Poset Representations for Sets of Elementary Triplets

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Abstract

Semi-graphoid independence relations, composed of independence triplets, are typically exponentially large in the number of variables involved. For compact representation of such a relation, just a subset of its triplets, called a basis, are listed explicitly, while its other triplets remain implicit through a set of derivation rules. Two types of basis were defined for this purpose, which are the dominant-triplet basis and the elementary-triplet basis, of which the latter is commonly assumed to be significantly larger in size in general. In this paper we introduce the *elementary po-triplet* as a compact representation of multiple elementary triplets, by using separating posets. By exploiting this new representation, the size of an elementary-triplet basis can be reduced considerably. For computing the elementary closure of a starting set of po-triplets, we present an elegant algorithm that operates on the least and largest elements of the separating posets involved.

Keywords: Independence relations, Efficiency of representation, Elementary closure computation.

1. Introduction

The notion of independence plays a key role in practical systems of uncertainty, since effective use of knowledge about independences allows these systems to deal with the computational complexity of their problem-solving tasks; probabilistic graphical models in fact build explicitly on this observation. To allow a study of independence without the numerical context involved, classical probabilistic independence has been formulated in independence relations and associated axiomatic systems describing their properties. The statements of an independence relation are called *triplets*, and describe the independence of two sets of random variables given a third, so-called, *separating set*. The most often studied axiomatic system of independence includes four axioms, called the *semi-graphoid axioms*. Any set of triplets over a set of variables that is closed under these axioms, is called a *semi-graphoid independence relation* (Geiger et al., 1991; Pearl, 1988).

Semi-graphoid independence relations are typically exponentially large in the number of variables involved, and representing them by enumeration of their triplets is infeasible from a practical point of view. A more concise representation of a semi-graphoid relation is arrived at by explicitly listing a small subset of its triplets, called a *basis*, and letting its other triplets be defined implicitly through the four semi-graphoid axioms (Studený, 1998); the full relation can then be generated by exploiting the axioms as derivation rules. Two types of basis have been proposed to this end: the *dominant-triplet* basis, composed of triplets such that any remaining triplet can be derived directly from *one* triplet from this basis (Studený, 1998), and the *elementary-triplet* basis. The latter type of basis is composed of all *elementary triplets* of a semi-graphoid relation, where an elementary triplet captures an independence between two individual variables. The dominant-triplet basis has been studied more intensively, since elementary-triplet bases are generally assumed to be larger in size than dominant-triplet bases for semi-graphoid relations.

We focus in this paper on elementary-triplet bases of semi-graphoid independence relations. We will introduce the *elementary po-triplet* as a compact representation of multiple elementary triplets. The triplets captured by a single po-triplet share a fixed pair of (conditionally) independent variables, and their separating sets form a partially ordered set. For the manipulation of po-triplets, we define various concepts and operators, among which are intersection and complete-union operators. Depending on the structure of a given semi-graphoid relation, a po-triplet representation of its set of elementary triplets will generally be more compact than the set of elementary triplets itself. Yet, a po-triplet representation of the full elementary-triplet basis of a relation can typically be further reduced, based on the observation that an elementary-triplet basis includes redundancies (Matúš, 1992; Peña, 2017; Bolt and van der Gaag, 2019). These redundancies originate from two axiomatic properties of elementary independence. A smaller elementary-triplet basis can therefore be constructed by letting specific triplets be defined implicitly through these two axioms; the full elementary-triplet basis can then be generated by exploiting these axioms as derivation rules. Building upon this observation, we will address the generation of a full elementary-triplet basis in po-triplet representation from a starting set of po-triplets. We show, more specifically, that the closure of a starting set of elementary po-triplets can be efficiently computed by means of a dedicated operator that manipulates just the least and largest elements of the separating posets of the po-triplets involved, without the need to explicitly generate all represented elementary triplets.

The paper is organised as follows. In Section 2, we provide some preliminaries on semi-graphoid independence relations in general and on the two most commonly studied types of basis. Section 3 then introduces our concept of elementary po-triplet as a compact representation of multiple elementary triplets. Section 4 addresses the computation of a full elementary-triplet basis in po-triplet representation from a given starting set of elementary po-triplets. The paper ends in Section 5 with our conclusions and envisioned further research.

2. Preliminaries

We consider a finite, non-empty set V of discrete random variables, with $|V| = n$, $n \geq 2$. We will use small letters to indicate separate variables from V and capital letters to denote sets of variables; when indicating individual variables in a set, we slightly abuse notational conventions and write xyz instead of $\{x, y, z\}$, as long as ambiguity cannot occur. An (ordered) *triplet* over V is a statement of the form $\theta = \langle X, Y \,|\, Z \rangle$, where $X, Y, Z \subseteq V$ are pairwise disjoint subsets with $X, Y \neq \varnothing$. The triplet θ indicates that the sets of variables X and Y are (conditionally) independent given the set Z; Z is called the *separating set* of X and Y in θ. A triplet $\theta = \langle X, Y \,|\, Z \rangle$ over V has an associated triplet $\theta^T = \langle Y, X \,|\, Z \rangle$, called its (*symmetric*) *transpose*, which is considered different from θ itself.

