Algebras of Sets and Coherent Sets of Gambles

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Abstract. In a recent work we have shown how to construct an information algebra of coherent sets of gambles defined on general possibility spaces. Here we analyze the connection of such an algebra with the *set algebra* of sets of its *atoms* and the *set algebra* of subsets of the possibility space on which gambles are defined. Set algebras are particularly important information algebras since they are their *prototypical* structures. Furthermore, they are the algebraic counterparts of classical propositional logic. As a consequence, this paper also details how propositional logic is naturally embedded into the theory of *imprecise probabilities*.

Keywords: Desirability · Information algebras · Order theory · Imprecise probabilities · Coherence.

1 Introduction and Overview

While analysing the compatibility problem of coherent sets of gambles, Miranda and Zaffalon [14] have recently remarked that their main results could be obtained also using the theory of information algebras [7]. This observation has been taken up and deepened in some of our recent works [13, 18]: we have shown that the founding properties of desirability can in fact be abstracted into properties of information algebras. Stated differently, desirability makes up an information algebra of coherent sets of gambles.

Information algebras are algebraic structures composed by 'pieces of information' that can be manipulated by operations of *combination*, to aggregate them, and *extraction*, to extract information regarding a specific question. From the point of view of information algebras, sets of gambles defined on a possibility space Ω are pieces of information about Ω. It is well known that coherent sets of gambles are ordered by inclusion and, in this order, there are maximal elements [4]. In the language of information algebras such elements are called *atoms*. In particular, any coherent set of gambles is contained in a maximal set (an atom) and it is the intersection (meet) of all the atoms it is contained in. An information algebra with these properties is called atomistic. Atomistic information algebras have the universal property of being embedded in a set algebra, which is an information algebra whose elements are sets. This is an important representation theorem for information algebras, since set algebras are a special kind of algebras based on the usual set operations. Set algebras are a kind of prototype information algebras, as fields of sets are prototypes of Boolean algebras, it matters to know that the information algebra of coherent sets of gambles is also of this kind, although this is not evident at the first sight. This result is similar to the well-known representation theorems for Boolean algebras and Priestley duality for distributive lattices, see [1]. Similar general representation theorems for a special type of general information algebras (commutative ones) have been presented in [7, 12]. It is conjectured (but remains to be proved) that these results carry over to the more general type of information algebras proposed in [8] and to which the algebra of coherent sets of gambles belongs. Conversely, any such set algebra of subsets of Ω is embedded in the algebra of coherent sets of gambles defined on Ω. These links between set algebras and the algebra of coherent sets of gambles are the main topic of the present work. Since set algebras are algebraic counterparts of classical propositional logic, the results of this paper detail as well how the latter is formally part of the theory of imprecise probabilities [17]. We refer also to [3] for another aspect of this issue.

After recalling the main concepts introduced in our previous work in Sections 2–4, in Section 5 we establish some results about atoms needed for the subsequent sections. In Section 6 we define the concept of *embedding* for the information algebras of interest and finally, in Section 7, we show the links between set algebras (of subsets of Ω and of sets of atoms) and the algebra of coherent sets of gambles.

2 Desirability

Consider a set Ω of possible worlds. A gamble over this set is a bounded function $f : \Omega \to \mathbb{R}$. It is interpreted as an uncertain reward in a linear utility scale. A subject might desire a gamble or not, depending on the information they have about the experiment whose possible outcomes are the elements of Ω. We denote the set of all gambles on Ω by $\mathcal{L}(\Omega)$, or more simply by $\mathcal{L}$, when there is no possible ambiguity. We also introduce $\mathcal{L}^+(\Omega) := \{f \in \mathcal{L}(\Omega) : \ f \geq 0, f \neq 0\}$, or simply $\mathcal{L}^+$ when no ambiguity is possible, the set of non-negative non-vanishing gambles. These gambles should always be desired, since they may increase the wealth with no risk of decreasing it. As a consequence of the linearity of our utility scale, we assume also that a subject disposed to accept the transactions represented by f and g, is disposed to accept also $\lambda f + \mu g$ with $\lambda, \mu \geq 0$ not both equal to 0. More generally speaking, we consider the notion of a coherent set of gambles [17]:

Definition 1 (Coherent set of gambles). *We say that a subset $\mathcal{D}$ of $\mathcal{L}$ is a* coherent *set of gambles if and only if $\mathcal{D}$ satisfies the following properties:*

D1. $\mathcal{L}^+ \subseteq \mathcal{D}$ *[Accepting Partial Gains];*
D2. $0 \notin \mathcal{D}$ *[Avoiding Status Quo];*
D3. $f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$ *[Additivity];*

D4. $f \in \mathcal{D}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{D}$ *[Positive Homogeneity].*

So, $\mathcal{D}$ is a convex cone. Let us denote with $C(\Omega)$, or simply with C, the family of coherent sets of gambles on Ω. This leads to the concept of natural extension.

Definition 2 (Natural extension for gambles). *Given a set $\mathcal{K} \subseteq \mathcal{L}$, we call $\mathcal{E}(\mathcal{K}) := \mathsf{posi}(\mathcal{K} \cup \mathcal{L}^+)$, where $\mathsf{posi}(\mathcal{K}') := \left\{\sum_{j=1}^{r} \lambda_j f_j : f_j \in \mathcal{K}', \lambda_j > 0, r \geq 1\right\}$, for every set $\mathcal{K}' \subseteq \mathcal{L}$, its* natural extension.

$\mathcal{E}(\mathcal{K})$ of a set of gambles $\mathcal{K}$ is coherent if and only if $0 \notin \mathcal{E}(\mathcal{K})$.

In [18] we showed that $\Phi(\Omega) := C(\Omega) \cup \{\mathcal{L}(\Omega)\}$, or simply Φ when there is no possible ambiguity, is a complete lattice under inclusion [1], where meet is intersection and join is defined for any family of sets $\{\mathcal{D}_i\}_{i \in I} \in \Phi$ as

$$\bigvee_{i \in I} \mathcal{D}_i := \bigcap \left\{ \mathcal{D} \in \Phi : \bigcup_{i \in I} \mathcal{D}_i \subseteq \mathcal{D} \right\}.$$

Note that, if the family of coherent sets $\mathcal{D}_i$ has no upper bound in C, then its join is simply $\mathcal{L}$. Moreover, we defined the following closure operator [1] on subsets of gambles $\mathcal{K} \subseteq \mathcal{L}$

$$\mathcal{C}(\mathcal{K}) := \bigcap \{\mathcal{D} \in \Phi : \mathcal{K} \subseteq \mathcal{D}\}. \tag{1}$$

It is possible to notice that $\mathcal{C}(\mathcal{K}) = \mathcal{E}(\mathcal{K})$ if $0 \notin \mathcal{E}(\mathcal{K})$, that is if $\mathcal{E}(\mathcal{K})$ is coherent, otherwise $\mathcal{C}(\mathcal{K}) = \mathcal{L}$ and we may have $\mathcal{E}(\mathcal{K}) \neq \mathcal{C}(\mathcal{K})$.

