Information algebras in the theory of imprecise probabilities

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Abstract

In this paper we create a bridge between desirability and information algebras: we show how coherent sets of gambles, as well as coherent lower previsions, induce such structures. This allows us to enforce the view of such imprecise-probability objects as algebraic and logical structures; moreover, it enforces the interpretation of probability as information, and gives tools to manipulate them as such.

Keywords: desirability, information algebras, compatibility, imprecise probabilities, coherence.

1. Introduction and Overview

In a recent paper Miranda and Zaffalon derived some results about the *compatibility* problem, i.e., the problem of establishing if some probabilistic assessments have a common joint probabilistic model, in the framework of *desirability* [1]. They remarked that these results could be obtained also using the theory of *information algebras* [2]. This issue is taken up and anlyzed in this paper.

Desirability, or the theory of *coherent sets of gambles*, is a very general theory of uncertainty introduced by Peter Williams in 1975 as a generalization of de Finetti's theory [3, 4, 5]. In particular, it provides a very general setting for compatibility because it allows to work with any possibility space, unrestricted domains and imprecise probabilities, here represented by lower and upper expectation called *lower and upper previsions* [1, 4]. On the other hand, information algebras are algebraic structures composed by 'pieces of information' that can be manipulated by operations of *combination*, to aggregate them, and *extraction*, to extract information regarding a specific domain [2, 6]. They were initially introduced as axiomatic systems sufficient to generalize the local computation scheme introduced by Lauritzen-Spiegelhalter for probabilistic networks to a multitude of other uncertainty formalisms like Dempster-Shafer belief functions, possibility theory and many others [7, 8].

There are two different versions of information algebras, a *domain-free* one and a *labeled* one. They are closely related and each one can be derived or reconstructed from the other. Roughly speaking, the domain-free version is better suited for theoretical studies, whereas the labeled version is better adapted to computational purposes, since it provides more efficient storage structures.

In this paper we analyse more in depth the connections between these two theories. In Section 2 and in Section 3, we provide some preliminary notions about each theory. Hence, in Section 4 and in Section 5, we prove the possibility of building domain-free and labeled information algebras starting from coherent sets of gambles (in Section 4) or coherent lower previsions (in Section 5), both interpreted as pieces of information about values of a group of variables. This creates a bridge between desirability and information algebras theory that allows to improve the two theories. On the one hand indeed, desirability, through its link with imprecise probabilities, enriches the view of probability as information and shows a way to integrate it into information algebras. On the other hand, information algebras allow to abstract away properties

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of desirability that can be regarded as properties of the more general algebraic structures of information algebras rather than the special ones of desirability. As a first example of the advantages of our results, in Section 6 we show that the main compatibility result of [1] for unconditional assessments follows directly from properties of information algebras.

2. Desirability

We start by introducing the necessary notation and basic definitions from desirability theory. For additional comments, we refer to [4, 5].

2.1. Coherent sets of gambles

Consider a non-empty set Ω describing the possible and mutually exclusive outcomes of some experiment. We call it *space of possibilities*. In this paper we let its cardinality be general, so Ω can be infinite.

Definition 1 (Gamble). Given a possibility space Ω, a *gamble* on Ω is a bounded real-valued function $f : \Omega \to \mathbb{R}$.

A gamble is interpreted as an uncertain reward in a linear utility scale. A subject might desire a gamble or not depending on the information they have about the experiment whose possible outcomes are the elements of the possibility space on which the gamble is defined.

We denote the set of all gambles on Ω by $\mathcal{L}(\Omega)$. We also let $\mathcal{L}^+(\Omega) := \{f \in \mathcal{L}(\Omega) : f \geq 0, f \neq 0\}$ denote the subset of non-vanishing, non-negative gambles on Ω. Similarly, we let $\mathcal{L}^-(\Omega) := \{f \in \mathcal{L}(\Omega) : f \leq 0, f \neq 0\}$ denote the subset of non-vanishing, non-positive gambles on Ω. We shall simplify the notation whenever possible by omitting the possibility space Ω. Thus, we shall write $\mathcal{L}, \mathcal{L}^+, \mathcal{L}^-$ in place of $\mathcal{L}(\Omega)$, $\mathcal{L}^+(\Omega), \mathcal{L}^-(\Omega)$ respectively.

Gambles in $\mathcal{L}^+$ should always be desired, since they may increase the utility with no risk of decreasing it. Gambles in $\mathcal{L}^-$ instead, should never be desired. As a consequence of the linearity of the utility scale, we assume also that a subject disposed to accept the transactions represented by gambles f and g, is disposed to accept also the transactions $\lambda f + \mu g$ for every $\lambda, \mu \geq 0$ not both equal to 0. More generally, we can consider the notion of a coherent set of desirable gambles, also called coherent set of gambles for simplicity.

Definition 2 (Coherence for sets of gambles). We say that a subset D of $\mathcal{L}(\Omega)$ is a *coherent* set of desirable gambles, or more simply a *coherent* set of gambles, if and only if D satisfies the following properties:

D1. $\mathcal{L}^+(\Omega) \subseteq D$ [Accepting Partial Gains],

D2. $0 \notin D$ [Avoiding Null Gain],

D3. $f, g \in D \Rightarrow f + g \in D$ [Additivity],

D4. $f \in D, \lambda > 0 \Rightarrow \lambda f \in D$ [Positive Homogeneity].

So D is a convex cone. In what follows let $C(\Omega) := \{D \subseteq \mathcal{L}(\Omega) : D \text{ is coherent}\}$, or simply C when there is no possible ambiguity, denote the set of all coherent sets of gambles.

This definition leads to the concept of natural extension.

Definition 3 (Natural extension for sets of gambles). Given a set $\mathcal{K} \subseteq \mathcal{L}(\Omega)$, we call $\mathcal{E}(\mathcal{K}) := \mathsf{posi}(\mathcal{K} \cup \mathcal{L}^+(\Omega))$, where

$$\mathsf{posi}(\mathcal{K}') := \left\{ \sum_{j=1}^{r} \lambda_j f_j : f_j \in \mathcal{K}', \lambda_j > 0, r \geq 1 \right\}$$

for every set $\mathcal{K}' \subseteq \mathcal{L}(\Omega)$, its *natural extension*.

We do not indicate the dependency of the natural extension operator on the possibility set Ω. However, it has to be intended that the operator applied to a set of gambles $\mathcal{K} \subseteq \mathcal{L}$, depends on the possibility set on which gambles in $\mathcal{K}$ are defined.[1] The natural extension of a set of gambles $\mathcal{K} \subseteq \mathcal{L}$, $\mathcal{E}(\mathcal{K})$, is coherent if and only if $0 \notin \mathcal{E}(\mathcal{K})$.

Coherent sets are closed under intersection, that is, they form a topless $\cap$-structure [9]. By standard order theory [9], they are ordered by inclusion, intersection is meet in this order and a family of coherent sets of gambles $\{D_j\}_{j \in J}$, where J is an index set, have a join, indicated with $\bigvee_{j \in J} D_j$, if they have an upper bound among coherent sets:

$$\bigvee_{j \in J} D_j := \bigcap \{D' \in C(\Omega) : \bigcup_{j \in J} D_j \subseteq D'\}.$$

By construction if $\mathcal{E}(\mathcal{K})$, for some $\mathcal{K} \subseteq \mathcal{L}$, is coherent, it is the smallest coherent set containing $\mathcal{K}$:

$$\mathcal{E}(\mathcal{K}) = \bigcap \{D' \in C(\Omega) : \mathcal{K} \subseteq D'\}.$$

So that, given the previous family of coherent sets of gambles $\{D_j\}_{j \in J}$, if $\mathcal{E}(\bigcup_{j \in J} D_j)$ is coherent, we have:

$$\bigvee_{j \in J} D_j = \mathcal{E}(\bigcup_{j \in J} D_j).$$

In view of the following section, it is convenient to add $\mathcal{L}(\Omega)$ to $C(\Omega)$ and let $\Phi(\Omega) := C(\Omega) \cup \{\mathcal{L}(\Omega)\}$. In what follows we refer to it also with Φ when there is no ambiguity. The family of sets in Φ is still a $\cap$-structure, but now a topped one. So, again by standard results of order theory [9], $(\Phi, \subseteq)$ induces a complete lattice where meet is intersection and join is defined for any family of sets $\{D_j\}_{j \in J}$ with $D_j \in \Phi(\Omega)$ for every $j \in J$, as:

$$\bigvee_{j \in J} D_j := \bigcap \{D' \in \Phi(\Omega) : \bigcup_{j \in J} D_j \subseteq D'\}.$$

Note that, if a family of coherent sets $\{D_j\}_{j \in J}$ has no upper bound in $C(\Omega)$, then its join is simply $\mathcal{L}(\Omega)$. In this topped $\cap$-structure the operator $\mathcal{C} : \mathcal{P}(\mathcal{L}(\Omega)) \to \mathcal{P}(\mathcal{L}(\Omega))$, where $\mathcal{P}(\mathcal{L}(\Omega))$ is the power-set of $\mathcal{L}(\Omega)$, defined as follows for every $\mathcal{K} \subseteq \mathcal{L}$:

$$\mathcal{C}(\mathcal{K}) := \bigcap \{D' \in \Phi(\Omega) : \mathcal{K} \subseteq D'\},$$

is a closure (or consequence) operator on $(\mathcal{P}(\mathcal{L}(\Omega)), \subseteq)$ [9]. Also in this case, it depends on the possibility set on which gambles in $\mathcal{K}$ are defined.

Definition 4 (Closure operator on $(\mathcal{P}(\mathcal{L}), \subseteq)$). A closure operator on the ordered set $(\mathcal{P}(\mathcal{L}), \subseteq)$ is a function $\mathcal{C} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ that satisfies the following conditions for all sets $\mathcal{K}, \mathcal{K}' \in \mathcal{P}(\mathcal{L})$:

- $\mathcal{K} \subseteq \mathcal{C}(\mathcal{K})$,
- $\mathcal{K} \subseteq \mathcal{K}'$ implies $\mathcal{C}(\mathcal{K}) \subseteq \mathcal{C}(\mathcal{K}')$,
- $\mathcal{C}(\mathcal{C}(\mathcal{K})) = \mathcal{C}(\mathcal{K})$.

Given that we always consider inclusion as the order relation on $\mathcal{P}(\mathcal{L})$, we will refer to the operator $\mathcal{C}$, more simply, as a consequence operator on $\mathcal{P}(\mathcal{L})$.

Note that also the natural extension operator is a consequence operator on $\mathcal{P}(\mathcal{L})$. Moreover, given a subset $\mathcal{K} \subseteq \mathcal{L}$, we have $\mathcal{C}(\mathcal{K}) = \mathcal{E}(\mathcal{K})$ if $0 \notin \mathcal{E}(\mathcal{K})$, that is if $\mathcal{E}(\mathcal{K})$ is coherent. Otherwise we have $\mathcal{C}(\mathcal{K}) = \mathcal{L}$

[1] More explicitly, let us consider two different possibility spaces Ω, Ω'. Let us consider also two sets of gambles defined respectively on Ω and Ω': $\mathcal{K} \subseteq \mathcal{L}(\Omega)$ and $\mathcal{K}' \subseteq \mathcal{L}(\Omega')$. Then $\mathcal{E}(\mathcal{K}) := \mathsf{posi}(\mathcal{K} \cup \mathcal{L}^+(\Omega))$ and $\mathcal{E}(\mathcal{K}') := \mathsf{posi}(\mathcal{K}' \cup \mathcal{L}^+(\Omega'))$.

and we may have $\mathcal{E}(\mathcal{K}) \neq \mathcal{L} = \mathcal{C}(\mathcal{K})$.[2] We refer to [10] for a similar order-theoretic view on belief models. These results prepare the way to an information algebra of coherent sets of gambles (see Section 4).