A set of triplets constitutes a *semi-graphoid independence relation* if it is closed under the well-known axioms of *symmetry, decomposition, weak union* and *contraction* (for further details, see (Pearl, 1988)). Semi-graphoid relations typically are exponentially large in the number of variables involved, which makes representing them by enumeration of their element triplets infeasible for practical purposes. By taking the four axioms as derivation rules however, semi-graphoid independence relations can be represented more concisely by enumerating a tailored subset of their triplets, called a *basis*, and letting all other triplets be defined implicitly through these rules. More formally, given a starting set of triplets J, we write $J \vdash^* \theta$ if the triplet θ can be derived from J by finite application of the semi-graphoid derivation rules. The *closure* of J, denoted by $\overline{J}$, then is the set of all triplets θ such that $J \vdash^* \theta$. A triplet set J is a *basis* for a semi-graphoid independency relation K if $\overline{J} = K$; we note that any triplet set J thereby constitutes a basis for its own closure $\overline{J}$.

Although different types of basis were proposed for independence relations, most attention has focused on the *dominant-triplet basis* introduced by Studený (1998). This basis of so-called dominant triplets builds on the convenient property that any triplet of a given semi-graphoid relation can be derived from one of its dominant triplets by means of the semi-graphoid derivation rules. A dominant-triplet basis thereby provides for efficiently solving the *membership problem* on semi-graphoid independence relations, which is the problem of deciding whether a specific triplet θ is an element of the closure $\overline{J}$ of a given triplet set J, or phrased alternatively, whether θ can be derived from J. For computing a dominant-triplet basis from a given starting set of triplets, an elegant algorithm is available (Studený, 1998; Baioletti et al., 2009; Lopatatzidis and van der Gaag, 2017), which has a dedicated operator for deriving (possibly) new dominant triplets at its core.

In this paper, we focus on another type of basis for semi-graphoid independence relations, called the *elementary-triplet basis*. An *elementary triplet* is a triplet of the form $\langle x, y \mid Z \rangle$ where x, y are individual variables (Matúš, 1992; Peña, 2017). Any non-empty semi-graphoid relation $\overline{J}$ includes two or more such elementary triplets. The set $\overline{J}^E$ of *all* elementary triplets of a semi-graphoid independence relation $\overline{J}$ is closed under the symmetry and equivalence axioms, which are defined for all variables $x, y, w \in V$ and separating sets $Z \subseteq V$ as:

E1: if $\langle x, y \mid Z \rangle \in \overline{J}^E$, then $\langle y, x \mid Z \rangle \in \overline{J}^E$ (*Symmetry*);

E2: if $\langle x, y \mid Z \rangle \in \overline{J}^E$ and $\langle x, w \mid Z \cup \{y\} \rangle \in \overline{J}^E$, then $\langle x, y \mid Z \cup \{w\} \rangle \in \overline{J}^E$ and $\langle x, w \mid Z \rangle \in \overline{J}^E$ (*Equivalence*).

The set $\overline{J}^E$ of all elementary triplets of a semi-graphoid independence relation $\overline{J}$ is known to constitute a basis for $\overline{J}$ (Matúš, 1992; Studený, 2005; Peña, 2017), and we will refer to this basis as the *full elementary-triplet basis* of $\overline{J}$. Full elementary-triplet bases have been shown to provide for efficiently solving some common problems on semi-graphoid independence relations, such as the above-mentioned membership problem and the problem of finding the intersection of two independence relations (Peña, 2017). A full elementary-triplet basis may include redundant triplets however, and a smaller elementary-triplet basis may exist (Peña, 2017; Bolt and van der Gaag, 2019), from which the full basis can be generated by taking the symmetry and equivalence axioms as derivation rules. We will call the closure, under the axioms E1–E2, of an arbitrary set of elementary triplets J^E, the *elementary closure* of J^E.

3. Introducing elementary po-triplets

For introducing our concept of *elementary po-triplet* as a compact representation of multiple elementary triplets, we focus on the two random variables x, y. We consider the powerset of $V \setminus \{x, y\}$ and take set inclusion as a partial order over its elements, to arrive at a poset with the empty set $\varnothing$ for its least element and $V \setminus \{x, y\}$ for its largest element; slightly abusing notation, we will denote this poset by $\mathcal{P}(V \setminus \{x, y\})$. We define a *full interval* $I_{\mathcal{P}}$ of $\mathcal{P}(V \setminus \{x, y\})$ to be a poset of elements such that, for any $A, C \in I_{\mathcal{P}}$ and $B \in \mathcal{P}(V \setminus \{x, y\})$, we have that: if $A \subseteq B \subseteq C$, then $B \in I_{\mathcal{P}}$. A full interval of $\mathcal{P}(V \setminus \{x, y\})$ with the least element Z and greatest element Z', will be indicated by $I_{\mathcal{P}} = [Z; Z']$; a full interval composed of a single element Z will be written as $[Z; Z]$. Figure 1 illustrates a commonly used graphical representation of posets, called a *Hasse diagram*: shown is the