The most informative cases of coherent sets of gambles, i.e., coherent sets that are not proper subsets of other coherent sets, are called *maximal*. The following proposition provides a characterisation of such maximal elements [4, Proposition 2].

Proposition 1 (Maximal coherent set of gambles). *A coherent set of gambles $\mathcal{D}$ is* maximal *if and only if $(\forall f \in \mathcal{L} \setminus \{0\})\ f \notin \mathcal{D} \Rightarrow -f \in \mathcal{D}$.*

We shall denote maximal sets with M to differentiate them from the general case of coherent sets. These sets play an important role with respect to information algebras (see Section 5). Another important class is that of *strictly desirable* sets of gambles [17].[3]

Definition 3 (Strictly desirable set of gambles). *A coherent set of gambles $\mathcal{D}$ is said to be* strictly desirable *if and only if it satisfies $(\forall f \in \mathcal{D} \setminus \mathcal{L}^+)(\exists \delta > 0)\ f - \delta \in \mathcal{D}$.*

For strictly desirable sets, we shall employ the notation $\mathcal{D}^+$.

[3] Strictly desirable sets of gambles are important since they are in a one-to-one correspondence with *coherent lower previsions*, a generalization of the usual expectation operator on gambles. Given a coherent lower prevision $\underline{P}(\cdot)$, $\mathcal{D}^+ := \{f \in \mathcal{L} : \underline{P}(f) > 0\} \cup \mathcal{L}^+$ is a strictly desirable set of gambles [17, Section 3.8.1].

3 Stucture of Questions and Possibilities

In this section we review the main results about the structure of Ω [8, 9, 18]. With reference to our previous work [18], we recall that coherent sets of gambles are understood as pieces of information describing beliefs about the elements in Ω. Beliefs may be originally expressed relative to different questions or variables that we identify by families of equivalence relations $\equiv_x$ on Ω for x in some index set Q [5, 8, 9].[4] A question $x \in Q$ has the same answer in possible worlds $\omega \in \Omega$ and $\omega' \in \Omega$, if $\omega \equiv_x \omega'$. There is a partial order between questions capturing granularity: question y is finer than question x if $\omega \equiv_y \omega'$ implies $\omega \equiv_x \omega'$. This can be expressed equivalently considering partitions $\mathcal{P}_x$, $\mathcal{P}_y$ of Ω whose blocks are respectively, the equivalence classes $[\omega]_x$, $[\omega]_y$ of the equivalence relations $\equiv_x$, $\equiv_y$, representing possible answers to x and y. Then $\omega \equiv_y \omega'$ implying $\omega \equiv_x \omega'$, means that any block $[\omega]_y$ of partition $\mathcal{P}_y$ is contained in some block $[\omega]_x$ of partition $\mathcal{P}_x$. If this is the case, we say that: $x \leq y$ or $\mathcal{P}_x \leq \mathcal{P}_y$.[5] Partitions $Part(\Omega)$ of any set Ω, form a lattice under this order [6]. In particular, the partition $\sup\{\mathcal{P}_x, \mathcal{P}_y\} \coloneqq \mathcal{P}_x \vee \mathcal{P}_y$ of two partitions $\mathcal{P}_x, \mathcal{P}_y$ is, in this order, the partition obtained as the non-empty intersections of blocks of $\mathcal{P}_x$ with blocks of $\mathcal{P}_y$. It can be equivalently expressed also as $\mathcal{P}_{x \vee y}$. The definition of the meet $\mathcal{P}_x \wedge \mathcal{P}_y$, or equivalently $\mathcal{P}_{x \wedge y}$, is somewhat involved [6]. We usually assume that the set of questions Q analyzed, considered together with their associated partitions denoted with $\mathcal{Q} \coloneqq \{\mathcal{P}_x : x \in Q\}$, is a join-sub-semilattice of $(Part(\Omega), \leq)$ [1]. In particular, we assume often that the top partition in $Part(\Omega)$, i.e. $\mathcal{P}_\top$ (where the blocks are singleton sets $\{\omega\}$ for $\omega \in \Omega$), belongs to $\mathcal{Q}$. A gamble f on Ω is called *x-measurable*, iff for all $\omega \equiv_x \omega'$ we have $f(\omega) = f(\omega')$, that is, if f is constant on every block of $\mathcal{P}_x$. It could then also be considered as a function (a gamble) on the set of blocks of $\mathcal{P}_x$. We denote with $\mathcal{L}_x(\Omega)$, or more simply with $\mathcal{L}_x$ when no ambiguity is possible, the set of all x-measurable gambles. We recall also the logical independence and conditional logical independence relation between partitions [8, 9].

Definition 4 (Independent Partitions). *For a finite set of partitions $\mathcal{P}_1, \dots, \mathcal{P}_n \in Part(\Omega)$, $n \geq 2$, let us define*

$$R(\mathcal{P}_1, \dots, \mathcal{P}_n) \coloneqq \{(B_1, \dots, B_n) : B_i \in \mathcal{P}_i, \cap_{i=1}^n B_i \neq \emptyset\}.$$

We call the partitions independent*, if $R(\mathcal{P}_1, \dots, \mathcal{P}_n) = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$.*

Definition 5 (Conditionally Independent Partitions). *Consider a finite set of partitions $\mathcal{P}_1, \dots \mathcal{P}_n \in Part(\Omega)$, and a block B of a partition $\mathcal{P}$ (contained or not in the list $\mathcal{P}_1, \dots, \mathcal{P}_n$), then define for $n \geq 1$,*

$$R_B(\mathcal{P}_1, \dots, \mathcal{P}_n) \coloneqq \{(B_1, \dots, B_n) : B_i \in \mathcal{P}_i, \cap_{i=1}^n B_i \cap B \neq \emptyset\}.$$

[4] This view generalises the most often used *multivariate model*, where questions are represented by families of variables and theirs domains [7, 8].

[5] In the literature usually the inverse order between partitions is considered. However, this order better corresponds to our natural order of questions by granularity.

We call $\mathcal{P}_1, \ldots, \mathcal{P}_n$ conditionally independent given $\mathcal{P}$, *if for all blocks* B *of* $\mathcal{P}$, $R_B(\mathcal{P}_1, \ldots, \mathcal{P}_n) = R_B(\mathcal{P}_1) \times \cdots \times R_B(\mathcal{P}_n)$.