The most informative cases of coherent sets of gambles, i.e., coherent sets that are not proper subsets of other coherent sets, are called *maximal*. The following proposition provides a characterisation of such maximal elements.

Proposition 1 (Maximal set of gambles). *A coherent set of gambles $D \subseteq \mathcal{L}(\Omega)$ is* maximal *if and only if*

$$(\forall f \in \mathcal{L}(\Omega) \setminus \{0\})\ f \notin D \Rightarrow -f \in D.$$

We shall indicate maximal sets of gambles with M to differentiate them from the general case of coherent sets. These sets play an important role because of the following facts proved in [11]:

1. any coherent set of gambles is a subset of a maximal one,
2. any coherent set of gambles is the intersection of all maximal coherent sets it is contained in.

For a discussion of the importance of maximal coherent sets of gambles in relation to information algebras, see Section 4.3.

A further important class of coherent sets of gambles are the *strictly desirable* ones. Their importance will be highlighted in Section 2.2.

Definition 5 (Strictly desirable set of gambles). A coherent set of gambles D is said to be *strictly desirable* if and only if it satisfies $(\forall f \in D \setminus \mathcal{L}^+)(\exists \delta > 0)\ f - \delta \in D$.

We shall use the notation D^+ for strictly desirable sets of gambles to differentiate them from the general case of coherent sets of gambles.

So strictly desirable sets of gambles are coherent and form a subfamily of coherent sets of gambles. In what follows we will indicate with $C^+(\Omega)$, or simply C^+, the set of all strictly desirable sets of gambles. Moreover, similarly to before, we define $\Phi^+(\Omega) \coloneqq C^+(\Omega) \cup \{\mathcal{L}(\Omega)\}$, which we can indicate also with Φ^+ when there is no ambiguity.

Another important class of sets, which plays an important role highlighted again in Section 2.2, is the class of *almost desirable sets of gambles* [4].

Definition 6 (Almost desirable set of gambles). We say that a subset $\overline{D}$ of $\mathcal{L}(\Omega)$ is an *almost desirable* set of gambles if and only if $\overline{D}$ satisfies the following properties:

D1'. $f \in \mathcal{L}(\Omega)$ and $\inf f > 0$ implies $f \in \overline{D}$ [Accepting Sure Gains],

D2'. $f \in \overline{D}$ implies $\sup f \geq 0$ [Avoiding Sure Loss],

D3'. $f, g \in \overline{D} \Rightarrow f + g \in \overline{D}$ [Additivity],

D4'. $f \in \overline{D}, \lambda > 0 \Rightarrow \lambda f \in \overline{D}$ [Positive Homogeneity],

D5'. $f + \delta \in \overline{D}$ for all $\delta > 0$ implies $f \in \overline{D}$ [Closure].

A set of this type is no coherent since it contains $f = 0$. However, it is subjected to a weaker notion of coherence, in fact it has empty intersection with $\{f \in \mathcal{L} :\ \sup f < 0\}$. We shall use the notation $\overline{D}$ for almost desirable sets of gambles to differentiate them from coherent ones.

[2] For example, if $\mathcal{K}$ is an *almost desirable set of gambles*, see the following Definition 6, then $0 \in \mathcal{E}(\mathcal{K})$. However, $\mathcal{E}(\mathcal{K}) \neq \mathcal{L} = \mathcal{C}(\mathcal{K})$.

2.2. Lower previsions, upper previsions and credal sets

Coherent sets of gambles, strictly desirable sets of gambles and almost desirable ones encompass a probabilistic model for Ω, made of lower and upper expectations, called *previsions* after de Finetti [12]. The correspondence between sets of gambles and lower and upper previsions is bijective in the cases of strictly and almost desirable sets but not for coherent sets in general.

A *lower prevision* $\underline{P}$, is a function with values in $\mathbb{R} \cup \{+\infty\}$ defined on some class of gambles $dom(\underline{P})$, called the *domain* of $\underline{P}$.[3] It is also possible to think of $\underline{P}(f)$ as the supremum buying price that a subject is willing to spend for the gamble f. Following this interpretation, it is possible to define it starting from every non-empty generic (not necessarily coherent, strictly desirable or almost desirable) set of gambles $\mathcal{K}$ that the subject is willing to accept.

Definition 7 (Lower and upper prevision). Given a non-empty set $\mathcal{K} \subseteq \mathcal{L}(\Omega)$, we can associate to it the *lower prevision* (operator) $\underline{P} : dom(\underline{P}) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\underline{P}(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in \mathcal{K}\} \tag{1}$$

for every $f \in dom(\underline{P})$, and the *upper prevision* (operator) $\overline{P} : dom(\overline{P}) \to \mathbb{R} \cup \{-\infty\}$ defined as

$$\overline{P}(f) \coloneqq -\underline{P}(-f) \tag{2}$$

for every $f \in dom(\overline{P})$, where $dom(\underline{P}), dom(\overline{P}) = -dom(\underline{P}) \subseteq \mathcal{L}(\Omega)$.

So, $dom(\underline{P})$ is constituted by all the gambles f for which $\{\mu \in \mathbb{R} : f - \mu \in \mathcal{K}\}$ is not empty.

Given the fact that it is always possible to express upper previsions in terms of lower ones, in what follows we will concentrate only on lower previsions. In the definition above we have not made explicit the dependence on $\mathcal{K}$. However, when it is important to indicate it, we can see $\underline{P}$ also as the outcome of a function σ applied to a set of gambles $\mathcal{K}$ and write $\underline{P} = \sigma(\mathcal{K})$. We can also denote the set of gambles for which $\underline{P}$ is defined as $dom(\sigma(\mathcal{K}))$.

If $\mathcal{K} \subseteq \mathcal{L}(\Omega)$ is a coherent set of gambles, the associated functional $\underline{P}$, constructed from $\mathcal{K}$ through equation (1), is called *coherent lower prevision*. Its domain coincides with the whole set $\mathcal{L}(\Omega)$, see Lemma 6 in Appendix A, and it is characterized by the following properties. For every $f, g \in \mathcal{L}(\Omega)$:

1. $\underline{P}(f) \geq \inf_{\omega \in \Omega} f(\omega)$,
2. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$, $\forall \lambda > 0$,
3. $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g)$.

In this case, it is possible to introduce also the following definition.

Definition 8. Consider a lower prevision $\underline{P}$ and an upper prevision $\overline{P}$ constructed from a coherent set of gambles $D \subseteq \mathcal{L}(\Omega)$ through Eq. (1) and Eq. (2) respectively. If $\underline{P}(f) = \overline{P}(f)$ for some $f \in \mathcal{L}(\Omega)$, then we call the common value the *prevision* of f and we denote it by $P(f)$. If this happens for all $f \in \mathcal{L}(\Omega)$, then we call the functional P a *linear prevision*.

From its definition and from the second and third coherence properties of lower previsions it follows that a linear prevision is a linear functional on $\mathcal{L}$. In what follows we denote the set of all coherent lower previsions defined on $\mathcal{L}(\Omega)$ as $\underline{\mathbb{P}}(\Omega)$, or $\underline{\mathbb{P}}$ when there is no ambiguity. Analogously, we denote the set of all linear previsions as $\mathbb{P}(\Omega)$, or $\mathbb{P}$ when there is no ambiguity.

Coherent lower previsions, and therefore also linear ones, form a particular important class of lower previsions. Every coherent lower prevision $\underline{P}$ indeed, has a set of dominating linear previsions:

$$\mathcal{M}(\underline{P}) \coloneqq \{P \in \mathbb{P} : (\forall f \in \mathcal{L})\ P(f) \geq \underline{P}(f)\}, \tag{3}$$

[3] Usually lower previsions are functions with values in $\mathbb{R}$. We consider here an extended version of this concept.

which turns out to be non-empty, convex and closed under the *weak* topology* [4]. Each linear prevision is in a one-to-one correspondence with a finitely additive probability, which can be obtained by making the restriction of the linear prevision to indicators of events. As a consequence, it is possible to regard $\mathcal{M}(\underline{P})$ also as a set of probabilities (a so-called *credal set*).

There is a one-to-one correspondence between coherent lower previsions and strictly desirable sets of gambles. Given a coherent lower prevision $\underline{P}$, the set:

$$D^+ \coloneqq \mathcal{L}^+ \cup \{f \in \mathcal{L} : \underline{P}(f) > 0\}, \tag{4}$$

is coherent and strictly desirable and moreover induces $\underline{P}$ through Eq. (1).

There is also a one-to-one correspondence between coherent lower previsions and almost desirable sets of gambles. Given a coherent lower prevision $\underline{P}$, the set:

$$\overline{D} \coloneqq \{f \in \mathcal{L} : \underline{P}(f) \geq 0\}, \tag{5}$$

is an almost desirable set of gambles and induces again $\underline{P}$ through Eq. (1).

These one-to-one correspondences do not hold for arbitrary coherent sets of gambles, in the sense that several different coherent sets of gambles D may induce the same coherent lower prevision $\underline{P}$ by means of Eq. (1).

Let us define the maps τ and $\overline{\tau}$ from coherent lower previsions to strictly desirable sets of gambles and almost desirable sets of gambles accordingly by:

$$\tau(\underline{P}) \coloneqq \{f \in \mathcal{L} : \underline{P}(f) > 0\} \cup \mathcal{L}^+(\Omega), \quad \overline{\tau}(\underline{P}) \coloneqq \{f \in \mathcal{L} : \underline{P}(f) \geq 0\},$$

for every coherent lower prevision $\underline{P}$. Then τ and $\overline{\tau}$ are the inverses of the map σ restricted to strictly desirable and almost desirable sets of gambles respectively.

On lower previsions in general is then possible to introduce a partial order relation.

Definition 9. Given two lower (not necessarily coherent) previsions $\underline{P}, \underline{Q}$ defined respectively on $dom(\underline{P}), dom(\underline{Q}) \subseteq \mathcal{L}(\Omega)$, we say that $\underline{Q}$ *dominates* $\underline{P}$, if $dom(\underline{P}) \subseteq dom(\underline{Q})$ and $\underline{P}(f) \leq \underline{Q}(f)$ for all $f \in dom(\underline{P})$.

This is a partial order on the whole set of lower previsions.

From Lemma 6 in Appendix A follows in particular that, if $\mathcal{K} \subseteq D \subseteq \mathcal{L}$ such that $0 \notin \mathcal{E}(\mathcal{K})$ and D is coherent, then $\sigma(\mathcal{K}) \leq \sigma(D)$. So, in particular, $\sigma(\mathcal{K}) \leq \sigma(\mathcal{E}(\mathcal{K}))$. Vice versa, if instead we consider only coherent lower previsions, then $\tau, \overline{\tau}$ also preserve order.

As for sets of gambles then, it is possible to introduce a natural extension operator for lower previsions. Its definition however, is a bit more involved [4]. Here therefore, as for sets of gambles, we consider a slightly different closure operator.

Definition 10. Given a lower prevision $\underline{P}$ defined on a domain $dom(\underline{P}) \subseteq \mathcal{L}(\Omega)$, we define

$$\underline{E}^*(\underline{P})(f) \coloneqq \begin{cases} \underline{E}(\underline{P})(f) \coloneqq \min\{\underline{P}' \in \underline{\mathbb{P}}(\Omega) : \underline{P} \leq \underline{P}'\}(f) & \text{if } \exists \underline{P}' \in \underline{\mathbb{P}}(\Omega) : \underline{P} \leq \underline{P}', \\ \infty & \text{otherwise,} \end{cases}$$

for every $f \in \mathcal{L}(\Omega)$.

