $\gamma P_1 + (1-\gamma)P_2 \notin \mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B})))$ for any $\gamma \neq 0$, whence $\mathcal{M}(\underline{P}(\underline{P}(\cdot|\mathcal{B}))) = \{P_2\}$; but then it would be $\underline{P}(\underline{P}(\cdot|\mathcal{B})) \geq \underline{P}$, and therefore $\underline{Q} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$. This is a contradiction. □

Thus, under the hypotheses of the previous result, for $\underline{Q}$ to be coherent, either $\underline{P}$ dominates $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ (which means that $\underline{P}$ is itself $\mathcal{B}$-conglomerable and coincides with $\underline{Q}$) or $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ dominates $\underline{P}$. This second possibility is characterised in the following result:

Proposition 22. *Let* $\underline{P} := \min\{P_1, P_2\}$ *be a coherent lower prevision on* $\mathcal{L}(\Omega)$ *and let* $\mathcal{B}$ *be a partition of* Ω *such that* $\underline{P}(B) > 0$ *for every* B*. Define* $\underline{P}(\cdot|\mathcal{B})$ *by Eq.* (2.2)*. Then* $\underline{P}(\underline{P}(\cdot|\mathcal{B})) \gneq \underline{P} \Rightarrow \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ *is linear.*

Proof. If $\underline{P}(\underline{P}(\cdot|\mathcal{B})) \gneq \underline{P}$, Proposition 21 implies that $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ is the $\mathcal{B}$-conglomerable natural extension of $\underline{P}$. It also follows from the proof of this proposition that the restriction of $\underline{P}$ to the gambles that are constant on the elements of $\mathcal{B}$ is linear.

On the other hand, if $\underline{P}(\cdot|\mathcal{B})$ were not linear, then there would be some gamble f and some $B \in \mathcal{B}$ such that $P_1(f|B) \neq P_2(f|B)$. Since $P_1(B) = P_2(B) = \underline{P}(B) > 0$, it follows that for every $\alpha \in (0,1)$,

$$(\alpha P_1 + (1-\alpha)P_2)(f|B) = \frac{\alpha P_1(fB) + (1-\alpha)P_2(fB)}{\underline{P}(B)}$$

belongs to the interval $\in (\min\{P_1(f|B), P_2(f|B)\}, \max\{P_1(f|B), P_2(f|B)\})$; now, if $\mathcal{M}(\underline{Q}) \supsetneq \mathcal{M}(\underline{P})$ then there is some $[a,b] \subsetneq [0,1]$ such that $\mathcal{M}(\underline{Q}) = \{\alpha P_1 + (1-\alpha)P_2 : \alpha \in [a,b]\}$, and then we deduce that $\underline{Q}(\cdot|B) = \min\{(\alpha P_1 + (1-\alpha)P_2)(\cdot|B) : \alpha \in [a,b]\}$ does not agree with $\underline{P}(\cdot|B)$ in either f or $-f$. This is a contradiction. As a consequence, $P_1(\cdot|B) = P_2(\cdot|B)$ for every B and therefore $\underline{P}(\underline{P}(\cdot|B))$ is linear. □

The above result does not hold when $\underline{P}$ is the lower envelope of more than two coherent lower previsions: we can take for instance a linear conditional prevision $P(\cdot|\mathcal{B})$, two different $\mathcal{B}$-marginals P_1, P_2 and a third linear prevision P_3 that is not $\mathcal{B}$-conglomerable but has marginal P_2 and conditional $P(\cdot|\mathcal{B})$. Then given $\underline{P} := \min\{P_1(P(\cdot|\mathcal{B})), P_2(P(\cdot|\mathcal{B})), P_3\}$, it will hold that $\underline{P}(\underline{P}(\cdot|\mathcal{B})) = \min\{P_1(P(\cdot|\mathcal{B})), P_2(P(\cdot|\mathcal{B}))\} \gneq \underline{P}$, even if it is not linear.

7. Conclusions

In this paper, we have explored the notion of full conglomerability for coherent lower previsions, using the definition considered by Walley [1991, Section 6.8]. We have considered both the precise and imprecise cases.

First of all, we have investigated if full conglomerability can be characterised in terms of some continuity or super-additivity conditions. We have considered a number of possibilities, and have shown that, although there are some necessary or sufficient conditions, there is none that is at the same time necessary *and* sufficient. This seems to indicate that there is not immediate advantage to use the general notion of full conglomerability if our goal is to have models that are regular enough to be somewhat easier to deal with in practice. Still, our results may help to simplify the verification of full conglomerability of a coherent lower prevision.

Since countably additive models and their envelopes are in particular fully conglomerable, we have also investigated their connection with continuity and super-additivity. Our results show that these models have good mathematical properties; although the connection with continuity in the precise case is well known, as it follows almost immediately from existing results from probability theory, in the imprecise case we have given a necessary and a sufficient condition, as well as a characterisation in terms of the natural extension from gambles with a finite range. In our view, this indicates that envelopes of countably additive linear previsions may be an interesting special class of fully conglomerable models for practical use. This is perhaps the main message that springs from the technical analysis in this paper in terms of embedding conglomerability in a viable way in probability.

The definition of joint coherence of a conditional and an unconditional lower prevision has led us to define the functional $\underline{Q}$ as a supremum of marginal extensions. We have shown that this functional can be used to characterise the full conglomerability, and that in general it provides a conservative approximation of the fully conglomerable natural extension, whenever the latter exists. We have also shown that $\underline{Q}$ is not coherent in general, and that in some particular cases its coherence is equivalent to its equality with the fully conglomerable natural extension.

A deeper study of this functional is one of the main open problems for future work; in particular, we would like to determine whether the existence of the fully conglomerable natural extension is equivalent to (and not only sufficient for) $\underline{Q}$ to avoid sure loss, and whether the coherence of $\underline{Q}$ is sufficient (and not only necessary) for its equality with the conglomerable natural extension of $\underline{P}$. These two problems are related to the extension of some of the results we have established in Section 6.1 to the general case.

More generally, it would be interesting to make a deeper comparison between our results and the ones established by Seidenfeld et al. for the precise case by means of full conditional measures. These are particularly interesting because they have been established also for unbounded random variables [Schervisch et al., 2014], while most work on coherent lower previsions only applies to bounded random variables, with the notable exception of Troffaes and de Cooman [2014].

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