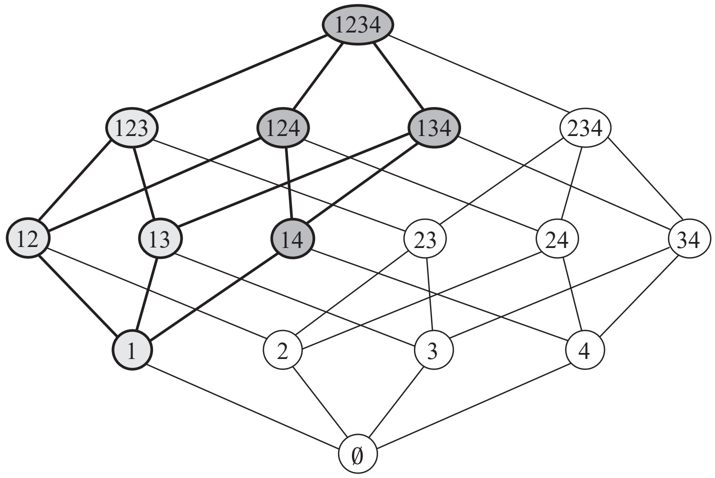

Figure 1: A Hasse diagram of the poset $\mathcal{P}(1234)$, with the full interval $[1; 1234]$ indicated in bold; within this interval, the full intervals $[1; 123]$ and $[14; 1234]$ are indicated by different shadings.

example poset $\mathcal{P}(1234)$, in which the full interval $[1; 1234]$ is indicated in bold. We now formally introduce our concept of elementary po-triplet.

Definition 1 *An* elementary po-triplet *over V is a statement of the form $\langle x, y \mid [Z; Z'] \rangle$ where $x, y \in V$ and $[Z; Z']$ is a full interval of the poset $\mathcal{P}(V \setminus \{x, y\})$.*

An elementary po-triplet $\sigma = \langle x, y \mid [Z; Z'] \rangle$ is taken to represent the set of elementary triplets $\{\langle x, y \mid W \rangle \mid Z \subseteq W \subseteq Z'\}$. To distinguish σ from the set of elementary triplets it describes, we will use the notation $\hat{\sigma}$ to indicate the projection of σ onto this set. We further use the notation σ^T to denote the *symmetric transpose* $\langle y, x \mid [Z; Z'] \rangle$ of the po-triplet $\sigma = \langle x, y \mid [Z; Z'] \rangle$, which captures the set of elementary triplets $\{\theta^T \mid \theta \in \hat{\sigma}\}$. The interval $[Z; Z']$ of a po-triplet $\sigma = \langle x, y \mid [Z; Z'] \rangle$ is called the *separating interval* for the variables x and y in σ. We note that, if the separating interval of σ has the set $Z' = V \setminus \{x, y\}$ for its largest element, then the elementary triplet $\theta = \langle x, y \mid Z \rangle$ is a *stable* triplet, with stability indicating that the variables x and y are independent given Z and any possible superset of Z (cf. de Waal and van der Gaag (2004)).

We now define various concepts and operators for elementary po-triplets. We say that an elementary triplet $\theta = \langle x, y \mid W \rangle$ is *included* in the po-triplet $\sigma = \langle x, y \mid [Z; Z'] \rangle$ if the separating set W of θ is an element of the separating interval of σ, or alternatively, if $\theta \in \hat{\sigma}$. We generalise this concept of inclusion to elementary po-triplets as follows.

Definition 2 *Let $\sigma_i = \langle x, y \mid [Z_i; Z'_i] \rangle$, $i = 1, 2$, be elementary po-triplets. Then, $\sigma_1 = \langle x, y \mid [Z_1; Z'_1] \rangle$ is* po-included *in $\sigma_2 = \langle x, y \mid [Z_2; Z'_2] \rangle$, denoted $\sigma_1 \trianglelefteq \sigma_2$, if $Z_2 \subseteq Z_1$ and $Z'_1 \subseteq Z'_2$.*

From the definition above, it is readily seen that $\sigma_1 \trianglelefteq \sigma_2$ if and only if $\hat{\sigma}_1 \subseteq \hat{\sigma}_2$, that is, σ_1 is po-included in σ_2 if and only if each elementary triplet described by σ_1 is also captured by σ_2.

We define the intersection of two elementary po-triplets.

Definition 3 *Let* $\sigma_i = \langle x, y \mid [Z_i; Z'_i]\,\rangle$, $i = 1, 2$, *be elementary po-triplets with* $Z_1 \cup Z_2 \subseteq Z'_1 \cap Z'_2$. *Then, the* po-intersection *of* σ_1 *and* σ_2, *denoted by* $\sigma_1 \sqcap \sigma_2$, *is the po-triplet defined as*

$$\sigma_1 \sqcap \sigma_2 = \langle x, y \mid [Z_1 \cup Z_2; Z'_1 \cap Z'_2]\,\rangle$$

From the definition above, we have that the po-intersection $\sigma_1 \sqcap \sigma_2$ of two elementary po-triplets σ_1, σ_2 is defined if and only if the condition $Z_1 \cup Z_2 \subseteq Z'_1 \cap Z'_2$ is met. The po-triplet $\sigma = \sigma_1 \sqcap \sigma_2$ then has $\hat{\sigma} = \hat{\sigma}_1 \cap \hat{\sigma}_2$, and by Definition 2 we further have that $\sigma \trianglelefteq \sigma_i$, $i = 1, 2$. If the condition $Z_1 \cup Z_2 \subseteq Z'_1 \cap Z'_2$ is not met for two po-triplets σ_1, σ_2, then there does not exist a separating set $W \subseteq V \setminus \{x, y\}$ and associated triplet $\theta = \langle x, y \mid W \rangle$ such that θ is included in both σ_1 and σ_2. The set $\hat{\sigma}_1 \cap \hat{\sigma}_2$ then is empty and the po-intersection of σ_1, σ_2 is not defined. We stress that the concept of po-intersection is stated in terms of *just* the least and largest elements of the separating intervals of the two po-triplets being intersected. From an algorithmic perpective therefore, using the above definition of po-intersection is more efficient than taking the intersection of the set projections of the two po-triplets and constructing a po-triplet representation of the result. We illustrate the concept of po-intersection by means of an example.