It coincides with the natural extension operator on lower previsions, which we denote with the symbol $\underline{E}$, if $\underline{P}$ is a lower prevision for which there exists at least a dominating coherent lower prevision [4]. In this case $\underline{E}^*(\underline{P}) = \underline{E}(\underline{P})$ is the minimal coherent lower prevision that dominates $\underline{P}$. Moreover, if $\underline{P} = \underline{P}'_{\mathcal{K}}$ where $\underline{P}'$ is coherent and $\underline{P}'_{\mathcal{K}}$ is the restriction of $\underline{P}'$ to gambles in $\mathcal{K} \subseteq \mathcal{L}$ where $\mathcal{K}$ is a linear space of gambles, then $\underline{E}^*(\underline{P})_{\mathcal{K}} = \underline{E}(\underline{P}'_{\mathcal{K}})_{\mathcal{K}} = \underline{P}'_{\mathcal{K}}$, i.e., the natural extension agrees with $\underline{P}'$ on its domain or a linear subspace of it (see 4, Theorem 3.1.2). This becomes particularly important in Section 5.2.

As usual then, the definition of $\underline{E}^*$ operator, depends on the possibility set on which gambles over that $\underline{P}$ operates are defined.

Definition 9 moreover implies that given a lower prevision $\underline{P}$, if there is $\underline{P}' \in \underline{\mathbb{P}} : \underline{P} \leq \underline{P}'$ we have:

$$\underline{E}^*(\underline{P})(f) = \min\{\underline{P}'(f) : \underline{P}' \in \underline{\mathbb{P}},\ \underline{P} \leq \underline{P}'\}$$

for every $f \in \mathcal{L}$. This fact will be used in what follows.

Finally, in view of Section 5, similarly to sets of gambles, it is convenient to consider $\underline{\Phi}(\Omega) \coloneqq \underline{\mathbb{P}}(\Omega) \cup \{\sigma(\mathcal{L}(\Omega))\}$, where $\sigma(\mathcal{L}(\Omega))(f) = \infty$ for all $f \in \mathcal{L}(\Omega)$. We can refer to it also with $\underline{\Phi}$ if there is no possible ambiguity. Notice that the equivalence: $\underline{E}^*(\underline{P}'_{\mathcal{K}})_{\mathcal{K}} = \underline{P}'_{\mathcal{K}}$, where $\mathcal{K}$ is a linear space of gambles, remains valid for all $\underline{P}' \in \underline{\Phi}$.

More new results preliminary to the rest of the work can be found in Appendix A.

Example 1. To conclude this section we provide an example to clarify some of the notions above. Suppose $\Omega = \bigtimes_{i \in I} \Omega_i$, where $I = \{1, 2, 3, 4\}$ and where each Ω_i is the set of the possible values of a binary variable X_i which can assume only values 0 and 1, therefore $\Omega_i = \{0, 1\}$ for every $i = 1, ..., 4$. In this context, suppose to have a set of agents that express their beliefs about the possible values of the binary variables considered by means of the following sets of gambles.

$$D_1^+ \coloneqq \{f \in \mathcal{L} : \min_{\{x_3, x_4 \in \{0,1\}\}} \{f(1, 0, x_3, x_4), f(0, 1, x_3, x_4)\} > 0\} \cup \mathcal{L}^+;$$

$$D_2^+ \coloneqq \{f \in \mathcal{L} : \min_{\{x_1, x_2, x_4 \in \{0,1\}\}} f(x_1, x_2, 1, x_4) > 0\} \cup \mathcal{L}^+;$$

$$D_3 \coloneqq D_2^+ \cup$$
$$\{f \in \mathcal{L} : \min_{\{x_1, x_2, x_4 \in \{0,1\}\}} f(x_1, x_2, 1, x_4) = 0 < \min_{\{x_1, x_2 \in \{0,1\}\}} f(x_1, x_2, 0, 0)\};$$

$$D_3' \coloneqq D_2^+ \cup$$
$$\{f \in \mathcal{L} : \min_{\{x_1, x_2, x_4 \in \{0,1\}\}} f(x_1, x_2, 1, x_4) = 0 < \min_{\{x_1, x_2 \in \{0,1\}\}} f(x_1, x_2, 0, 1)\}.$$

We can observe that that D_1^+, D_2^+ are the strictly desirable sets of gambles constructed through Eq. (4) starting respectively from the coherent lower previsions $\underline{P}_1, \underline{P}_2$ defined as:

$$\underline{P}_1(f) = \sigma(D_1^+)(f) \coloneqq \min_{\{x_3, x_4 \in \{0,1\}\}} \{f(1, 0, x_3, x_4), f(0, 1, x_3, x_4)\},$$

$$\underline{P}_2(f) = \sigma(D_2^+)(f) \coloneqq \min_{\{x_1, x_2, x_4 \in \{0,1\}\}} f(x_1, x_2, 1, x_4),$$

for every $f \in \mathcal{L}$. D_3 and D_3' instead are coherent but not strictly desirable sets of gambles such that $\sigma(D_3) = \sigma(D_3') = \underline{P}_2$.

We can observe moreover that $\underline{P}_1$ is equivalent to the probabilistic assessment $P(X_1 \neq X_2) = 1$, $\underline{P}_2$ instead is equivalent to $P(X_3 = 1) = 1$.

3. Information algebras

Let us give now also a short introduction to those concepts from the theory of information algebras that we shall use in this paper. We refer to [2] and [6] for a much more complete introduction and treatment.

Information algebras are algebraic structures to manage information that involves several formalisms in computer science, such as for example relational databases, multiple systems of formal logic, numerical problems of linear algebra and so on. In particular, they provide the necessary abstract framework for generic inference procedures allowing their application to a large variety of different formalisms for representing information.

The starting point of their formulation goes back to Shenoy and Shafer [8, 13], who introduced for the first time an axiomatic system to generalize the local computation scheme for probabilities proposed in [7]. It is pointed out in particular that many other formalisms satisfy the same set of axioms. A slightly changed version of the axiomatic formulation of Shenoy and Shafer is the starting point for [2], in which it is introduced a mathematical structure identified by those axioms, called *valuation algebra*. The idempotent variant of a valuation algebra is called *information algebra*.

The key observation is that many different formalisms for representing and treating knowledge or information have in common some elementary features: information comes in pieces, possibly referring to different questions, which must be aggregated to represent the whole of the information, or from which the part relevant to a particular question must be extracted. This naturally leads to consider an algebraic structure, precisely a valuation or an information algebra, composed of a set of 'pieces of information' that can be manipulated by operations of *combination*, to aggregate pieces of information, and *focusing* to extract part of information related to a certain domain. There are moreover, two equivalent axiomatic formulations of these structures: the *domain-free* one, more suitable for theoretical considerations, and the *labeled one*, more suitable instead for computational considerations.

3.1. Domain-free information algebras

Let us denote with Φ the set whose elements are considered to represent pieces of information, which are in turn denoted by lower case Greek letters. We assume that Φ is equipped with a binary operation:

$$\cdot : \Phi \times \Phi \to \Phi \textit{ (Combination).}$$

For every $\phi, \psi \in \Phi$, $\phi \cdot \psi$ represents the aggregated or combined information of the two pieces ϕ and ψ. Mimicking the intuitive properties of "aggregation", combination is assumed to be associative, commutative and idempotent. This makes $(\Phi, \cdot)$ a commutative semigroup. Further we assume a unit element 1 and a null element 0 to belong to Φ such that:

$$\phi \cdot 1 = \phi, \;\; \phi \cdot 0 = 0$$

for every $\phi \in \Phi$. The unit element represents vacuous information, combined with any piece of information it gives nothing new. The null element represents contradiction, combined with any information it nullifies it. Summing up, we assume $(\Phi, \cdot, 0, 1)$ to be a commutative idempotent semigroup with a null and, respectively, a neutral element.

Turning to questions, often we consider reasoning and inference to be concerned with variables with unknown values. Therefore here, we assume that the questions of interest regard the values of a group of variables $\{X_i : i \in I\}$, where I is a non-empty index set. In practice, I is often assumed to be finite or countable. But we need not make this restriction. So Φ is the set of pieces of information about the values of these variables. In this context, we consider a binary operation which, given a pair $(\phi, S) \in \Phi \times \mathcal{P}(I)$, extracts from ϕ the part of information relating to the unknown values of the group S of variables:

$$\epsilon : \Phi \times \mathcal{P}(I) \to \Phi \textit{ (Extraction).}$$

We ask then more properties to be satisfied to extraction and combination operation in order to mimic other important characteristics of information and simplify calculations. In summary, we can give the following definition of a domain-free information algebra in our context.

Definition 11 (Domain-free information algebra). A *domain-free information algebra* is a two-sorted structure $(\Phi, \mathcal{P}(I), \cap, \cup, \cdot, 0, 1, \epsilon)$, where:

- $(\Phi, \cdot, 0, 1)$ is a commutative semigroup with $\cdot : \Phi \times \Phi \to \Phi$, where $\cdot((\phi, \psi))$ is denoted by $\phi \cdot \psi$ for each $(\phi, \psi) \in \Phi \times \Phi$, and with 0 and 1 as the null and unit elements respectively,
- $(\mathcal{P}(I), \cap, \cup)$ is the lattice constructed from the power-set of a non-empty index set of variables I ordered by inclusion,

- $\epsilon : \Phi \times \mathcal{P}(I) \to \Phi$, where $\epsilon((\phi, S))$ is denoted by $\epsilon_S(\phi)$ for each $(\phi, S) \in \Phi \times \mathcal{P}(I)$,

satisfying moreover the following properties:

- *Transitivity*: for $\phi \in \Phi$ and $S, T \in \mathcal{P}(I)$,
$$\epsilon_S(\epsilon_T(\phi)) = \epsilon_T(\epsilon_S(\phi)) = \epsilon_{S\cap T}(\phi);$$
- *Combination*: for $\phi, \psi \in \Phi$ and $S \in \mathcal{P}(I)$,
$$\epsilon_S(\epsilon_S(\phi) \cdot \psi) = \epsilon_S(\phi) \cdot \epsilon_S(\psi);$$
- *Nullity*: for $S \in \mathcal{P}(I)$,
$$\epsilon_S(0) = 0;$$
- *Support*: for $\phi \in \Phi$,
$$\epsilon_I(\phi) = \phi;$$
- *Idempotency*: for $\phi \in \Phi$ and $S \in \mathcal{P}(I)$,
$$\epsilon_S(\phi) \cdot \phi = \phi.$$

Note that, by Idempotency we also have $\epsilon_S(1) = \epsilon_S(1) \cdot 1 = 1$ and, if $\epsilon_S(\phi) = 0$, $\phi = \epsilon_S(\phi) \cdot \phi = 0 \cdot \phi = 0$ for every $S \subseteq I$.

Since from an index set I we can always construct the lattice $(\mathcal{P}(I), \cap, \cup)$, we indicate more simply a domain-free information algebra $(\Phi, \mathcal{P}(I), \cap, \cup, \cdot, 0, 1, \epsilon)$ with $(\Phi, I, \cdot, 0, 1, \epsilon)$.

Transitivity says that the order of successive extractions does not matter. The property of Combination is for a domain-free information algebra the most important requirement: if we combine the part of a piece of information relating to the group $S \subseteq I$ of variables with any other piece of information and extract the part relating to S from the aggregated information, then we may as well *first* extract the part regrading S from the second piece of information and then combine. This is most important for so-called local computation schemes. Nullity says that extraction from the contradiction still gives a contradiction. Support is useful to construct an equivalent labeled version of a domain-free information algebra, see Section 3.4. Finally Idempotency says that combining a piece of information with part of it gives nothing new. It has important consequences, especially, it allows to introduce an information order among pieces of information, see next sections.