This relation holds if and only if $B_i \cap B \neq \emptyset$ for all $i = 1, \ldots, n$, implies $B_1 \cap \ldots \cap B_n \cap B \neq \emptyset$. In this case we write $\perp\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}|\mathcal{P}$ or, for $n = 2$, $\mathcal{P}_1 \perp \mathcal{P}_2|\mathcal{P}$. $\mathcal{P}_x \perp \mathcal{P}_y|\mathcal{P}_z$ can be indicated also with $x \perp y|z$. We may also say that $\mathcal{P}_1 \perp \mathcal{P}_2|\mathcal{P}$ if and only if $\omega \equiv_{\mathcal{P}} \omega'$ implies the existence of an element $\omega'' \in \Omega$ such that $\omega \equiv_{\mathcal{P}_1 \vee \mathcal{P}} \omega''$ and $\omega' \equiv_{\mathcal{P}_2 \vee \mathcal{P}} \omega''$. The three-place relation $\mathcal{P}_1 \perp \mathcal{P}_2|\mathcal{P}$, in particular, satisfies the following properties:

Theorem 1. *Given* $\mathcal{P}, \mathcal{P}', \mathcal{P}_1, \mathcal{P}_2 \in Part(\Omega)$, *we have:*

C1 $\mathcal{P}_1 \perp \mathcal{P}_2|\mathcal{P}_2$;
C2 $\mathcal{P}_1 \perp \mathcal{P}_2|\mathcal{P}$ *implies* $\mathcal{P}_2 \perp \mathcal{P}_1|\mathcal{P}$;
C3 $\mathcal{P}_1 \perp \mathcal{P}_2|\mathcal{P}$ *and* $\mathcal{P}' \leq \mathcal{P}_2$ *imply* $\mathcal{P}_1 \perp \mathcal{P}'|\mathcal{P}$;
C4 $\mathcal{P}_1 \perp \mathcal{P}_2|\mathcal{P}$ *implies* $\mathcal{P}_1 \perp \mathcal{P}_2 \vee \mathcal{P}|\mathcal{P}$.

From these properties it follows also: $\mathcal{P}_x \perp \mathcal{P}_y|\mathcal{P}_z \iff \mathcal{P}_{x \vee z} \perp \mathcal{P}_{y \vee z}|\mathcal{P}_z$. A join semilattice $(Q, \leq)$ together with a relation $x \perp y|z$ with $x, y, z \in Q$, $(Q, \leq, \perp)$, satisfying conditions C1 to C4 is called a *quasi-separoid* (or also *q-separoid*) [8], a retract of the concept of *separoid* [1].

4 Information Algebra of Coherent Sets of Gambles

In [18] we showed that $\Phi(\Omega)$ with the following operations:

1. Combination: $\mathcal{D}_1 \cdot \mathcal{D}_2 \coloneqq \mathcal{C}(\mathcal{D}_1 \cup \mathcal{D}_2)$,[6]
2. Extraction: $\epsilon_x(\mathcal{D}) \coloneqq \mathcal{C}(\mathcal{D} \cap \mathcal{L}_x)$ for $x \in Q$,

where $(Q, \leq, \perp)$ is a q-separoid of questions on Ω in which $x \perp y|z$ with $x, y, z \in Q$ is the conditional independence relation introduced on them (see Section 3), is a *domain-free information algebra* that we call the *domain-free information algebra of coherent sets of gambles*. Combination captures aggregation of pieces of belief, and extraction, which for *multivariate models* leads to marginalisation in the equivalent *labeled* view of information algebras [7, 13], describes filtering the part of information relative to a question $x \in Q$. Information algebras are particular *valuation algebras* as defined by [15] but with idempotent combination. Domain-free versions of valuation algebras have been proposed by Shafer [19]. Idempotency of combination has important consequences, such as the possibility to define an information order, atoms, approximation, and more [7, 8]. It also offers—the subject of the present paper—important connections to set algebras.

Here we remind the characterizing properties of the domain-free information algebra Φ together with the q-separoid $(Q, \leq, \perp)$ and with a family E of extraction operators $\epsilon_x : \Phi \to \Phi$ for $x \in Q$:

[6] Notice that combination of sets in Φ coincides with their join in the lattice $(\Phi, \subseteq)$, as defined in Section 2.

1. *Semigroup:* $(\Phi, \cdot)$ is a commutative semigroup with a null element $0 = \mathcal{L}(\Omega)$ and a unit $1 = \mathcal{L}^+(\Omega)$.
2. *Quasi-Separoid:* $(Q, \leq, \perp)$ is a quasi-separoid.
3. *Existential Quantifier:* For any $x \in Q$, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D} \in \Phi$:
   (a) $\epsilon_x(0) = 0$,
   (b) $\epsilon_x(\mathcal{D}) \cdot \mathcal{D} = \mathcal{D}$,
   (c) $\epsilon_x(\epsilon_x(\mathcal{D}_1) \cdot \mathcal{D}_2) = \epsilon_x(\mathcal{D}_1) \cdot \epsilon_x(\mathcal{D}_2)$.
4. *Extraction:* For any $x, y, z \in Q, \mathcal{D} \in \Phi$, such that $x \vee z \perp y \vee z | z$ and $\epsilon_x(\mathcal{D}) = \mathcal{D}$, we have: $\epsilon_{y \vee z}(\mathcal{D}) = \epsilon_{y \vee z}(\epsilon_z(\mathcal{D}))$.
5. *Support:* For any $\mathcal{D} \in \Phi$ there is an $x \in Q$ so that $\epsilon_x(\mathcal{D}) = \mathcal{D}$, i.e. a *support* of $\mathcal{D}$ [18], and for all $y \geq x,\ y \in Q,\ \epsilon_y(\mathcal{D}) = \mathcal{D}$.