Example 1 *Using Figure 2 as a reference for the separating intervals involved, we consider the following three elementary po-triplets:*

$$\sigma_1 = \langle 5, 6 \mid [\varnothing, 123]\,\rangle \quad \sigma_2 = \langle 5, 6 \mid [13, 1234]\,\rangle \quad \sigma_3 = \langle 5, 6 \mid [13, 123]\rangle$$

To establish the po-intersection $\sigma_1 \sqcap \sigma_2$ *of the two po-triplets* σ_1, σ_2, *we first verify the condition for its existence. As* $Z_1 \cup Z_2 = \{1, 3\} \subseteq \{1, 2, 3\} = Z'_1 \cap Z'_2$, *we find that the po-intersection is indeed defined and that its separating interval equals* $[13; 123]$. *We conclude that* $\sigma_1 \sqcap \sigma_2 = \sigma_3$. □

While taking the po-intersection of two elementary po-triplets is quite straightforward, taking their union is more subtle as a consequence of the following observation: if we would define the po-union of two elementary po-triplets σ_1, σ_2 to correspond with simply taking the set union of their projections, then the resulting set of elementary triplets $\hat{\sigma}_1 \cup \hat{\sigma}_2$ would not always be representable by a single po-triplet. We illustrate this observation by means of an example.

Example 2 *Referring to Figure 1 for the separating intervals involved, we consider the following four elementary po-triplets*

$$\sigma_1 = \langle 5, 6 \mid [1; 123]\,\rangle \quad \sigma_2 = \langle 5, 6 \mid [14; 1234]\,\rangle \quad \sigma_3 = \langle 5, 6 \mid [12; 1234]\,\rangle \quad \sigma_4 = \langle 5, 6 \mid [1; 1234]\rangle$$

with their set projections $\hat{\sigma}_i$, $i = 1, \ldots, 4$. *We adopt the concept of po-union as suggested above, that is, defined as taking the union of the set projections of the po-triplets being united. For taking the po-union of the two po-triplets* σ_1, σ_2, *we establish the union of their set projections* $\hat{\sigma}_1$, $\hat{\sigma}_2$, *and find that it equals the set projection of the po-triplet* σ_4, *that is, we find that* $\hat{\sigma}_1 \cup \hat{\sigma}_2 = \hat{\sigma}_4$. *For the two po-triplets* σ_1, σ_2 *therefore, taking their po-union as suggested would result in the single po-triplet* σ_4. *We now address the po-union of the po-triplets* σ_1, σ_3. *Taking the union of their set projections results in the set of all elementary triplets* $\langle 5, 6 \,|\, W \rangle$ *with* $W \in \{1, 12, 13, 123, 124, 1234\}$.

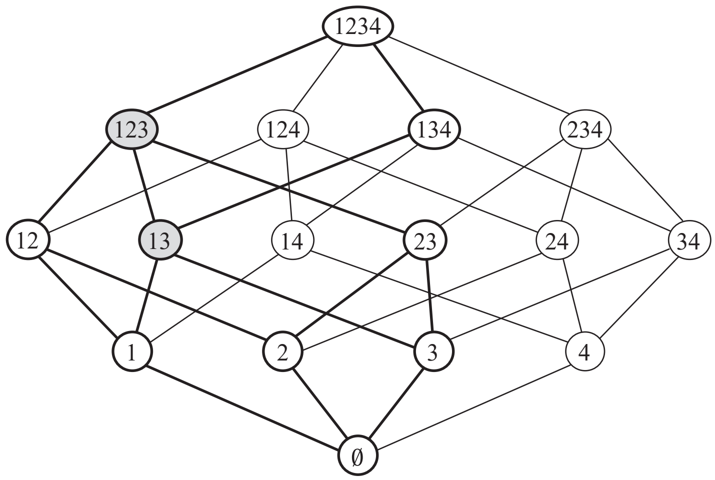

Figure 2: A Hasse diagram of the poset $\mathcal{P}(1234)$, with the full intervals $[\varnothing; 123]$ and $[13; 1234]$ indicated in bold; the elements shared by these intervals are shown in grey.

We observe that the separating sets W do not jointly constitute a full interval of the poset $\mathcal{P}(1234)$. More specifically, the set union $\hat{\sigma}_1 \cup \hat{\sigma}_3$ does not include the elementary triplets $\langle 5, 6 \mid W' \rangle$ with $W' = 14$ and $W' = 134$, respectively, which would be required to arrive at a full interval. We conclude that the set union $\hat{\sigma}_1 \cup \hat{\sigma}_3$ cannot be represented by a single po-triplet. □

The previous example demonstrated that defining the po-union of two po-triplets as taking the union of their set projections, does not always give a result that is representable by a single po-triplet. For now, we will nonetheless build on the concept of po-union as suggested above and define the *complete po-union* for two elementary po-triplets having the property that the union of their set projections can be described by a single po-triplet.