3.2. Labeled information algebras

Consider again a set of pieces of information $\mathbf{\Phi}$ and a group I of variables. Unlike before, here we suppose that every piece of information refers to some determined subset of variables called its *domain*. This is highlighted by the *Labeling* operation:

$$d : \mathbf{\Phi} \to \mathcal{P}(I) \text{ (Labeling)},$$

which associates to every piece of information the domain of the information contained. This leads also to a slightly different definition of the extraction operation that now is called *marginalisation* or *projection* and that operates focusing on the information captured by a piece of information $\boldsymbol{\phi} \in \mathbf{\Phi}$ for a domain smaller than $d(\boldsymbol{\phi})$:

$$\pi : (dom(\pi) \subseteq \mathbf{\Phi} \times \mathcal{P}(I)) \to \mathbf{\Phi} \text{ (Marginalisation)}.$$

Also here, we assume more other properties to manage information. In particular, we can give the following definition of a labeled information algebra in our context.

Definition 12. A *labeled information algebra* is a two-sorted structure $(\mathbf{\Phi}, \mathcal{P}(I), \cap, \cup, d, \cdot, \{\mathbf{0}_S\}_{S\in\mathcal{P}(I)}, \{\mathbf{1}_S\}_{S\in\mathcal{P}(I)}, \pi)$ where

- $d : \mathbf{\Phi} \to \mathcal{P}(I)$,
- $(\mathbf{\Phi}, \cdot)$ is a commutative semigroup with $\cdot : \mathbf{\Phi} \times \mathbf{\Phi} \to \mathbf{\Phi}$, where $\cdot((\boldsymbol{\phi}, \boldsymbol{\psi}))$ is denoted by $\boldsymbol{\phi} \cdot \boldsymbol{\psi}$ for every $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{\Phi} \times \mathbf{\Phi}$. For all $S \in \mathcal{P}(I)$ there exist an element $\mathbf{0}_S$ and an element $\mathbf{1}_S$ with $d(\mathbf{0}_S) = S$ and $d(\mathbf{1}_S) = S$ such that for all $\boldsymbol{\phi} \in \mathbf{\Phi}$ with $d(\boldsymbol{\phi}) = S$, $\mathbf{0}_S \cdot \boldsymbol{\phi} = \boldsymbol{\phi} \cdot \mathbf{0}_S = \mathbf{0}_S$ and $\mathbf{1}_S \cdot \boldsymbol{\phi} = \boldsymbol{\phi} \cdot \mathbf{1}_S = \boldsymbol{\phi}$,
- $(\mathcal{P}(I), \cap, \cup)$ is the lattice constructed from the power-set of a non-empty index set of variables I ordered by inclusion,
- $\pi : (dom(\pi) \subseteq \mathbf{\Phi} \times \mathcal{P}(I)) \to \mathbf{\Phi}$, where $\pi((\boldsymbol{\phi}, S))$ is denoted by $\pi_S(\boldsymbol{\phi})$ and it is defined for every $\boldsymbol{\phi} \in \mathbf{\Phi}$ and $S \subseteq d(\boldsymbol{\phi})$,

satisfying moreover the following properties:

1. *Labeling*: for $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathbf{\Phi}$,
$$d(\boldsymbol{\phi} \cdot \boldsymbol{\psi}) = d(\boldsymbol{\phi}) \cup d(\boldsymbol{\psi});$$
2. *Marginalisation*: for $\boldsymbol{\phi} \in \mathbf{\Phi}$ and $S \subseteq d(\boldsymbol{\phi})$,
$$d(\pi_S(\boldsymbol{\phi})) = S;$$
3. *Transitivity*: for $\boldsymbol{\phi} \in \mathbf{\Phi}$ and $T \subseteq S \subseteq d(\boldsymbol{\phi})$,
$$\pi_T(\pi_S(\boldsymbol{\phi})) = \pi_T(\boldsymbol{\phi});$$
4. *Combination*: for $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathbf{\Phi}$ with $d(\boldsymbol{\phi}) = S$ and $d(\boldsymbol{\psi}) = T$,
$$\pi_S(\boldsymbol{\phi} \cdot \boldsymbol{\psi}) = \boldsymbol{\phi} \cdot \pi_{S \cap T}(\boldsymbol{\psi});$$
5. *Nullity* and *Neutrality*: for $S, T \in \mathcal{P}(I)$,
$$\mathbf{0}_S \cdot \mathbf{0}_T = \mathbf{0}_{S \cup T};$$
$$\mathbf{1}_S \cdot \mathbf{1}_T = \mathbf{1}_{S \cup T};$$
6. *Idempotency*: for $\boldsymbol{\phi} \in \mathbf{\Phi}$ and $S \subseteq d(\boldsymbol{\phi})$,
$$\pi_S(\boldsymbol{\phi}) \cdot \boldsymbol{\phi} = \boldsymbol{\phi};$$
7. *Stability*: for $S, T \in \mathcal{P}(I)$ with $T \subseteq S$,
$$\pi_T(\mathbf{1}_S) = \mathbf{1}_T.$$

Note that, by Combination, Nullity and Marginalisation, we also have $\pi_T(\mathbf{0}_S) = \pi_T(\mathbf{0}_S \cdot \mathbf{0}_T) = \pi_T(\mathbf{0}_S) \cdot \mathbf{0}_T = \mathbf{0}_T$ for every $T \subseteq S \subseteq I$.

Since from an index set I we can always construct the lattice $(\mathcal{P}(I), \cap, \cup)$, we indicate more simply a labeled information algebra $(\mathbf{\Phi}, \mathcal{P}(I), \cap, \cup, d, \cdot, \{\mathbf{0}_S\}_{S \in \mathcal{P}(I)}, \{\mathbf{1}_S\}_{S \in \mathcal{P}(I)}, \pi)$ with $(\mathbf{\Phi}, I, d, \cdot, \{\mathbf{0}_S\}_{S \subseteq I}, \{\mathbf{1}_S\}_{S \subseteq I}, \pi)$.

Axioms required to a labeled information algebra are very similar to the ones required to a domain-free one. The main differences are due to the additional link between a piece of information and its domain. This link is what makes labeled information algebras more suitable for computational purposes. They limit the memory requirement to what is needed, whereas the domain-free versions waste memory as they contain a lot of redundancy. We shall show below, in Section 3.4, that labeled and domain-free versions of information algebras are equivalent, either one may be constructed from the other one.

3.3. Homomorphism, isomorphism and subalgebras

An important concept from universal algebra is the notion of *homomorphism*. In this context, a homomorphism is a map between information algebras which maintains operations. For the aims of this article, we introduce only the definition of homomorphism for domain-free and labeled information algebras constructed considering the same index set of variables of interest I and assuming that any subset $S \subseteq I$ is mapped on itself. More complex definitions of homomorphism can be given to treat more general cases.

Definition 13 (Homomorphism - Domain-free version). Given two domain-free information algebras $(\Phi^1, I, \cdot^1, 0^1, 1^1, \epsilon^1)$ and $(\Phi^2, I, \cdot^2, 0^2, 1^2, \epsilon^2)$, a pair (h, g) of maps $h : \Phi^1 \to \Phi^2$ and $g : \mathcal{P}(I) \to \mathcal{P}(I)$ with $g(S) = S$ for every $S \in \mathcal{P}(I)$,[4] is a *homomorphism* from the first domain-free information algebra to the second one iff, for $\phi^1, \psi^1 \in \Phi^1$ and $S \subseteq I$, we have:

1. $h(\phi^1 \cdot^1 \psi^1) = h(\phi^1) \cdot^2 h(\psi^1)$,
2. $h(0^1) = 0^2$ and $h(1^1) = 1^2$,
3. $h(\epsilon^1_S(\phi^1)) = \epsilon^2_{g(S)}(h(\phi^1)) = \epsilon^2_S(h(\phi^1))$.

Definition 14 (Homomorphism - Labeled version). Given two labeled information algebras $(\boldsymbol{\Phi}^1, I, d^1, \cdot^1, \{\mathbf{0}^1_S\}_{S\subseteq I}, \{1$ and $(\boldsymbol{\Phi}^2, I, d^2, \cdot^2, \{\mathbf{0}^2_S\}_{S\subseteq I}, \{\mathbf{1}^2_S\}_{S\subseteq I}, \pi^2)$, a pair (h, g) of maps $h : \boldsymbol{\Phi}^1 \to \boldsymbol{\Phi}^2$ and $g : \mathcal{P}(I) \to \mathcal{P}(I)$ with $g(S) = S$ for every $S \in \mathcal{P}(I)$, is called a *homomorphism* from the first labeled information algebra to the second one iff, for $\boldsymbol{\phi}^{\mathbf{1}}, \boldsymbol{\psi}^{\mathbf{1}} \in \boldsymbol{\Phi}^1$ and $S \subseteq d^1(\boldsymbol{\phi}^{\mathbf{1}})$, we have:

1. $h(\boldsymbol{\phi}^{\mathbf{1}} \cdot^1 \boldsymbol{\psi}^{\mathbf{1}}) = h(\boldsymbol{\phi}^{\mathbf{1}}) \cdot^2 h(\boldsymbol{\psi}^{\mathbf{1}})$,
2. $h(\mathbf{0}^1_S) = \mathbf{0}^2_S$ and $h(\mathbf{1}^1_S) = \mathbf{1}^2_S$, for all $S \subseteq I$,
3. $h(\pi^1_S(\boldsymbol{\phi}^{\mathbf{1}})) = \pi^2_{g(S)}(h(\boldsymbol{\phi}^{\mathbf{1}})) = \pi^2_S(h(\boldsymbol{\phi}^{\mathbf{1}}))$.

Note that in the definition of a homomorphism for labeled information algebras, we do not require that Labeling is conserved by h. This indeed follows already from the stability axiom of a labeled information algebra [2, p. 58].

Given that we consider g fixed in our definitions of homomorphisms, in what follows we refer only to the map h as a homomorphism, implying always the existence of a map $g : \mathcal{P}(I) \to \mathcal{P}(I)$ with $g(S) = S$ for every $S \in \mathcal{P}(I)$. If h is a homomorphism and it is bijective, it is called an *isomorphism* both for the domain-free and for the labeled case.

Another important concept from universal algebra is the notion of *subalgebra*. In terms of information algebras, this is a subset of an information algebra which is closed under its operations. More precisely, we have the following definition for the domain-free case.

Definition 15 (Subalgebras - Domain-free version). Let $(\Phi, I, \cdot, 0, 1, \epsilon)$ be a domain-free information algebra. $(\Phi', I, \cdot, 0, 1, \epsilon)$ is said to be a subalgebra of $(\Phi, I, \cdot, 0, 1, \epsilon)$, if

1. $\Phi' \subseteq \Phi$,
2. $\phi', \psi' \in \Phi'$ implies $\phi' \cdot \psi' \in \Phi'$,
3. $0, 1 \in \Phi'$,
4. $\phi' \in \Phi', S \subseteq I$ implies $\epsilon_S(\phi') \in \Phi'$.

A similar notion holds also for labeled information algebras [2].

[4] As previously noticed, the index set of variables is the same in both the signatures and g is the identity function.

3.4. Equivalence between domain-free and labeled information algebras

We have a full correspondence between domain-free and labeled information algebras.