When we need to specify the constructing elements of the information algebra, we can refer to it with the tuple $(\Phi, Q, \leq, \perp, \cdot, 0, 1, E)$ or equivalently with $(\Phi, \mathcal{Q}, \leq, \perp, \cdot, 0, 1, E)$, where $(\mathcal{Q}, \leq, \perp)$ is the q-separoid of partitions equivalent to $(Q, \leq, \perp)$.[7]When we do not need this degree of accuracy, we can refer to it simply as Φ. Similar considerations can be made for other domain-free information algebras. Notice that, in particular, $(\Phi, \cdot)$ is an idempotent, commutative semigroup. So, a partial order on Φ is defined by $\mathcal{D}_1 \leq \mathcal{D}_2$ if $\mathcal{D}_1 \cdot \mathcal{D}_2 = \mathcal{D}_2$. Then $\mathcal{D}_1 \leq \mathcal{D}_2$ if and only if $\mathcal{D}_1 \subseteq \mathcal{D}_2$. This order is called an *information order* [18]. This definition entails the following facts: $\epsilon_x(\mathcal{D}) \leq \mathcal{D}$ for every $\mathcal{D} \in \Phi,\ x \in Q$; given $\mathcal{D}_1, \mathcal{D}_2 \in \Phi$, if $\mathcal{D}_1 \leq \mathcal{D}_2$, then $\epsilon_x(\mathcal{D}_1) \leq \epsilon_x(\mathcal{D}_2)$ for every $x \in Q$ [7].

5 Atoms and Maximal Coherent Sets of Gambles

Atoms are a well-known concept in (domain-free) information algebras. In our previous work [18], we showed that maximal coherent sets M are atoms of Φ. We denote them with $At(\Phi)$. Moreover, for every $\mathcal{D} \in \Phi$, we define $At(\mathcal{D}) := \{M \in At(\Phi) : \mathcal{D} \subseteq M\}$. We remind here some elementary properties of atoms for the specific case of $At(\Phi)$ [7]. Given $M, M_1, M_2 \in At(\Phi)$ and $\mathcal{D} \in \Phi$:

1. $M \cdot \mathcal{D} = M$ or $M \cdot \mathcal{D} = 0$;
2. either $\mathcal{D} \leq M$ or $M \cdot \mathcal{D} = 0$;
3. either $M_1 = M_2$ or $M_1 \cdot M_2 = 0$.

Φ is in particular *atomic* [7], i.e. for any $\mathcal{D} \neq 0$ the set $At(\mathcal{D})$ is not empty, and *atomistic*, i.e. for any $\mathcal{D} \neq 0$, $\mathcal{D} = \bigcap At(\mathcal{D})$. It is a general result of atomistic information algebras that the subalgebras $\epsilon_x(\Phi) := \{\epsilon_x(\mathcal{D}),\ \mathcal{D} \in \Phi\}$ are also atomistic [12]. Moreover, in [18], we showed that $At(\epsilon_x(\Phi)) = \epsilon_x(At(\Phi)) = \{\epsilon_x(M) : M \in At(\Phi)\}$ for any $x \in Q$ and, therefore, we call $\epsilon_x(M)$ for $M \in At(\Phi)$ and $x \in Q$ *local atoms* for x. [8] Local atoms $M_x = \epsilon_x(M)$ for $x \in Q$ induce a partition At_x of $At(\Phi)$ with blocks $At(M_x)$, see Lemma 1 below. If M and M' belong

[7] We can use the same notation E for the extraction operators in the two signatures of the information algebra. Indeed, we can indicate ϵ_x also as $\epsilon_{\mathcal{P}_x}$, where $\mathcal{P}_x \in \mathcal{Q}$ is the partition associated to $x \in Q$.

[8] Since local atoms for x are atoms of the subalgebra $\epsilon_x(\Phi)$, they respect in particular all the elementary properties of atoms restricted to elements of $\epsilon_x(\Phi)$ [7].

to the same block, we say that $M \equiv_x M'$. Let us indicate with $Part_Q(At(\Phi)) \coloneqq \{At_x : x \in Q\}$ the set of these partitions. $Part_Q(At(\Phi))$ is in particular a subset of $Part(At(\Phi))$, the set of all the partitions of $At(\Phi)$. On $Part(At(\phi))$ we can introduce a partial order $\leq$, respect to which $Part(At(\Phi))$ is a lattice, and a conditional independence relation $At_1 \perp At_2 | At_3$, with $At_1, At_2, At_3 \in Part(At(\Phi))$ analogous to the ones introduced in Section 3. Their introduction, indeed, is independent of the set on which partitions are defined [6]. It can be shown moreover that $(Part(At(\Phi)), \leq, \perp)$ is a quasi-separoid [8, Theorem 2.6]. We claim now that partitions $At_x \in Part_Q(At(\Phi))$ mirror the partitions $\mathcal{P}_x \in \mathcal{Q}$. Before stating this main result, we need the following lemma.

Lemma 1. *Let us consider $M, M' \in At(\Phi)$ and $x \in Q$. Then*

$$M \equiv_x M' \iff \epsilon_x(M) = \epsilon_x(M') \iff M, M' \in At(\epsilon_x(M)) = At(\epsilon_x(M')).$$

Hence, $At_x \leq At_y$ if and only if $At(\epsilon_x(M)) \supseteq At(\epsilon_y(M))$ for every $M \in At(\Phi)$ and $x, y \in Q$.

Proof. If $M \equiv_x M'$, there exists a local atom M_x such that $M, M' \in At(M_x)$. Therefore, $M, M' \geq M_x$ and $\epsilon_x(M), \epsilon_x(M') \geq M_x$ [18, Lemma 15, item 3]. However, $\epsilon_x(M), \epsilon_x(M')$ and M_x are all local atoms, hence $\epsilon_x(M) = \epsilon_x(M') = M_x$. The converse is obvious. For the second part, let us suppose $At(\epsilon_y(M)) \subseteq At(\epsilon_x(M))$ for every $M \in At(\Phi)$ with $x, y \in Q$, and consider $M', M'' \in At(\Phi)$ such that $M' \equiv_y M''$. Then $M', M'' \in At(\epsilon_y(M'))$ and so $M', M'' \in At(\epsilon_x(M'))$, which implies $M' \equiv_x M''$. Vice versa, consider $At_x \leq At_y$ and $M' \in At(\epsilon_y(M))$ for some $M, M' \in At(\Phi)$. Then, $M \equiv_y M'$, hence $M \equiv_x M'$ and so $M' \in At(\epsilon_x(M))$.

Now we can state the main result of this section.