Definition 4 *Let $\sigma_i = \langle x, y \mid [Z_i; Z'_i] \rangle$, $i = 1, 2$, be elementary po-triplets. The po-triplets σ_1, σ_2 are said to have a* complete po-union*, denoted as $\sigma = \sigma_1 \sqcup \sigma_2$, if either of the following two conditions holds:*

- *if $\sigma_i \trianglelefteq \sigma_j$, $i, j = 1, 2$, then the complete po-union σ of σ_1, σ_2 equals $\sigma = \sigma_j$;*
- *if there exists a variable $z \in V \setminus (\{x, y\} \cup Z'_j)$ such that $Z_i = Z_j \cup \{z\}$ and $Z'_i = Z'_j \cup \{z\}$, $i, j = 1, 2$, $i \neq j$, then the complete po-union σ of σ_1, σ_2 equals $\sigma = \langle x, y \mid [Z_j; Z'_i] \rangle$.*

The above definition indicates the conditions under which two elementary po-triplets have a complete po-union. The first of these conditions covers the situation where the one po-triplet is included in the other one; in this situation, the complete po-union equals the including po-triplet. The second condition describes the situation where the separating intervals of the two po-triplets are order-isomorphic and the one interval is stacked directly on the other one, as illustrated in Figure 1 for the intervals $[1; 123]$ and $[14; 1234]$. In this situation, the order isomorphism implies that the two separating intervals have the same structure for their Hasse diagrams and, hence, are of equal size.

ELEMENTARY PO-CLOSURE

Input: a starting set J^{po} of elementary po-triplets;

Output: an elementary po-closure $\overline{J}^{po}$ of J^{po}.

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function *ElementaryPoClosure*(J^{po})

  $J_0 \leftarrow J^{po}$

  $i \leftarrow 0$

  repeat

    $i \leftarrow i + 1$

    $N_i \leftarrow \bigcup_{\sigma,\sigma' \in J_{i-1}} GE(\sigma, \sigma')$

    $J_i \leftarrow UnitePoTriplets(J_{i-1} \cup N_i)$

  until $J_i = J_{i-1}$

  $J_i \leftarrow UnitePoTriplets(\,AddTransposes(J_i)\,)$

  return $\overline{J}^{po} \leftarrow J_i$

end function

Figure 3: An outline of our algorithm for computing an elementary po-closure of a set of po-triplets.

As the two separating intervals moreover are stacked directly on top of one another, the order isomorphism is captured by a function that inserts a *single* new variable in each element of the interval of the one po-triplet, to give the elements of the separating interval of the other one. As this function specifies a partial order between the elements of the two intervals conform to the partial order of the poset $\mathcal{P}(V \setminus \{x, y\})$, the separating intervals are guaranteed to jointly constitute a full interval of the poset. We note that, like the concept of po-intersection, complete po-union is stated in terms of *just* the least and largest elements of the separating intervals of the elementary po-triplets being united.

Our concept of elementary po-triplet provides for a more compact representation of a semi-graphoid independence relation than the relation's set of elementary triplets itself, in the sense that the po-triplet representation generally includes fewer statements. While for a given pair of variables x, y, the maximum number of elementary triplets in an independence relation equals the size of the powerset $\mathcal{P}(V \setminus \{x, y\})$, a po-triplet representation of the elementary triplets of any such relation includes maximally half this number of statements. As the number of separating sets in an interval $[Z; Z']$, $Z, Z' \in \mathcal{P}(V \setminus \{x, y\})$, $Z \subseteq Z'$, is exponential in $|Z' \setminus Z|$, a single po-triplet can represent exponentially many elementary triplets. Depending on the structure of a given semi-graphoid relation therefore, the po-triplet representation of its set of elementary triplets is likely to be more compact than the representation by the elementary triplets themselves. For a given set of elementary triplets, typically multiple po-triplet representations exist, ranging from a separate po-triplet per elementary triplet, to a representation by the smallest possible number of po-triplets. In the remainder of this paper, we assume po-triplet representations in which no two po-triplets have a complete po-union. Such a representation may not be the smallest possible, as the separating intervals of multiple po-triplets may possibly be re-arranged to give a representation by a smaller number of statements; we leave the problem of constructing smaller po-triplet representations for our further research.

4. Computing elementary po-closures

In the previous section, we introduced the concept of elementary po-triplet for compact representation of sets of elementary triplets. As argued in Section 2, full sets of elementary triplets have been studied mostly in the context of representing independence relations. In this section, we address the computation of a full set of elementary triplets $\overline{J}^E$ from a defining subset J^E by means of the derivation rules E1–E2. We will argue more specifically that a po-triplet representation of $\overline{J}^E$ can be computed *directly* from a po-triplet representation of the set J^E, without the need to explicitly generate its full set projection. Figure 3 provides an outline of our algorithm for this purpose. The core of our algorithm is an iterative loop (lines 4–8), in which new po-triplets are constructed by means of a dedicated operator called *GE* (line 6) and a compact po-triplet representation is built from the po-triplets obtained thus far, by means of the *UnitePoTriplets* function (line 7). We discuss some of the steps of our algorithm in detail.