Let us consider a domain-free information algebra $(\Phi, I, \cdot, 0, 1, \epsilon)$. Consider then the set of pairs

$$\mathbf{\Phi} := \{(\phi, S), \phi \in \Phi, S \subseteq I : \; \epsilon_S(\phi) = \phi\}. \tag{6}$$

These pairs can be considered as pieces of information ϕ, labeled by their domains. Now, let us define on $\mathbf{\Phi}$ and $\mathcal{P}(I)$ the following operations expressed in terms of the ones defined on $(\Phi, I, \cdot, 0, 1, \epsilon)$.

- Labeling: $d : \mathbf{\Phi} \to \mathcal{P}(I)$, defined as
$$d(\phi, S) := S,$$
for every $(\phi, S) \in \mathbf{\Phi}$.
- Combination: $\cdot : \mathbf{\Phi} \times \mathbf{\Phi} \to \mathbf{\Phi}$, where $\cdot(((\phi, S)(\psi, T)))$ is denoted by $(\phi, S) \cdot (\psi, T)$ and defined as
$$(\phi, S) \cdot (\psi, T) := (\phi \cdot \psi, S \cup T),$$
for every $(\phi, S), (\psi, T) \in \mathbf{\Phi}$,
- Marginalisation: $\pi : (dom(\pi) \subseteq \mathbf{\Phi} \times \mathcal{P}(I)) \to \mathbf{\Phi}$, where $\pi(((\phi, S), T))$ is denoted by $\pi_T(\phi, S)$ and defined as
$$\pi_T(\phi, S) := (\epsilon_T(\phi), T),$$
for every $(\phi, S) \in \mathbf{\Phi}$ and $T \subseteq S \subseteq I$.

It can be shown that $(\mathbf{\Phi}, I, d, \cdot, \{(0, S)\}_{S \subseteq I}, \{(1, S)\}_{S \subseteq I}, \pi)$ where $d, \cdot$ and π are the operations just defined on $\mathbf{\Phi}$ and $\mathcal{P}(I)$, is a labeled information algebra [2]. From this labeled information algebra it is possible to reconstruct a domain-free information algebra that is isomorphic to the original one. Similarly, it is possible to start with a labeled algebra, construct the associated domain-free information algebra and then reconstruct by the procedure just introduced a labeled information algebra. And this labeled algebra is again essentially (up to isomorphism) the same as the original one.

3.5. Atoms

The axioms defining a domain-free and a labeled version of an information algebra lead to the definition of a partial order on pieces of information that is called *information order*.

Definition 16 (Information order - Domain-free version). Consider a domain-free information algebra $(\Phi, I, \cdot, 0, 1, \epsilon)$. Given $\phi, \psi \in \Phi$ we say that $\phi \leq \psi$ if and only if $\phi \cdot \psi = \psi$.

This means that ψ is more informative than ϕ, if adding ϕ to ψ gives nothing new. The same definition can be given for the labeled case.

In certain information algebras there are maximally informative elements, called *atoms*.

Definition 17 (Atoms - Domain-free version). Given a domain-free information algebra $(\Phi, I, \cdot, 0, 1, \epsilon)$, an element $\alpha \in \Phi$ is called an *atom* if and only if

- $\alpha \neq 0$,
- for all $\phi \in \Phi$, $\alpha \leq \phi$ implies either $\phi = \alpha$ or $\phi = 0$.

This says that no information, except the null information, can be more informative than an atom. A similar definition can be given also in the labeled case, where atoms are maximally informative elements with respect to their domains.

Definition 18 (Atoms - Labeled version). Given a labeled information algebra $(\mathbf{\Phi}, I, d, \cdot, \{\mathbf{0}_S\}_{S \subseteq I}, \{\mathbf{1}_S\}_{S \subseteq I}, \pi)$, an element $\boldsymbol{\alpha} \in \mathbf{\Phi}$ with $d(\boldsymbol{\alpha}) = S$ is called an *atom relative to* S, if and only if

- $\boldsymbol{\alpha} \neq \mathbf{0}_S$,
- for all $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ with $d(\boldsymbol{\phi}) = S$, $\boldsymbol{\alpha} \leq \boldsymbol{\phi}$ implies either $\boldsymbol{\phi} = \boldsymbol{\alpha}$ or $\boldsymbol{\phi} = \mathbf{0}_S$.

Now, let limit us to the domain-free case. Similar discussion can be recovered for the labeled one. We can recall some elementary results on atoms.

Lemma 1. *Consider a domain-free information algebra* $(\Phi, I, \cdot, 0, 1, \epsilon)$.

- *If* $\alpha \in \Phi$ *is an atom and* $\phi \in \Phi$*, then* $\phi \cdot \alpha = \alpha$ *or* $\phi \cdot \alpha = 0$.
- *If* $\alpha \in \Phi$ *is an atom and* $\phi \in \Phi$*, then either* $\phi \leq \alpha$ *or* $\phi \cdot \alpha = 0$.
- *If* $\alpha, \beta \in \Phi$ *are atoms, then* $\alpha = \beta$ *or* $\alpha \cdot \beta = 0$.

Denote with $At(\Phi)$ the set of all atoms in Φ. Let us define also, for every element $\phi \in \Phi$,

$$At(\phi) \coloneqq \{\alpha \in At(\Phi) : \phi \leq \alpha\}. \tag{7}$$

This motivates the following definitions.

Definition 19 (Atomic information algebras - Domain free version). A domain-free information algebra $(\Phi, I, \cdot, 0, 1, \epsilon)$ is called:

1. *atomic*, if and only if for all $\phi \in \Phi$, $\phi \neq 0$, $At(\phi)$ is not empty.
2. *atomistic* or *atomic composed*, if and only if it is atomic and for all $\phi \in \Phi$, $\phi \neq 0$,
$$\phi = \inf At(\phi).$$
3. *completely atomistic* or *atomic closed*, if and only if it is atomistic and for every non-empty subset $A \subseteq At(\phi)$ for some $\phi \in \Phi$, the infimum exists and belongs to Φ.

Atoms of any atomic labeled information algebra form a *tuple system*, which abstracts systems of concrete tuples as used in relational database systems [2]. It is possible to show moreover that a labeled information algebra can be constructed from sets of tuples [2].

In the rest of the paper we provide examples of domain-free and labeled information algebras that arise from coherent sets of gambles and coherent lower previsions. We prove in particular that they are completely atomistic, where atoms, or atoms relative to S for every $S \subseteq I$ in the labeled cases, are formed respectively by maximal coherent sets of gambles and linear previsions.

4. Information algebras of coherent sets of gambles

4.1. Domain-free version

As previously said, coherent sets of gambles represent beliefs of an agent about a possibility space. Therefore, they can be interpreted as pieces of information about it.

As in the previous section, we limit the analysis to the case in which the information one is interested in concerns the values of certain groups of variables. We assume therefore, a special form for the possibility space Ω, namely a *multivariate model*. Consider a group of variables $\{X_i : i \in I\}$ where I is a non-empty index set. Any variable X_i has a domain of possible values Ω_i. For any subset S of I let

$$\Omega_S \coloneqq \bigtimes_{i \in S} \Omega_i,$$

and $\Omega = \Omega_I$.[5] Coherent sets of gambles, or rather sets in $\Phi(\Omega)$, correspond therefore to pieces of information regarding the values of the variables considered.

[5] If needed, we assume the axiom of choice.

The elements ω in Ω can be seen as functions $\omega : I \to \Omega$, so that $\omega_i \in \Omega_i$, for any $i \in I$. A gamble f on Ω is called S-measurable if it depends only on the values of the group S of variables, i.e., if for all $\omega, \omega' \in \Omega$ with $\omega|S = \omega'|S$ we have $f(\omega) = f(\omega')$ (here $\omega|S$ is the restriction of the map ω to S). Let $\mathcal{L}_S(\Omega)$, or more simply $\mathcal{L}_S$, denote the set of all S-measurable gambles. If $I = \emptyset$, $\mathcal{L}(\Omega_I) = \mathbb{R}$, the set of constant gambles [14, Section 2.3], moreover, we can define also $\mathcal{L}_\emptyset := \mathbb{R}$, and note that $\mathcal{L}_I = \mathcal{L}(\Omega)$. For future we show the following result.

Lemma 2. *For any subsets S and T of I:*

$$\mathcal{L}_{S \cap T} = \mathcal{L}_S \cap \mathcal{L}_T.$$

Proof. Consider firstly $f \in \mathcal{L}_{S\cap T}$. Consider two elements $\omega, \mu \in \Omega$ so that $\omega|S = \mu|S$. Then we have also $\omega|S \cap T = \mu|S \cap T$ and $f(\omega) = f(\mu)$. So we see that $f \in \mathcal{L}_S$ and similarly $f \in \mathcal{L}_T$.

Conversely, assume $f \in \mathcal{L}_S \cap \mathcal{L}_T$. Consider two elements $\omega,\ \mu \in \Omega$. so that $\omega|S \cap T = \mu|S \cap T$. Consider then the element $\lambda \in \Omega$ defined as

$$\lambda_i := \begin{cases} \omega_i = \mu_i, & i \in (S \cap T), \\ \omega_i, & i \in (S \setminus T) \cup (S \cup T)^c, \\ \mu_i, & i \in T \setminus S \end{cases}$$

for every $i \in I$. Then $\lambda|S = \omega|S$ and $\lambda|T = \mu|T$. Since f is both S- and T-measurable we have $f(\omega) = f(\mu)$. It follows that $f \in \mathcal{L}_{S\cap T}$ and this concludes the proof.

Now, let us consider Φ and $\mathcal{P}(I)$ and define on them the following operations.

1. Combination. $\cdot : \Phi \times \Phi \to \Phi$, where $\cdot((D_1, D_2))$ is denoted by $D_1 \cdot D_2$ and defined as

   $$D_1 \cdot D_2 := D_1 \vee D_2 := \mathcal{C}(D_1 \cup D_2),$$

   for every $D_1, D_2 \in \Phi$.

2. Extraction. $\epsilon : \Phi \times \mathcal{P}(I) \to \Phi$, where $\epsilon((D, S))$ is denoted by $\epsilon_S(D)$ and defined as

   $$\epsilon_S(D) := \mathcal{C}(D \cap \mathcal{L}_S),$$

   for every $D \in \Phi$, $S \in \mathcal{P}(I)$.

Note that $D_1 \cdot D_2 = \mathcal{L}$ for some $D_1, D_2 \in \Phi$, means that the two sets D_1 and D_2 are not *consistent*, that is, $\mathcal{E}(D_1 \cup D_2)$ is not coherent (see Section 6). So, $\mathcal{L}$ is the null element of combination and represents *inconsistency*. The set $\mathcal{L}^+$ instead, is the unit element of combination, representing vacuous information.

We claim that Φ and $\mathcal{P}(I)$ equipped with these operations form a domain-free information algebra.

Theorem 1. *1. $(\Phi, \cdot, 0, 1)$ is a commutative semigroup with a null element $0 = \mathcal{L}$ and a unit element $1 = \mathcal{L}^+$.*

2. For any subset $S \subseteq I$ and $D, D_1, D_2 \in \Phi$:

E1 $\epsilon_S(0) = 0$,

E2 $\epsilon_S(D) \cdot D = D$,

E3 $\epsilon_S(\epsilon_S(D_1) \cdot D_2) = \epsilon_S(D_1) \cdot \epsilon_S(D_2)$.

3. For any $S, T \subseteq I$ and any $D \in \Phi$, $\epsilon_S(\epsilon_T(D)) = \epsilon_T(\epsilon_S(D)) = \epsilon_{S\cap T}(D)$.