Theorem 2. *The map $\mathcal{P}_x \mapsto At_x$, from $(\mathcal{Q}, \leq, \perp)$ to $(Part_Q(At(\Phi)), \leq, \perp)$, preserves order, that is $\mathcal{P}_x \leq \mathcal{P}_y$ implies $At_x \leq At_y$, and conditional independence relations, that is $\mathcal{P}_x \perp \mathcal{P}_y | \mathcal{P}_z$ implies $At_x \perp At_y | At_z$.*

Proof. If $x \leq y$, then $\epsilon_x(M) \leq \epsilon_y(M)$ for any atom $M \in At(\Phi)$ [18, Lemma 15, item 4]. Therefore, $At(\epsilon_x(M)) \supseteq At(\epsilon_y(M))$ for any $M \in At(\Phi)$, hence, by Lemma 1, $At_x \leq At_y$. For the second part, first of all notice that properties C2 and C3 of Theorem 1 are valid also for the conditional independence relation defined on $Part(At(\Phi))$, restricted to $Part_Q(At(\Phi))$. Then, recall that $x \perp y | z$ if and only if $x \vee z \perp y \vee z | z$. Consider then local atoms $M_{x \vee z}, M_{y \vee z}$ and M_z so that

$$At(M_{x \vee z}) \cap At(M_z) \neq \emptyset, \quad At(M_{y \vee z}) \cap At(M_z) \neq \emptyset.$$

Hence, there is an atom $M' \in At(M_{x \vee z}) \cap At(M_z)$ and an atom $M'' \in At(M_{y \vee z}) \cap At(M_z)$. Therefore, $M_{x \vee z} = \epsilon_{x \vee z}(M')$, $M_{y \vee z} = \epsilon_{y \vee z}(M'')$ and $M_z = \epsilon_z(M') = \epsilon_z(M'')$. Now, thanks to the Existential Quantifier axiom, we have:

$$\epsilon_z(M_{x \vee z} \cdot M_{y \vee z} \cdot M_z) = \epsilon_z(M_{x \vee z} \cdot M_{y \vee z}) \cdot M_z = \epsilon_z(\epsilon_{x \vee z}(M') \cdot \epsilon_{y \vee z}(M'')) \cdot M_z$$

Thanks to [18, Theorem 16] and [18, Lemma 15, item 3, item 6], we obtain

$$\epsilon_z(\epsilon_{x\vee z}(M') \cdot \epsilon_{y\vee z}(M'')) \cdot M_z = \epsilon_z(M') \cdot \epsilon_z(M'') \cdot M_z \neq 0.$$

Therefore $M_{x\vee z} \cdot M_{y\vee z} \cdot M_z \neq 0$ [18, Lemma 15, item 2] and hence, since the algebra is atomic, there is an atom $M''' \in At(M_{x\vee z} \cdot M_{y\vee z} \cdot M_z)$. Then $M_{x\vee z}, M_{y\vee z}, M_z \leq M'''$, whence $M''' \in At(M_{x\vee z}) \cap At(M_{y\vee z}) \cap At(M_z)$ and so $At_{x\vee z} \perp At_{y\vee z} | At_z$. From C2 and C3 of Theorem 1 and the first part of this theorem then, it follows that $At_x \perp At_y | At_z$.

6 Information Algebras Homomorphisms

We are interested in homomorphisms between domain-free information algebras. A domain-free information algebra is a two-sorted structure consisting of an idempotent commutative semigroup with a null and a unit element and a q-separoid, where the two sorts are linked by the extraction operators. Therefore an homomorphism should preserve operations and relations of both structures. This is achieved by the following definition.

Definition 6 (Domain-free information algebras homomorphism). *Let* $(\Psi, Q, \leq, \perp, \cdot, 0, 1, E)$ *and* $(\Psi', Q', \leq', \perp', \cdot', 0', 1', E')$ *be two domain-free information algebras, where* $E := \{\epsilon_x : x \in Q\}$ *and* $E' := \{\epsilon'_{x'} : x' \in Q'\}$ *are respectively the families of the extraction operators of the two structures. Consider a pair of maps* $f : \Psi \to \Psi'$ *and* $g : Q \to Q'$*, with the associated map* $h : E \to E'$ *defined by* $h(\epsilon_x) = \epsilon'_{g(x)}$*. Then the pair* (f, g) *is an information algebras homomorphism between* Ψ *and* Ψ' *if:*

1. $f(\psi \cdot \phi) = f(\psi) \cdot' f(\phi)$*, for every* $\psi, \phi \in \Psi$*;*
2. $f(0) = 0'$ *and* $f(1) = 1'$*;*
3. $x \leq y$ *implies* $g(x) \leq' g(y)$ *and* $x \perp y | z$ *implies* $g(x) \perp' g(y) | g(z)$*, for all* $x, y, z \in Q$*;*
4. $f(\epsilon_x(\psi)) = \epsilon'_{g(x)}(f(\psi))$*, for all* $\psi \in \Psi$ *and* $\epsilon_x \in E$*.*

We can give an analogous definition of homomorphism considering partitions equivalent to questions. We use the same notation for both the signatures.

If f and g are one-to-one, the pair (f, g) is an *information algebras embedding* and Ψ is said to be *embedded* into Ψ'; if they are bijective, (f, g) is an *information algebras isomorphism* and the two structures are called *isomorphic*.

7 Set Algebras

Archetypes of information algebras are so-called set algebras, where the elements are subsets of some universe, combination is intersection, and extraction is related to so-called saturation operators. We first specify the set algebra of subsets of Ω and show then that this algebra may be embedded into the information algebra of coherent sets. Conversely, we show that the algebra Φ may itself be embedded into a set algebra of its atoms, so is, in some precise sense, itself a set algebra. This is a general result for atomistic information algebras [7, 12].

To any partition $\mathcal{P}_x$ of Ω there corresponds a saturation operator defined for any subset $S \subseteq \Omega$ by

$$\sigma_x(S) \coloneqq \{\omega \in \Omega : (\exists \omega' \in S)\ \omega \equiv_x \omega'\}. \tag{2}$$

It corresponds to the union of the elements of blocks of $\mathcal{P}_x$ that have not empty intersection with S. The following are well-known properties of saturation operators. They can be shown analogously to the similar results in [8, 18].

Lemma 2. *For all $S, T \subseteq \Omega$ and any partition $\mathcal{P}_x$ of Ω:*

1. $\sigma_x(\emptyset) = \emptyset$,
2. $S \subseteq \sigma_x(S)$,
3. $\sigma_x(\sigma_x(S) \cap T) = \sigma_x(S) \cap \sigma_x(T)$,
4. $\sigma_x(\sigma_x(S)) = \sigma_x(S)$,
5. $S \subseteq T \Rightarrow \sigma_x(S) \subseteq \sigma_x(T)$,
6. $\sigma_x(\sigma_x(S) \cap \sigma_x(T)) = \sigma_x(S) \cap \sigma_x(T)$.