We begin the discussion of the concepts underlying our algorithm for po-closure computation by defining the *generalised equivalence operator*, or *ge-operator* for short, which basically implements the equivalence rule E2 tailored to po-triplets; this *ge*-operator will be extended shortly to the *GE*-operator used by our algorithm for constructing po-triplets.

Definition 5 *Let* $\sigma_1 = \langle x, y \mid [Z_1; Z_1'] \rangle$, $\sigma_2 = \langle x, w \mid [Z_2; Z_2'] \rangle$, $y \neq w$, *be elementary po-triplets. Then, the ge-*operator *is defined by*

- $ge(\sigma_1, \sigma_2) = \{\sigma_1', \sigma_2'\}$ *with*
  - $\sigma_1' = \langle x, y \mid [((Z_1 \cup Z_2) \setminus \{y\}) \cup \{w\};\ (Z_1' \cap Z_2') \cup \{w\}] \rangle$;
  - $\sigma_2' = \langle x, w \mid [(Z_1 \cup Z_2) \setminus \{y\};\ Z_1' \cap Z_2'] \rangle$;

  if $(Z_1 \cup \{y\}) \cup Z_2 \subseteq (Z_1' \cup \{y\}) \cap Z_2'$;
- $ge(\sigma_1, \sigma_2) = \varnothing$, *otherwise.*

The *ge*-operator essentially applies the equivalence rule E2 to each possible (ordered) pair of elementary triplets (θ_1, θ_2), $\theta_i \in \hat{\sigma}_i$, in the set projections of its argument po-triplet σ_1, σ_2; the set of elementary triplets resulting from these multiple applications is returned in po-triplet representation. The *ge*-operator does not require the set projections of the po-triplets σ_1, σ_2 to be generated explicitly, but instead establishes its result by direct manipulation of the least and largest elements of the separating intervals of its argument po-triplets. We note that the operator embeds po-intersection for this purpose, to guarantee that the equivalence rule is applied only to the relevant parts of the separating intervals of σ_1, σ_2. More specifically, the condition $(Z_1 \cup \{y\}) \cup Z_2 \subseteq (Z_1' \cup \{y\}) \cap Z_2'$ for the result to be non-empty, formulates the existence of separating sets Z that are shared by the intervals $[Z_1 \cup \{y\}; Z_1' \cup \{y\}]$ and $[Z_2; Z_2']$, as required by the equivalence rule. By these considerations, application of the equivalence rule to a set of po-triplets by means of the *ge*-operator, is more efficient from an algorithmic perspective than its application to the starting set of elementary triplets itself. We illustrate application of the *ge*-operator by means of an example.

Example 3 *We consider the two po-triplets* $\sigma_1 = \langle 6, 1 \mid [3; 345] \rangle$ *and* $\sigma_2 = \langle 6, 2 \mid [13, 1347] \rangle$, *and address application of the ge-operator to the ordered pair* (σ_1, σ_2). *We now first observe that the* *condition for $ge(\sigma_1, \sigma_2)$ to be non-empty is met, as*

$$(Z_1 \cup \{y\}) \cup Z_2 \; = \; \{1,3\} \; \subseteq \; \{1,3,4\} \; = \; (Z_1' \cup \{y\}) \cap Z_2'$$

The interval $[13;134]$ thus identified by the criterion, captures the separating sets of the second-argument triplets of the triplet pairs from $\hat{\sigma}_1$, $\hat{\sigma}_2$ to which the equivalence rule can be applied:

- *(θ_1, θ_2) with $\theta_1 = \langle 6,1 \mid 3\rangle \in \hat{\sigma}_1$ and $\theta_2 = \langle 6,2 \mid 13\rangle \in \hat{\sigma}_2$;*
- *(θ_1', θ_2') with $\theta_1' = \langle 6,1 \mid 34\rangle \in \hat{\sigma}_1$ and $\theta_2' = \langle 6,2 \mid 134\rangle \in \hat{\sigma}_2$.*

Application of the rule to these pairs returns the two triplet sets $\{\langle 6,1 \mid 23\rangle, \langle 6,2 \mid 3\rangle\}$ and $\{\langle 6,1 \mid 234\rangle, \langle 6,2 \mid 34\rangle\}$, respectively. For the ordered po-triplet pair (σ_1, σ_2), the ge-operator returns

$$ge(\sigma_1, \sigma_2) = \{\, \langle 6,1 \mid [23;234]\rangle,\ \langle 6,2 \mid [3;34]\rangle \,\}$$

We observe that this set of po-triplets describes the four elementary triplets that result from application of the equivalence rule to all possible ordered pairs of triplets from the set projections $\hat{\sigma}_1$, $\hat{\sigma}_2$, respectively. We emphasize that the ge-operator is applied to ordered *pairs of po-triplets. For the same po-triplets σ_1, σ_2, we find in fact that $ge(\sigma_2, \sigma_1) = \varnothing$ as the condition for a non-empty result is not met for the reversed pair.* □

We now show that the *ge*-operator, when applied to the ordered po-triplet pair (σ_1, σ_2), returns a set of po-triplets that include all elementary triplets that can be derived by application of the equivalence rule to the ordered pairs of triplets from the set projections $\hat{\sigma}_1, \hat{\sigma}_2$, and does not include any other triplet. In the proof, we use the notation $\vdash^{E2}$ to indicate an application of the equivalence rule E2, that is, we write $(\,\langle x, y \mid Z\rangle, \langle x, w \mid Z \cup \{y\}\rangle\,) \;\vdash^{E2}\; \{\langle x, y \mid Z \cup \{w\}\rangle, \langle x, w \mid Z\rangle\}$.