4. For any $D \in \Phi$, $\epsilon_I(D) = D$.

Proof. 1. That $(\Phi, \cdot)$ is a commutative semigroup follows from $D_1 \cdot D_2 := D_1 \vee D_2$, for any D_1, D_2 in the complete lattice $(\Phi, \subseteq)$. As stated above, $0 = \mathcal{L}$ is the null element and $1 = \mathcal{L}^+$ the unit element of the semilattice (null and unit in a semigroup are always unique).

2. For E1 we have

$$\epsilon_S(0) = \epsilon_S(\mathcal{L}) \coloneqq \mathcal{C}(\mathcal{L} \cap \mathcal{L}_S) = \mathcal{C}(\mathcal{L}_S) = \mathcal{L} = 0,$$

for any $S \subseteq I$.

E2 follows since $D \cap \mathcal{L}_S \subseteq D$ and $C(D \cap \mathcal{L}_S) \subseteq D$, for any $D \in \Phi$, $S \subseteq I$.

To prove E3 define, using Lemma 5 in Appendix A,

$$\begin{aligned} A &\coloneqq \mathcal{C}(\mathcal{C}(D_1 \cap \mathcal{L}_S) \cup D_2) \cap \mathcal{L}_S = \mathcal{C}((D_1 \cap \mathcal{L}_S) \cup D_2) \cap \mathcal{L}_S, \\ B &\coloneqq \mathcal{C}(\mathcal{C}(D_1 \cap \mathcal{L}_S) \cup (\mathcal{C}(D_2 \cap \mathcal{L}_S)) = \mathcal{C}((D_1 \cap \mathcal{L}_S) \cup (D_2 \cap \mathcal{L}_S)). \end{aligned}$$

Then we have $B \coloneqq \epsilon_S(D_1) \cdot \epsilon_S(D_2)$ and $\mathcal{C}(A) \coloneqq \epsilon_S(\epsilon_S(D_1) \cdot D_2)$. Note that $B \subseteq \mathcal{C}(A)$.

We claim first that:

$$\epsilon_S(D_1) \cdot \epsilon_S(D_2) = 0 \iff \epsilon_S(D_1) \cdot D_2 = 0. \tag{8}$$

Indeed, $\epsilon_S(D_1) \cdot \epsilon_S(D_2) = 0$ implies a fortiori $\epsilon_S(D_1) \cdot D_2 = 0$.

Assume therefore that $\epsilon_S(D_1) \cdot D_2 = 0$. This implies $0 = \mathcal{C}(\mathcal{C}(D_1 \cap \mathcal{L}_S) \cup D_2) = \mathcal{C}((D_1 \cap \mathcal{L}_S) \cup D_2)$, by Lemma 5 in Appendix A. Now, if $D_1 = \mathcal{L}$ or $D_2 = \mathcal{L}$ we have immediately the result, otherwise we claim that $0 = f + g'$ with $f \in D_1 \cap \mathcal{L}_S$ and $g' \in D_2 \cap \mathcal{L}_S$. Indeed, from $0 = \mathcal{C}((D_1 \cap \mathcal{L}_S) \cup D_2)$, we know that $0 \in \mathcal{E}((D_1 \cap \mathcal{L}_S) \cup D_2)$ therefore $0 = f + g + h'$ with $f \in D_1 \cap \mathcal{L}_S$, $g \in D_2$, $h' \in \mathcal{L}^+(\Omega) \subseteq D_2$ or $h' = 0$. Then, if we introduce $g' = g + h'$, we have $0 = f + g'$ with $f \in D_1 \cap \mathcal{L}_S$, $g' \in D_2$. However, this implies $g' = -f \in \mathcal{L}_S$ and then $g' \in D_2 \cap \mathcal{L}_S$. Notice that $\epsilon_S(D_1) \cdot \epsilon_S(D_2) =: B = \mathcal{C}((D_1 \cap \mathcal{L}_S) \cup (D_2 \cap \mathcal{L}_S))$. Therefore, we have the result.

So, if $\epsilon_S(D_1) \cdot D_2 = 0$ or $B = 0$, then $\mathcal{C}(A) = 0$ and $B = 0$. Therefore we have $\mathcal{C}(A) \subseteq B$.

Vice versa, assume both $\epsilon_S(D_1) \cdot D_2$ and $\epsilon_S(D_1) \cdot \epsilon_S(D_2)$ coherent. Therefore $\epsilon_S(D_1) \cdot D_2 = \mathcal{C}((D_1 \cap \mathcal{L}_S) \cup D_2) = \mathcal{E}((D_1 \cap \mathcal{L}_S) \cup D_2)$. Then we have

$$A = \{f \in \mathcal{L}_S : f \geq \lambda g + \mu h, g \in D_1 \cap \mathcal{L}_S, h \in D_2, \lambda, \mu \geq 0, f \neq 0\}.$$

Consider $f \in A$. Then $f = \lambda g + \mu h + h'$, where $h' \in \mathcal{L}^+ \cup \{0\}$. Since f and g are S-measurable, $\mu h + h'$ must be S-measurable. Now, if $\mu h + h' = 0$ then $f \in D_1 \cap \mathcal{L}_S \subseteq B$. Otherwise, $\mu h + h' \in D_2 \cap \mathcal{L}_S$. So in any case $f \in B$, hence we have $\mathcal{C}(A) \subseteq C(B) = B$.

3. Note first that $\epsilon_S(\epsilon_T(D)) = 0$ and $\epsilon_{S \cap T}(D) = 0$ if and only if $D = 0$. So assume D to be coherent. Then we have, by Lemma 2,

$$\begin{aligned} \epsilon_S(\epsilon_T(D)) &\coloneqq \mathcal{C}(C(D \cap \mathcal{L}_T) \cap \mathcal{L}_S), \\ \epsilon_{S \cap T}(D) &\coloneqq \mathcal{C}(D \cap \mathcal{L}_{S \cap T}) = \mathcal{C}(D \cap \mathcal{L}_T \cap \mathcal{L}_S). \end{aligned}$$

Obviously, $\epsilon_{S \cap T}(D) \subseteq \epsilon_S(\epsilon_T(D))$. Consider then $f \in C(D \cap \mathcal{L}_T) \cap \mathcal{L}_S$. If $f \in \mathcal{L}_S^+$ then clearly $f \in \epsilon_{S \cap T}(D)$. Otherwise,

$$f \in \mathcal{L}_S, \quad f \geq g, \quad g \in D \cap \mathcal{L}_T.$$

Define

$$g'(\omega) \coloneqq \sup_{\lambda|S = \omega|S} g(\lambda).$$

Then we have $f \geq g'$. Clearly g' is S-measurable and belongs to D, $g' \in D \cap \mathcal{L}_S$. We claim that g' is also T-measurable. Consider two elements ω and μ so that $\omega|S \cap T = \mu|S \cap T$. Note that we may write

$$g'(\omega) \coloneqq \sup_{\lambda|S = \omega|S} g(\lambda) = \sup_{\lambda|I \setminus S} g(\omega|S \cap T, \omega|S \setminus T, \lambda|T \setminus S, \lambda|R),$$

where $R = (S \cup T)^c$. Similarly, we have

$$g'(\mu) \coloneqq \sup_{\lambda'|S=\mu|S} g(\lambda') = \sup_{\lambda'|I\setminus S} g(\omega|S \cap T, \mu|S \setminus T, \lambda'|T \setminus S, \lambda'|R).$$

Since g is T-measurable, we have:

$$g'(\mu) = \sup_{\lambda'|I\setminus S} g(\omega|S \cap T, \omega|S \setminus T, \lambda'|T \setminus S, \lambda'|R),$$

that clearly coincides with $g'(\omega)$.

This shows that g' is $S \cap T$-measurable, therefore both S- and T-measurable by Lemma 2. So we have $g' \in D \cap \mathcal{L}_S \cap \mathcal{L}_T$, hence $f \in \mathcal{C}(D \cap \mathcal{L}_T \cap \mathcal{L}_S)$. And hence $\mathcal{C}(\mathcal{C}(D \cap \mathcal{L}_T) \cap \mathcal{L}_S) \subseteq \mathcal{C}(\mathcal{C}(D \cap \mathcal{L}_T \cap \mathcal{L}_S)) = \mathcal{C}(D \cap \mathcal{L}_T \cap \mathcal{L}_S)$.

Analogously, we can prove that $\epsilon_T(\epsilon_S(D)) = \epsilon_{S\cap T}(D)$.

4. It is obvious.

This result shows that $(\Phi(\Omega), I, \cdot, \mathcal{L}(\Omega), \mathcal{L}^+(\Omega), \epsilon)$ where $\cdot, \epsilon$ are defined above on $\Phi(\Omega)$ and $\mathcal{P}(I)$, is a domain-free information algebra that, with a little abuse of nomenclature, we call *domain-free information algebra of coherent sets of gambles defined on* Ω. The possibility set Ω is assumed to be fixed in this article, therefore, we also indicate it with $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ and we call it *the domain-free information algebra of coherent sets of gambles* when there is no possible ambiguity.

Two associated labeled versions will be derived in the next subsection.

Example 2. Consider again the framework of Example 1. We have assumed a multivariate model for Ω. Therefore, we can look at D_1^+, D_2^+, D_3, D_3' also as pieces of information regarding the values of the variables $X_1, ..., X_4$ defined in Example 1.

If we now want to combine together all the pieces of information represeted by D_1^+, D_2^+, D_3, D_3' we obtain inconsistence. Indeed:

$$D_1^+ \cdot D_2^+ \cdot D_3 \cdot D_3' \coloneqq \mathcal{C}(D_1^+ \cup D_2^+ \cup D_3 \cup D_3') = \mathcal{L}. \tag{9}$$

In fact, $0 = f + g \in \mathcal{E}(D_1^+ \cup D_2^+ \cup D_3 \cup D_3')$, where the gambles $f \in D_3$ and $g \in D_3'$ are defined as follows for every $\omega \in \Omega$:

$$f(\omega) \coloneqq \begin{cases} 1 & \text{if } \omega = (1,0,0,0) \\ 1 & \text{if } \omega = (1,1,0,0) \\ 1 & \text{if } \omega = (0,1,0,0) \\ 1 & \text{if } \omega = (0,0,0,0) \\ -1 & \text{if } \omega = (1,0,0,1) \\ -1 & \text{if } \omega = (1,1,0,1) \\ -1 & \text{if } \omega = (0,1,0,1) \\ -1 & \text{if } \omega = (0,0,0,1) \\ 0 & \text{otherwise} \end{cases}, \quad g(\omega) \coloneqq \begin{cases} -1 & \text{if } \omega = (1,0,0,0) \\ -1 & \text{if } \omega = (1,1,0,0) \\ -1 & \text{if } \omega = (0,1,0,0) \\ -1 & \text{if } \omega = (0,0,0,0) \\ 1 & \text{if } \omega = (1,0,0,1) \\ 1 & \text{if } \omega = (1,1,0,1) \\ 1 & \text{if } \omega = (0,1,0,1) \\ 1 & \text{if } \omega = (0,0,0,1) \\ 0 & \text{otherwise} \end{cases}.$$

Suppose now that some reasons come to consider unreliable the agent who provided the piece of information D_3'. The remaining pieces of information can then be combined together in a consistent way:

$$D \coloneqq D_1^+ \cdot D_2^+ \cdot D_3 \coloneqq \mathcal{C}(D_1^+ \cup D_3) = \mathcal{E}(D_1^+ \cup D_3) =: D_1^+ \cdot D_3.$$

$\mathcal{E}(D_1^+ \cup D_3)$ is coherent indeed because $\mathcal{E}(D_1^+ \cup D_3) \subseteq \{f \in \mathcal{L} : f(1,0,1,0) > 0\} \cup \{f \in \mathcal{L} : f(1,0,0,0) > 0\} \cup \mathcal{L}^+ \not\ni 0$. It represents the whole reliable information about the variables considered. From it, we can extract the information regarding every subset of variables. Let us consider for example the extraction of the information about the subset $S_3 := \{3, 4\}$ of variables.

$$\epsilon_{S_3}(D) = \epsilon_{S_3}(D_1^+ \cdot D_3) = \epsilon_{S_3}(D_1^+ \cdot \epsilon_{S_3}(D_3)) =$$
$$\epsilon_{S_3}(D_1^+) \cdot \epsilon_{S_3}(D_3) = \mathcal{L}^+ \cdot D_3 = D_3.$$

Indeed D_3 contains only information regarding variables $\{X_3, X_4\}$, therefore $\epsilon_{S_3}(D_3) = D_3$. Vice versa, D_1^+ does not contain any information on these variables, therefore $\epsilon_{S_3}(D_1^+) = \mathcal{L}^+$.