Note that the first three items of this theorem imply that σ_x is an existential quantifier relative to intersection as combination. This is a first step to construct a domain-free information algebra of subsets of Ω. Now, consider the q-separoid $(Q, \leq, \perp)$ sort of the information algebra $(\Phi, Q, \leq, \perp, \cdot, 0, 1, E)$ considered in Section 4. We need the support axiom to be satisfied. If $\mathcal{P}_\top \in \mathcal{Q}$, the set of partitions equivalent to questions in Q, then $\sigma_\top(S) = S$ for all $S \subseteq \Omega$. Otherwise, we must limit ourselves to the subsets of Ω for which there exists a support $x \in Q$. We call these sets *saturated* with respect to some $x \in Q$, and we indicate them with $P_Q(\Omega)$ or more simply with P_Q when no ambiguity is possible. Clearly, if the top partition belongs to Q, $P_Q(\Omega) = P(\Omega)$, the power set of Ω. So in what follows we can refer more generally to sets in $P_Q(\Omega)$. Note that in particular $\Omega, \emptyset \in P_Q(\Omega)$. At this point the support axiom is satisfied for every element of $P_Q(\Omega)$. Indeed, if $x \leq y$ with $x, y \in Q$, then $\omega \equiv_y \omega'$ implies $\omega \equiv_x \omega'$, so that $\sigma_y(S) \subseteq \sigma_x(S)$. Then, if x is a support of S, we have $S \subseteq \sigma_y(S) \subseteq \sigma_x(S) = S$, hence $\sigma_y(S) = S$. Moreover $(P_Q(\Omega), \cap)$ is a commutative semigroup with the empty set as the null element and Ω as the unit. Indeed, the only property we need to prove, is that $P_Q(\Omega)$ is closed under intersection. So, let us consider S and T, two subsets of Ω with support $x \in Q$ and $y \in Q$ respectively. They have also both supports $x \vee y$ that belongs to Q, because $(Q, \leq)$ is a join semilattice. Therefore, thanks to Lemma 2, we have

$$\sigma_{x\vee y}(S \cap T) = \sigma_{x\vee y}(\sigma_{x\vee y}(S) \cap \sigma_{x\vee y}(T)) = \sigma_{x\vee y}(S) \cap \sigma_{x\vee y}(T) = S \cap T.$$

So, $P_Q(\Omega)$ is closed under intersection. Lemma 2 moreover, implies that it is closed also under saturation. It remains only to verify the extraction property to conclude that P_Q forms a domain-free information algebra. We show it in the next result.

Theorem 3. *Given $x, y, z \in Q$, suppose $x \vee z \perp y \vee z | z$. Then, for any $S \in P_Q(\Omega)$,*

$$\sigma_{y\vee z}(\sigma_x(S)) = \sigma_{y\vee z}(\sigma_z(\sigma_x(S))).$$

Proof. From $\sigma_z(\sigma_x(S)) \supseteq \sigma_x(S)$ we obtain $\sigma_{y\vee z}(\sigma_z(\sigma_x(S)) \supseteq \sigma_{y\vee z}(\sigma_x(S))$. Consider therefore an element $\omega \in \sigma_{y\vee z}(\sigma_z(\sigma_x(S)))$. Then there are elements μ, μ' and ω' so that $\omega \equiv_{y\vee z} \mu \equiv_z \mu' \equiv_x \omega'$ and $\omega' \in S$. This means that ω, μ belong to some block $B_{y\vee z}$ of partition $\mathcal{P}_{y\vee z}$, μ, μ' to some block B_z of partition $\mathcal{P}_z$ and μ', ω' to some block B_x of partition $\mathcal{P}_x$. It follows that $B_x \cap B_z \neq \emptyset$ and $B_{y\vee z} \cap B_z \neq \emptyset$. Then $x \vee z \perp y \vee z | z$ implies, thanks to properties of a quasi-separoid, that $x \perp y \vee z | z$. Therefore, we have $B_x \cap B_{y\vee z} \cap B_z \neq \emptyset$, and in particular, $B_x \cap B_{y\vee z} \neq \emptyset$. So there is a $\lambda \in B_x \cap B_{y\vee z}$ such that $\omega \equiv_{y\vee z} \lambda \equiv_x \omega' \in S$, hence $\omega \in \sigma_{y\vee z}(\sigma_x(S))$. So we have $\sigma_{y\vee z}(\sigma_x(S)) = \sigma_{y\vee z}(\sigma_z(\sigma_x(S)))$.

So, we have the domain-free information algebra $(P_Q(\Omega), Q, \leq, \perp, \cap, \emptyset, \Omega, \{\sigma_x : x \in Q\})$. It is in particular an algebra of sets, with intersection as combination and saturation as extraction. Such type of information algebra will be called *set algebra*. In particular, we claim that this set algebra of subsets of Ω can be embedded into the information algebra of coherent sets of gambles $\Phi(\Omega)$. Indeed, for any set $S \in P_Q(\Omega)$, define

$$\mathcal{D}_S \coloneqq \{f \in \mathcal{L}(\Omega) : \inf_{\omega \in S} f(\omega) > 0\} \cup \mathcal{L}^+(\Omega).$$

If $S \neq \emptyset$, this is clearly a coherent set, otherwise it corresponds to $\mathcal{L}(\Omega)$. The next theorem shows that the map $f : S \mapsto \mathcal{D}_S$ together with the identity map g, and the associated map $h : \epsilon_x \mapsto \sigma_x$, form an information algebras homomorphism between $P_Q(\Omega)$ and $\Phi(\Omega)$. In this case the q-separoid considered in the two algebras is the same and $g = id$, therefore item 3 in the definition of a domain-free information algebras homomorphism is trivially satisfied.

Theorem 4. *Let $S, T \in P_Q(\Omega)$ and $x \in Q$. Then*

1. $\mathcal{D}_S \cdot \mathcal{D}_T = \mathcal{D}_{S\cap T}$,
2. $\mathcal{D}_\emptyset = \mathcal{L}(\Omega)$, $\mathcal{D}_\Omega = \mathcal{L}^+(\Omega)$,
3. $\epsilon_x(\mathcal{D}_S) = \mathcal{D}_{\sigma_x(S)}$.