Proposition 6 *Let $\sigma_1 = \langle x, y \mid [Z_1; Z_1']\rangle$, $\sigma_2 = \langle x, w \mid [Z_2; Z_2']\rangle$, $y \neq w$, be elementary po-triplets, and let $ge(\sigma_1, \sigma_2) = \{\sigma_1', \sigma_2'\}$ be conform Definition 5. Then,*

(i) for all ordered triplet pairs (θ_1, θ_2), $\theta_i \in \hat{\sigma}_i$, $i = 1, 2$, it holds that: if $(\theta_1, \theta_2) \;\vdash^{E2}\; \{\theta_1', \theta_2'\}$, then $\theta_i' \in \hat{\sigma}_i'$;

(ii) for each $\theta' \in \hat{\sigma}_1' \cup \hat{\sigma}_2'$, there exists an ordered triplet pair (θ_1, θ_2), $\theta_i \in \hat{\sigma}_i$, $i = 1, 2$, such that $(\theta_1, \theta_2) \vdash^{E2} \theta'$.

Proof To prove the first property of the proposition, we consider an ordered pair of elementary triplets (θ_1, θ_2), $\theta_i \in \hat{\sigma}_i$, $i = 1, 2$, with $(\theta_1, \theta_2) \vdash^{E2} \{\theta_1', \theta_2'\}$. Since application of the equivalence rule to the pair (θ_1, θ_2) yields valid triplets, there must exist a separating set Z such that $\theta_1 = \langle x, y \,|\, Z\rangle \in \hat{\sigma}_1$ and $\theta_2 = \langle x, w \,|\, Z \cup \{y\}\rangle \in \hat{\sigma}_2$. By definition, we have that this set Z does not include the variables y or w; we further know that $Z_1 \subseteq Z \subseteq Z_1'$ and $Z_2 \subseteq Z \cup \{y\} \subseteq Z_2'$. By the equivalence rule, we conclude that $\theta_1' = \langle x, y \,|\, Z \cup \{w\}\rangle$ and $\theta_2' = \langle x, w \,|\, Z\rangle$. Now, to show that $\theta_i' \in \hat{\sigma}_i'$, we first verify that the *ge*-operator, when applied to the po-triplet pair (σ_1, σ_2), yields a non-empty set of po-triplets. By the existence of the separating set Z with the inclusion properties above, we find that $(Z_1 \cup \{y\}) \cup Z_2 \;\subseteq\; Z \cup \{y\} \;\subseteq\; (Z_1' \cup \{y\}) \cap Z_2'$ and, hence, that the condition for the operator to yield a non-empty result is met. It follows that $ge(\sigma_1, \sigma_2) = \{\sigma_1', \sigma_2'\}$ with σ_i', $i = 1, 2$, conform

Definition 5. From the observation that $((Z_1 \cup Z_2) \setminus \{y\}) \cup \{w\} \subseteq Z \cup \{w\} \subseteq (Z'_1 \cap Z'_2) \cup \{w\}$, we conclude that $\theta'_1 \in \hat{\sigma}'_1$. By an analogous argument, we find that $\theta'_2 \in \hat{\sigma}'_2$.

To prove property (*ii*) of the proposition, we consider an arbitrary triplet $\theta'_1 \in \hat{\sigma}'_1$. As θ'_1 must have resulted from application of the *ge*-operator to the po-triplet pair (σ_1, σ_2), there must exist an associated triplet $\theta'_2 \in \hat{\sigma}'_2$. By the definition of the set projections $\hat{\sigma}'_1$, $\hat{\sigma}'_2$, we know that the triplet θ'_1 is of the form $\theta'_1 = \langle x, y \mid Q \cup \{w\} \rangle$ for some separating set Q with $(Z_1 \cup Z_2) \setminus \{y\} \subseteq Q \subseteq Z'_1 \cap Z'_2$ and that θ'_2 then must be of the form $\theta'_2 = \langle x, w \mid Q \rangle$. We now show that the po-triplets σ_1, σ_2 with $ge(\sigma_1, \sigma_2) = \{\hat{\sigma}_1, \hat{\sigma}_2\}$ include triplets θ_1, θ_2 of the forms $\theta_1 = \langle x, y \mid Q \rangle$ and $\theta_2 = \langle x, w \mid Q \cup \{y\} \rangle$, respectively. From the inclusion property $(Z_1 \cup Z_2) \setminus \{y\} \subseteq Q \subseteq Z'_1 \cap Z'_2$ of the set Q, we find that $Z_1 \subseteq (Z_1 \cup Z_2) \setminus \{y\} \subseteq Q$ as $y \notin Z_1$ and, moreover, that $Q \subseteq Z'_1 \cap Z'_2 \subseteq Z'_1$. It follows that Q is an element of the interval $[Z_1; Z'_1]$ and, hence, that $\theta_1 \in \hat{\sigma}_1$. From the inclusion property of $Q \cup \{y\}$, we further find that $Z_2 \subseteq Z_1 \cup Z_2 \cup \{y\} \subseteq Q \cup \{y\} \subseteq (Z'_1 \cap Z'_2) \cup \{y\} \subseteq Z'_2$, where the latter property follows from the condition of the *ge*-operator being met. It follows that $Q \cup \{y\}$ is an element of the interval $[Z_2; Z'_2]$ and, hence, that $\theta_2 \in \hat{\sigma}_2$. It is now easily verified that $(\theta_1, \theta_2) \vdash^{E2} \theta'_1$ as stated in property (*ii*) of the proposition. By an analogous line of arguments, a similar property is found for an arbitrary triplet $\theta'_2 \in \hat{\sigma}'_2$. ■