As we have seen in Section 3.5, in any information algebra can be introduced a partial order, called information order, on pieces of information. In the case of coherent sets of gambles, this order translates in $D_1 \leq D_2$ if and only if $D_1 \cdot D_2 = D_2$. This means that D_2 is more informative than D_1 if and only if adding D_1 gives nothing new; D_1 is already contained in D_2. It is easy to verify in fact that $D_1 \leq D_2$ if and only if $D_1 \subseteq D_2$. So, in particular, $(\Phi, \leq)$ is a complete lattice, since information order corresponds to set inclusion. Moreover, combination corresponds to join with respect to this order, $D_1 \cdot D_2 = D_1 \vee D_2$, vacuous information $\mathcal{L}^+$ is the least information and $\mathcal{L}$ is the top element (although strictly speaking it is no more a piece of information, since it represents inconsistency).

Example 3. Considering again Example 1., we can observe that $D_2^+ \leq D_3, D_3'$.

Conditions E1 to E3 in Theorem 1 can also be rewritten using this order as the following. For any subset $S \subseteq I$ and $D, D_1, D_2 \in \Phi$:

E1 $\epsilon_S(0) = 0$,

E2 $\epsilon_S(D) \leq D$,

E3 $\epsilon_S(\epsilon_S(D_1) \vee D_2) = \epsilon_S(D_1) \vee \epsilon_S(D_2)$.

In algebraic logic such an operator is also called an *existential quantifier*.[6]

We further claim that extraction distributes over intersection (or meet in the complete lattice $(\Phi, \leq)$).

Theorem 2. *Let $\{D_j\}_{j \in J}$ be any family of sets of gambles in Φ and $S \subseteq I$. Then*

$$\epsilon_S(\bigcap_{j \in J} D_j) = \bigcap_{j \in J} \epsilon_S(D_j). \tag{10}$$

Proof. We may assume that $D_j \in C$ for all $j \in J$ since if some or all $D_j = \mathcal{L}$, then we may restrict the intersection on both sides over the set $D_j \in C$, or the intersection over both sides equals $\mathcal{L}$. If $D_j \in C$ for all $j \in J$, we have

$$\begin{aligned}
\epsilon_S(\bigcap_{j \in J} D_j) &= \mathcal{E}((\bigcap_{j \in J} D_j) \cap \mathcal{L}_S) := \mathsf{posi}(\mathcal{L}^+(\Omega) \cup ((\bigcap_{j \in J} D_j) \cap \mathcal{L}_S)), \\
\bigcap_{j \in J} \epsilon_S(D_j) &= \bigcap_{j \in J} \mathcal{E}(D_j \cap \mathcal{L}_S) := \bigcap_{j \in J} \mathsf{posi}(\mathcal{L}^+(\Omega) \cup (D_j \cap \mathcal{L}_S)).
\end{aligned}$$

Consider first a gamble $f \in \epsilon_S(\bigcap_{j \in J} D_j)$ so that $f = \lambda g + \mu h$, where λ, μ nonnegative and not both equal zero, $g \in (\bigcap_{j \in J} D_j) \cap \mathcal{L}_S$ and $h \in \mathcal{L}^+$. But then $g \in D_j \cap \mathcal{L}_S$ for all $j \in J$, so that $f \in \bigcap_{j \in J} \epsilon_S(D_j)$.

Conversely, assume $f \in \bigcap_{j \in J} \epsilon_S(D_j)$. If $f \in \mathcal{L}^+$, then $f \in \epsilon_S(\bigcap_{j \in J} D_j)$. Otherwise, $f \geq g_j$ for some $g_j \in (D_j \cap \mathcal{L}_S)$, for all $j \in J$. Hence, $f(\omega) \geq \sup_{k \in J} g_k(\omega)$ for every $\omega \in \Omega$. However, $\sup_{k \in J} g_k \in \bigcap_{j \in J} D_j$ because $\sup_{k \in J} g_k(\omega) \geq g_j(\omega)$ for all $j \in J$, for all $\omega \in \Omega$. Moreover, $\sup_{k \in J} g_k \in \mathcal{L}_S$, thanks to the fact that $g_j \in \mathcal{L}_S$ for all $j \in J$. Therefore, $f \in \epsilon_S(\bigcap_{j \in J} D_j)$.

So $(\Phi, \leq)$ is a lattice under information order and satisfies Eq. (10). An information algebra with this property, is called a *lattice* information algebra.

The family of strictly desirable sets of gambles enlarged with $\mathcal{L}$ is also closed under combination and extraction in $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$. Therefore, $(\Phi^+, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ forms a subalgebra of $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$.

[6] Although usually operators on a Boolean lattice are considered and the order is inverse to the information order.

4.2. Labeled versions

The domain-free information algebra of coherent sets of gambles treats the general case of gambles defined on Ω.[7] However, it is well known that, if a coherent set of gambles D is such that $D = \mathcal{E}(D \cap \mathcal{L}_S)$ for some $S \subset I$, it is essentially determined by values of gambles defined on a smaller set of possibilities than Ω, namely on blocks $[\omega]_S$ of the equivalence relation $\equiv_S$ defined as $\omega \equiv_S \omega' \iff \omega|S = \omega'|S$ for every $\omega, \omega' \in \Omega$. Indeed, it contains the same information of the set $D \cap \mathcal{L}_S$ that is in a one-to-one correspondence with a set $\tilde{D}$ directly defined on blocks $[\omega]_S$, see for example [1]. This view leads to a labeled version of $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ which is better suited for computational purposes.

We start deriving a labeled view of $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ using the general method for domain-free information algebras to derive corresponding labeled ones seen in Section 3.4. In this case, as well as in the case of coherent lower previsions, there is a second isomorphic version of the labeled algebra, which is nearer to the intuition explained before and which will be introduced after this general construction. As noticed in Section 3.4 then, from the labeled information algebra that derives from the general method, the domain-free one may be reconstructed, so the two views are equivalent.

We begin introducing the concept of *support* for sets of gambles in Φ.

Definition 20 (Support for sets of gambles). A subset S of I is called *support* of a set of gambles $D \in \Phi$, if $\epsilon_S(D) = D$.

This means that the information contained in D concerns, or is focused on, the group S of variables.

Here are a few well-known results on supports in domain-free information algebras, for the proofs see [2].

Lemma 3. *Let $D, D_1, D_2 \in \Phi$, $S, T \subseteq I$. The following are true:*

1. *any S is a support of the null 0 $(\mathcal{L}(\Omega))$ and the unit 1 $(\mathcal{L}^+(\Omega))$ elements,*
2. *S is a support of $\epsilon_S(D)$,*
3. *if S is a support of D, then S is a support of $\epsilon_T(D)$,*
4. *if S and T are supports of D, then so is $S \cap T$,*
5. *if S is a support of D, then $\epsilon_T(D) = \epsilon_{S \cap T}(D)$,*
6. *if T is a support of D and $T \subseteq S$, then S is a support of D,*
7. *if S is a support of D_1 and D_2, then it is also a support of $D_1 \cdot D_2$,*
8. *if S is a support of D_1 and T a support of D_2, then $S \cup T$ is a support of $D_1 \cdot D_2$.*

Example 4. Let us consider again Example 1. It is possible to notice that:

- $S_1 \coloneqq \{1, 2\}$ is a support of D_1^+;
- $S_2 \coloneqq \{3\}$ is a support of D_2^+;
- $S_3 \coloneqq \{3, 4\}$ is a support of D_3, D_3'.

However, item 6. of Lemma 3 guarantees that S_3 is also a support of D_2^+.

Now, we proceed with the procedure illustrated in Section 3.4 to construct a labeled information algebra from a domain-free one. Therefore, let us suppose again a multivariate model for the possibility set Ω, i.e. suppose $\Omega = \bigtimes_{i \in I} \Omega_i$ for some non-empty index set I, as in the previous subsection. Consider then the sets

[7] If we assume a multivariate model for it.

$\mathbf{\Phi}_S(\Omega)$, or $\mathbf{\Phi}_S$ when there is no possible ambiguity, of pairs (D, S), where $S \subseteq I$ is a support of $D \in \Phi(\Omega)$. That is, we collect together pieces of information concerning the same set of variables. Let

$$\mathbf{\Phi}(\Omega) \coloneqq \bigcup_{S \subseteq I} \mathbf{\Phi}_S(\Omega).$$

As usual, we can refer to it also with $\mathbf{\Phi}$, when there is no possible ambiguity. On $\mathbf{\Phi}$ and $\mathcal{P}(I)$ we define the following operations in terms of the ones defined on the information algebra $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$.

1. Labeling. $d : \mathbf{\Phi} \to \mathcal{P}(I)$, defined as
$$d(D, S) \coloneqq S,$$
for every $(D, S) \in \mathbf{\Phi}$.

2. Combination. $\cdot : \mathbf{\Phi} \times \mathbf{\Phi} \to \mathbf{\Phi}$, where $\cdot(((D_1, S), (D_2, T)))$ is denoted by $(D_1, S) \cdot (D_2, T)$ and defined as
$$(D_1, S) \cdot (D_2, T) \coloneqq (D_1 \cdot D_2, S \cup T),$$
for every $(D_1, S), (D_2, T) \in \mathbf{\Phi}$.

3. Marginalisation. $\pi : (dom(\pi) \subseteq \mathbf{\Phi} \times \mathcal{P}(I)) \to \mathbf{\Phi}$, where $\pi(((D, S), T))$ is denoted by $\pi_T(D, S)$ and defined as
$$\pi_T(D, S) \coloneqq (\epsilon_T(D), T),$$
for every $(D, S) \in \mathbf{\Phi}$, $T \subseteq S \subseteq I$.

We previously saw in Section 3.4 that $(\mathbf{\Phi}(\Omega), I, d, \cdot, \{(\mathcal{L}(\Omega), S)\}_{S \subseteq I}, \{(\mathcal{L}^+(\Omega), S)\}_{S \subseteq I}, \pi)$, or more simply $(\mathbf{\Phi}, I, d, \cdot, \{(\mathcal{L}, S)\}_{S \subseteq I}, \{(\mathcal{L}^+, S)\}_{S \subseteq I}, \pi)$, where $d, \cdot$ and π are defined above on $\mathbf{\Phi}$ and $\mathcal{P}(I)$, is a labeled information algebra that corresponds to a labeled view of $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$.