Proof. 1. Note that $\mathcal{D}_S = \mathcal{L}^+$ or $\mathcal{D}_T = \mathcal{L}^+$ if and only if $S = \Omega$ or $T = \Omega$. Clearly in this case we have immediately the result. The same is true if $\mathcal{D}_S = \mathcal{L}$ or $\mathcal{D}_T = \mathcal{L}$, which is equivalent to having $S = \emptyset$ or $T = \emptyset$. Now suppose $\mathcal{D}_S, \mathcal{D}_T \neq \mathcal{L}^+$ and $\mathcal{D}_S, \mathcal{D}_T \neq \mathcal{L}$. If $S \cap T = \emptyset$, then $\mathcal{D}_{S\cap T} = \mathcal{L}(\Omega)$. Consider $f \in \mathcal{D}_S$ and $g \in \mathcal{D}_T$. Since S and T are disjoint, we have $\tilde{f} \in \mathcal{D}_S$ and $\tilde{g} \in \mathcal{D}_T$, where $\tilde{f}$, $\tilde{g}$ are defined in the following way for every $\omega \in \Omega$:

$$\tilde{f}(\omega) \coloneqq \begin{cases} f(\omega) & \text{for } \omega \in S, \\ -g(\omega) & \text{for } \omega \in T, \\ 0 & \text{for } \omega \in (S \cup T)^c, \end{cases} \qquad \tilde{g}(\omega) \coloneqq \begin{cases} -f(\omega) & \text{for } \omega \in S, \\ g(\omega) & \text{for } \omega \in T, \\ 0 & \text{for } \omega \in (S \cup T)^c. \end{cases}$$

However, $\tilde{f} + \tilde{g} = 0 \in \mathcal{E}(\mathcal{D}_S \cup \mathcal{D}_T)$, hence $\mathcal{D}_S \cdot \mathcal{D}_T \coloneqq \mathcal{C}(\mathcal{D}_S \cup \mathcal{D}_T) = \mathcal{L}(\Omega) = \mathcal{D}_{S\cap T}$. Assume then that $S \cap T \neq \emptyset$. Note that $\mathcal{D}_S \cup \mathcal{D}_T \subseteq \mathcal{E}(\mathcal{D}_S \cup \mathcal{D}_T) \subseteq \mathcal{D}_{S\cap T}$ so that

$\mathcal{E}(\mathcal{D}_S \cup \mathcal{D}_T)$ is coherent and $\mathcal{D}_S \cdot \mathcal{D}_T = \mathcal{E}(\mathcal{D}_S \cup \mathcal{D}_T) \subseteq \mathcal{D}_{S\cap T}$. Consider then a gamble $f \in \mathcal{D}_{S\cap T}$. Select a $\delta > 0$ and define two functions

$$f_1(\omega) \coloneqq \begin{cases} 1/2f(\omega) & \text{for } \omega \in (S \cap T), \\ \delta & \text{for } \omega \in S \setminus T, \\ f(\omega) - \delta & \text{for } \omega \in T \setminus S, \\ 1/2f(\omega) & \text{for } \omega \in (S \cup T)^c, \end{cases} \qquad f_2(\omega) \coloneqq \begin{cases} 1/2f(\omega) & \text{for } \omega \in (S \cap T), \\ f(\omega) - \delta & \text{for } \omega \in S \setminus T, \\ \delta & \text{for } \omega \in T \setminus S, \\ 1/2f(\omega) & \text{for } \omega \in (S \cup T)^c, \end{cases}$$

for every $\omega \in \Omega$. Then $f = f_1 + f_2$ and $f_1 \in \mathcal{D}_S$, $f_2 \in \mathcal{D}_T$. Therefore $f \in \mathcal{E}(\mathcal{D}_S \cup \mathcal{D}_T) = \mathcal{D}_S \cdot \mathcal{D}_T$, hence $\mathcal{D}_S \cdot \mathcal{D}_T = \mathcal{D}_{S\cap T}$.

2. Both have been noted above.

3. If S is empty, then $\epsilon_x(\mathcal{D}_\emptyset) = \mathcal{L}(\Omega)$ so that item 3 holds in this case. Hence, assume $S \neq \emptyset$. Then we have that $\mathcal{D}_S$ is coherent, and therefore:

$$\epsilon_x(\mathcal{D}_S) \coloneqq \mathcal{C}(\mathcal{D}_S \cap \mathcal{L}_x) = \mathcal{E}(\mathcal{D}_S \cap \mathcal{L}_x) \coloneqq \mathsf{posi}(\mathcal{L}^+(\Omega) \cup (\mathcal{D}_S \cap \mathcal{L}_x)).$$

Consider a gamble $f \in \mathcal{D}_S \cap \mathcal{L}_x$. If $f \in \mathcal{L}^+(\Omega) \cap \mathcal{L}_x$ then clearly $f \in \mathcal{D}_{\sigma_x(S)}$. Otherwise, $\inf_S f > 0$ and f is x-measurable. If $\omega \equiv_x \omega'$ for some $\omega' \in S$ and $\omega \in \Omega$, then $f(\omega) = f(\omega')$. Therefore $\inf_{\sigma_x(S)} f = \inf_S f > 0$, hence $f \in \mathcal{D}_{\sigma_x(S)}$. Then $\mathcal{C}(\mathcal{D}_S \cap \mathcal{L}_x) \subseteq \mathcal{C}(\mathcal{D}_{\sigma_x(S)}) = \mathcal{D}_{\sigma_x(S)}$. Conversely, consider a gamble $f \in \mathcal{D}_{\sigma_x(S)}$. $\mathcal{D}_{\sigma_x(S)}$ is a strictly desirable set of gambles.[9] Hence, if $f \in \mathcal{D}_{\sigma_x(S)}$, $f \in \mathcal{L}^+(\Omega)$ or there is $\delta > 0$ such that $f - \delta \in \mathcal{D}_{\sigma_x(S)}$. If $f \in \mathcal{L}^+(\Omega)$, then $f \in \epsilon_x(\mathcal{D}_S)$. Otherwise, let us define for every $\omega \in \Omega$, $g(\omega) \coloneqq \inf_{\omega' \equiv_x \omega} f(\omega') - \delta$. If $\omega \in S$, then $g(\omega) > 0$ since $\inf_{\sigma_x(S)}(f - \delta) > 0$. So we have $\inf_S g \geq 0$ and g is x-measurable. However, $\inf_S(g + \delta) = \inf_S g + \delta > 0$ hence $(g + \delta) \in \mathcal{D}_S \cap \mathcal{L}_x$ and $f \geq g + \delta$. Therefore $f \in \mathcal{E}(\mathcal{D}_S \cap \mathcal{L}_x) = \mathcal{C}(\mathcal{D}_S \cap \mathcal{L}_x) \eqqcolon \epsilon_x(\mathcal{D}_S)$.

Item 3. guarantees that, if $S \in P_Q(\Omega)$, then there exists an $x \in Q$ such that $\epsilon_x(\mathcal{D}_S) = \mathcal{D}_S$. Moreover, f and g are one-to-one,[10]so P_Q is embedded into Φ.