The *ge*-operator defined above takes for its argument an ordered pair of po-triplets that have a fixed order for their separated variables. For constructing, from an ordered pair of elementary po-triplets (σ_1, σ_2), *all* possible po-triplets that can be derived not just by the equivalence derivation rule but also by the rule of symmetrythe derivation rules, we define an enhanced operator, called the *GE-operator*. The operator takes a pair of po-triplets for its argument, and applies the basic *ge*-operator to the pair's po-triplets and all their symmetric transposes:

$$GE(\sigma_1, \sigma_2) \quad = \quad ge(\sigma_1, \sigma_2) \ \cup \ ge(\sigma_1^T, \sigma_2) \ \cup \ ge(\sigma_1, \sigma_2^T) \ \cup \ ge(\sigma_1^T, \sigma_2^T)$$

This enhanced *GE*-operator is the operator used in our algorithm for po-closure of Figure 3. We note that the selection of the reversed po-triplet pair (σ_2, σ_1) is covered by the algorithm's loop.

As discussed above, the core of our algorithm for elementary po-closure is an iterative loop in which new po-triplets are constructed by means of the *GE*-operator. The set of po-triplets that results after applying this operator to all possible po-triplet pairs thus far, may already be smaller in size than its set projection. The size of the resulting set of po-triplets can often be further reduced however, by uniting po-triplets that share the same pair of conditionally independent variables. For this purpose, the *UnitePoTriplets* function, called from the main loop of our algorithm (line 7), implements iteratively taking complete po-unions until no further changes are induced in the po-triplet set at hand. In the final step of our algorithm, subsequent to the main loop, the construction of the elementary po-closure is completed by adding to the representation, the symmetric transposes that were not derived explicitly. We conclude the discussion of our algorithm for elementary po-closure, by briefly commenting on our algorithm for uniting po-triplets, oulined in Figure 4. The core of this algorithm is an iterative loop in which complete po-unions are constructed (line 6) and po-included po-triplets are removed (line 7). We emphasize that the *UnitePoTriplets* function builds upon the concept of *complete* po-union only for combining po-triplets, and thereby is not guaranteed to result in a smallest possible set of po-triplets describing the elementary po-closure of a given starting set. Also the order in which po-triplets are addressed by the *UnitePoTriplets* function will

UNITING PO-TRIPLETS

Input: a set J^{po} of elementary po-triplets;

Output: a set J^{po}_{-} in which no two elements have a complete po-union.

function *UnitePoTriplets*(J^{po})
  $J_0 \leftarrow J^{po}$
  $i \leftarrow 0$
  repeat
    $i \leftarrow i + 1$
    $N_i \leftarrow \bigcup_{\sigma, \sigma' \in J_{i-1}} \ \sigma \sqcup \sigma'$
    $J_i \leftarrow$ *RemovePoIncluded*$(\mathrm{J}_{i-1} \cup N_i)$
  until $J_i = J_{i-1}$
  return $J^{po}_{-} \leftarrow J_i$
end function

Figure 4: An outline of our algorithm for iteratively uniting po-triplets.

affect the size of the po-triplet representation returned for an elementary closure. These observations suggest that our algorithm for uniting elementary po-triplets leaves ample room for optimization.

5. Conclusions

In this paper we proposed the compact representation of sets of elementary triplets by *elementary po-triplets*. Using separating posets instead of separating sets to describe independence of individual variables, a single po-triplet has the potential to represent an exponential number of elementary triplets. A po-triplet representation may thus be exponentially smaller in size than the represented set of elementary triplets itself. For manipulating sets of po-triplets, we defined intersection and complete-union operators. Application of these operators involves the manipulation of just the least and largest elements of separating posets, without the need to generate the set projections of the po-triplets at hand. We further outlined a basic algorithm for computing the po-closure of a starting set of elementary po-triplets under the symmetry and equivalence axioms; we introduced the dedicated *ge-operator* for parallel application of the equivalence rule to the multiple pairs of elementary triplets described by its argument po-triplets.

Elementary-triplet bases have become of interest because they allow specific problems on independence relations to be solved efficiently, such as the problem of finding the intersection of two semi-graphoid relations. We are in the process of looking into the efficiency of solving this and various other well-known problems on semi-graphoid relations in po-triplet representation. We further intend to optimize our basic algorithm for computing elementary po-closures of arbitrary sets of po-triplets and hope to report the results from our further investigations in the near future.

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