Regarding instead the relation between Φ and a set algebra of sets of its atoms, consider the set algebra:

$$(P(At(\Phi)), Part(At(\Phi)), \leq, \perp, \cap, \emptyset, At(\Phi), \Sigma), \tag{3}$$

where $P(At(\Phi))$ is the power set of $At(\Phi)$, which clearly leads to the commutative semigroup $(P(At(\Phi)), \cap)$ with $\emptyset$ as the null element and $At(\Phi)$ as the unit element, $(Part(At(\Phi)), \leq, \perp)$ is the q-separoid introduced in Section 5 and Σ is the set of all saturation operators associated with any partition $At \in Part(At(\Phi))$, defined similarly to (2). It is clearly a set algebra. In fact, the *Semigroup*, the *Quasi-Separoid* and the *Support* axioms are clearly satisfied. The *Existential Quantifier* axiom follows from Lemma 2 applied to saturation operators in Σ and subsets of $At(\Phi)$ and the *Extraction* axiom follows from Theorem 3 applied again to Σ and elements in $P(At(\Phi))$. We claim moreover that $(\Phi, \mathcal{Q}, \leq, \perp, \cdot, 0, 1, E)$ can be embedded into $(P(At(\Phi)), Part(At(\Phi)), \leq, \perp, \cap, \emptyset, At(\Phi), \Sigma)$. Consider the maps $f : \Phi \to P(At(\Phi))$ and $g : \mathcal{Q} \to Part_Q(At(\Phi))$ defined as $f(\mathcal{D}) = At(\mathcal{D})$

[9] $\underline{P}(f) \coloneqq \inf_S(f)$ for every $f \in \mathcal{L}$ with $S \neq \emptyset$, is a coherent lower prevision [17].

[10] Indeed g is clearly one-to-one and regarding f, if $\mathcal{D}_S = \mathcal{D}_T$ it means that $S = T$.

and $g(\mathcal{P}_x) = At_x$, where $At_x \in Part_Q(At(\Phi))$ is the partition defined by the equivalence relation between atoms $M \equiv_x M'$ if and only if $\epsilon_x(M) = \epsilon_x(M')$ with $M, M' \in At(\Phi)$. As noted before, associated with the partition At_x, there is the saturation operator σ_x, defined for any subset $S \subseteq At(\Phi)$ by $\sigma_x(S) := \{M \in At(\Phi) : (\exists M' \in S)\ M \equiv_x M'\}$. Thus, we have the map h defined as $h(\epsilon_x) = \sigma_x$. The next result, together with Theorem 2, shows that the pair of maps (f, g) is an information algebra homomorphism between Φ and $P(At(\Phi))$.

Theorem 5. *For any element $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}$ of Φ and all $x \in Q$,*

1. $At(\mathcal{D}_1 \cdot \mathcal{D}_2) = At(\mathcal{D}_1) \cap At(\mathcal{D}_2)$,
2. $At(\mathcal{L}(\Omega)) = \emptyset$, $At(\mathcal{L}^+(\Omega)) = At(\Phi)$,
3. $At(\epsilon_x(\mathcal{D})) = \sigma_x(At(\mathcal{D}))$.

Proof. Item 2 is obvious. If there is a an atom $M \in At(\mathcal{D}_1 \cdot \mathcal{D}_2)$, then $M \geq \mathcal{D}_1 \cdot \mathcal{D}_2 \geq \mathcal{D}_1, \mathcal{D}_2$ and thus $M \in At(\mathcal{D}_1) \cap At(\mathcal{D}_2)$. Conversely, if $M \in At(\mathcal{D}_1) \cap At(\mathcal{D}_2)$, then $\mathcal{D}_1, \mathcal{D}_2 \leq M$, hence $\mathcal{D}_1 \cdot \mathcal{D}_2 \leq M$ and $M \in At(\mathcal{D}_1 \cdot \mathcal{D}_2)$. Furthermore, if $\epsilon_x(\mathcal{D}) = 0$, then $\mathcal{D} = 0$ and $At(\mathcal{D}) = \emptyset$, hence $\sigma_x(\emptyset) = \emptyset$ and vice versa [18, Lemma 15, item 2]. Assume therefore $At(\mathcal{D}) \neq \emptyset$ and consider $M \in \sigma_x(At(\mathcal{D}))$. There is then a $M' \in At(\mathcal{D})$ so that $\epsilon_x(M) = \epsilon_x(M')$. But $\mathcal{D} \leq M'$, hence $\epsilon_x(\mathcal{D}) \leq \epsilon_x(M') = \epsilon_x(M) \leq M$. Thus $M \in At(\epsilon_x(\mathcal{D}))$. Conversely consider $M \in At(\epsilon_x(\mathcal{D}))$. We claim that $\epsilon_x(M) \cdot \mathcal{D} \neq 0$. Because otherwise $0 = \epsilon_x(\epsilon_x(M) \cdot \mathcal{D}) = \epsilon_x(M) \cdot \epsilon_x(\mathcal{D}) = \epsilon_x(M \cdot \epsilon_x(\mathcal{D}))$, by Existential Quantifier axiom, which is not possible since $\epsilon_x(\mathcal{D}) \leq M$. So there is an $M' \in At(\epsilon_x(M) \cdot \mathcal{D})$ so that $\mathcal{D} \leq \epsilon_x(M) \cdot \mathcal{D} \leq M'$. We conclude that $M' \in At(\mathcal{D})$. Furthermore, $\epsilon_x(\epsilon_x(M) \cdot \mathcal{D}) = \epsilon_x(M) \cdot \epsilon_x(\mathcal{D}) \leq \epsilon_x(M')$. It follows that $\epsilon_x(M') \cdot \epsilon_x(M) \cdot \epsilon_x(\mathcal{D}) \neq 0$ and therefore $\epsilon_x(M) \cdot \epsilon_x(M') \neq 0$. But they are both local atoms, hence $\epsilon_x(M) = \epsilon_x(M')$, which together with $M' \in At(\mathcal{D})$ tells us that $M \in \sigma_x(At(\mathcal{D}))$.

Φ is atomistic, then f is one-to-one. Further, g is also one-to-one since $\epsilon_x \neq \epsilon_y$ implies $At_x \neq At_y$. Therefore Φ is embedded into $P(At(\Phi))$ and we can say that it is in fact a set algebra.

8 Conclusions

This paper presents an extension of our work on information algebras related to gambles on a possibility set that is not necessarily multivariate [18]. In the multivariate case, we showed also that lower previsions and strictly desirable sets of gambles form isomorphic information algebras [18]. It can be expected to be valid also in this case. Since then convex *credal sets* are sets of atoms associated with a lower prevision [17, 18], we claim that the algebra of lower previsions, hence of strictly desirable sets of gambles, can be embedded into the set algebra of credal sets. This is left for future work, along with other aspects such as the question of conditioning and the equivalent expression of the results shown in this paper for the *labeled view* of information algebras, better suited for computational purposes [7, 18].

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