We may associate to this labeled algebra another, isomorphic one. For a subset S of I, let $C(\Omega_S)$ be the family of coherent sets of gambles defined on $\Omega_S \coloneqq \bigtimes_{i \in S} \Omega_i$. Furthermore, let $\tilde{\mathbf{\Phi}}_S(\Omega)$ be the set of pairs $(\tilde{D}, S)$, where $S \subseteq I$ and $\tilde{D} \in \Phi(\Omega_S) \coloneqq C(\Omega_S) \cup \{\mathcal{L}(\Omega_S)\}$, and

$$\tilde{\mathbf{\Phi}}(\Omega) \coloneqq \bigcup_{S \subseteq I} \tilde{\mathbf{\Phi}}_S(\Omega).$$

We will refer to them also with $\tilde{\mathbf{\Phi}}$ and $\tilde{\mathbf{\Phi}}_S$ for every $S \subseteq I$, when there is no ambiguity.

It is well known that there is a one-to-one correspondence between gambles $f \in \mathcal{L}_S(\Omega_R)$ with $S \subseteq R \subseteq I$, and gambles $f' \in \mathcal{L}(\Omega_S)$. So, in what follows, given a gamble $f \in \mathcal{L}_S(\Omega_R)$, we indicate with $f^{\downarrow S}$ the corresponding gamble in $\mathcal{L}(\Omega_S)$ defined, for all $\omega_S \in \Omega_S$, as $f^{\downarrow S}(\omega_S) \coloneqq f(\omega_R)$ for all $\omega_R \in \Omega_R$ such that $\omega_R|S = \omega_S$. Vice versa, given a gamble $f' \in \mathcal{L}(\Omega_S)$ we indicate with $(f')^{\uparrow R}$ the corresponding gamble in $\mathcal{L}_S(\Omega_R)$ defined as $(f')^{\uparrow R}(\omega_R) \coloneqq f'(\omega_R|S)$, for all $\omega_R \in \Omega_R$. Clearly, if $f \in \mathcal{L}_S(\Omega_R)$, then $(f^{\downarrow S})^{\uparrow R} = f$. Vice versa, if $f' \in \mathcal{L}(\Omega_S)$, then $((f')^{\uparrow R})^{\downarrow S} = f'$.

We extend these maps also to sets of gambles in the following way. For every $\mathcal{K} \subseteq \mathcal{L}_S(\Omega_R)$:

$$\mathcal{K}^{\downarrow S} \coloneqq \{f' \in \mathcal{L}(\Omega_S) : f' = f^{\downarrow S} \text{ for some } f \in \mathcal{K}\}.$$

For every $\mathcal{K}' \subseteq \mathcal{L}(\Omega_S)$:

$$(\mathcal{K}')^{\uparrow R} \coloneqq \{f \in \mathcal{L}_S(\Omega_R) : f = (f')^{\uparrow R} \text{ for some } f' \in \mathcal{K}'\}.$$

Analogously given $f \in \mathcal{L}_T(\Omega_R)$ with $T \subseteq S \subseteq R$, we can also define, for every $\omega_S \in \Omega_S$, $f^{\downarrow S}(\omega_S) \coloneqq f(\omega_R)$, where $\omega_R \in \Omega_R$ is such that $\omega_R|S = \omega_S$. Moreover, for every $f' \in \mathcal{L}(\Omega_T)$ we can define $(f')^{\uparrow S}(\omega_S) \coloneqq f'(\omega_S|T)$, for every $\omega_S \in \Omega_S$. These definitions can then be translated to sets of gambles in the usual way. We define similarly also $(f'')^{\uparrow R}$ for $f'' \in \mathcal{L}_T(\Omega_S)$ and the associate concept for sets of gambles.

Lemma 4. *Consider $T \subseteq S \subseteq R \subseteq I$. The following properties are valid.*

1. *If $\mathcal{K} \subseteq \mathcal{L}_T(\Omega_R)$, then $\mathcal{K}^{\downarrow S} = (\mathcal{K}^{\downarrow T})^{\uparrow S}$. So, in particular, if $S = R$, we have $\mathcal{K} = (\mathcal{K}^{\downarrow T})^{\uparrow R}$.*
2. *If $\mathcal{K} \subseteq \mathcal{L}_T(\Omega_S)$, then $\mathcal{K}^{\downarrow T} = (\mathcal{K}^{\uparrow R})^{\downarrow T}$.*
3. *If $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{L}_T(\Omega_R)$, then $\mathcal{K}_1^{\downarrow T} \cap \mathcal{K}_2^{\downarrow T} = (\mathcal{K}_1 \cap \mathcal{K}_2)^{\downarrow T}$.*
4. *If $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{L}_T(\Omega_R)$, then $\mathcal{K}_1^{\downarrow T} \cup \mathcal{K}_2^{\downarrow T} = (\mathcal{K}_1 \cup \mathcal{K}_2)^{\downarrow T}$.*
5. *If $\mathcal{K} \subseteq \mathcal{L}_T(\Omega_R)$, then $(\mathcal{C}(\mathcal{K}) \cap \mathcal{L}_T)^{\downarrow T} = \mathcal{C}(\mathcal{K}^{\downarrow T})$.*

Proof. Items 1,2,3 and 4 are obvious. Regarding item 5, $0 \in \mathcal{E}(\mathcal{K}) \iff 0 \in \mathcal{E}(\mathcal{K}^{\downarrow T})$. Therefore, we need to show only that $(\mathcal{E}(\mathcal{K}) \cap \mathcal{L}_T)^{\downarrow T} = \mathcal{E}(\mathcal{K}^{\downarrow T})$ with $0 \notin \mathcal{E}(\mathcal{K}), \mathcal{E}(\mathcal{K}^{\downarrow T})$. So, consider $f' \in (\mathcal{E}(\mathcal{K}) \cap \mathcal{L}_T)^{\downarrow T}$. Then $f' = f^{\downarrow T}$, for some $f \in \mathcal{E}(\mathcal{K}) \cap \mathcal{L}_T$, so, for every $\omega_T \in \Omega_T$, $f'(\omega_T) = f(\omega_R) = \sum_{i=1}^r \lambda_i g_i(\omega_R) + \mu h(\omega_R)$, with $\lambda_i, \mu \geq 0$, $\forall i$ not all equal to 0, $r \geq 0$, $g_i \in \mathcal{K} \subseteq \mathcal{L}_T(\Omega_R)$, $h \in \mathcal{L}^+$, for every $\omega_R \in \Omega_R$ such that $\omega_R|T = \omega_T$. Therefore $h \in \mathcal{L}_T^+$. So, $f' = \sum_{i=1}^r \lambda_i g_i^{\downarrow T} + \mu h^{\downarrow T}$, therefore $f' \in \mathcal{E}(\mathcal{K}^{\downarrow T})$. The other inclusion can be proven analogously, therefore we have the thesis.

On $\tilde{\mathbf{\Phi}}$ and $\mathcal{P}(I)$ we define the following operations.

1. Labeling. $d : \tilde{\mathbf{\Phi}} \to \mathcal{P}(I)$, defined as
$$d(\tilde{D}, S) \coloneqq S,$$
for every $(\tilde{D}, S) \in \tilde{\mathbf{\Phi}}$.
2. Combination. $\cdot : \tilde{\mathbf{\Phi}} \times \tilde{\mathbf{\Phi}} \to \tilde{\mathbf{\Phi}}$, where $\cdot(((\tilde{D}_1, S), (\tilde{D}_2, T)))$ is denoted by $(\tilde{D}_1, S) \cdot (\tilde{D}_2, T)$ and defined as
$$(\tilde{D}_1, S) \cdot (\tilde{D}_2, T) \coloneqq (\mathcal{C}(\tilde{D}_1^{\uparrow S \cup T}) \cdot \mathcal{C}(\tilde{D}_2^{\uparrow S \cup T}), S \cup T),$$
for every $(\tilde{D}_1, S), (\tilde{D}_2, T) \in \tilde{\mathbf{\Phi}}$, where $\mathcal{C}(\tilde{D}_1^{\uparrow S \cup T}) \cdot \mathcal{C}(\tilde{D}_2^{\uparrow S \cup T}) \coloneqq \mathcal{C}(\mathcal{C}(\tilde{D}_1^{\uparrow S \cup T}) \cup \mathcal{C}(\tilde{D}_2^{\uparrow S \cup T}))$ is the combination defined for sets in $\Phi(\Omega_{S \cup T})$ defined analogously to combination for sets in $\Phi(\Omega)$.
3. Marginalisation. $\pi : (dom(\pi) \subseteq \tilde{\mathbf{\Phi}} \times \mathcal{P}(I)) \to \tilde{\mathbf{\Phi}}$, where $\pi(((\tilde{D}, S), T))$ is denoted by $\pi_T(\tilde{D}, S)$ and defined as
$$\pi_T(\tilde{D}, S) \coloneqq ((\epsilon_T(\tilde{D}) \cap \mathcal{L}_T(\Omega_S))^{\downarrow T}, T),$$
for every $(\tilde{D}, S) \in \tilde{\mathbf{\Phi}}$, $T \subseteq S \subseteq I$, where $\epsilon_T(\tilde{D})$ is the extraction defined for sets in $\Phi(\Omega_S)$ and $\mathcal{P}(I)$, defined analogously to extraction for sets in $\Phi(\Omega)$ and $\mathcal{P}(I)$.

Consider now the map $h : \mathbf{\Phi} \to \tilde{\mathbf{\Phi}}$, where $h((D, S))$ is denoted by $h(D, S)$ and defined as

$$h(D, S) \coloneqq ((\epsilon_S(D) \cap \mathcal{L}_S(\Omega))^{\downarrow S}, S) = ((D \cap \mathcal{L}_S(\Omega))^{\downarrow S}, S),$$

for every $(D, S) \in \mathbf{\Phi}$. The map is clearly well defined. Moreover it is bijective and maintains operations as the following theorem shows.

Theorem 3. *The map h has the following properties.*

1. *Let $(D, S),\ (D_1, S),\ (D_2, T) \in \mathbf{\Phi}$:*
$$\begin{aligned} h((D_1, S) \cdot (D_2, T)) &= h(D_1, S) \cdot h(D_2, T), \\ h(\mathcal{L}(\Omega), S) &= (\mathcal{L}(\Omega_S), S),\ \forall S \subseteq I, \\ h(\mathcal{L}^+(\Omega), S) &= (\mathcal{L}^+(\Omega_S), S),\ \forall S \subseteq I, \\ h(\pi_T(D, S)) &= \pi_T(h(D, S)),\ \text{if } T \subseteq S. \end{aligned}$$
2. *h is bijective.*

Proof. 1. Recall that D_1 has support S and D_2 has support T and hence $D_1 \cdot D_2$ has support $S \cup T$. Therefore, we have by definition

$$\begin{aligned} h((D_1,S)\cdot(D_2,T)) &:= h(D_1\cdot D_2, S\cup T) \\ &:= ((D_1\cdot D_2 \cap \mathcal{L}_{S\cup T})^{\downarrow S\cup T}, S\cup T) \\ &= ((\mathcal{C}((D_1\cap\mathcal{L}_S)\cup(D_2\cap\mathcal{L}_T))\cap\mathcal{L}_{S\cup T})^{\downarrow S\cup T}, S\cup T) \\ &= (\mathcal{C}(((D_1\cap\mathcal{L}_S)\cup(D_2\cap\mathcal{L}_T))^{\downarrow S\cup T}), S\cup T) \end{aligned}$$

thanks to Lemma 5 in Appendix A, and item 5 of Lemma 4. On the other hand, thanks again to Lemma 5 in Appendix A, we have

$$h(D_1,S)\cdot h(D_2,T) := ((D_1\cap\mathcal{L}_S)^{\downarrow S}, S)\cdot((D_2\cap\mathcal{L}_T)^{\downarrow T}, T) :=$$

$$(\mathcal{C}(((D_1\cap\mathcal{L}_S)^{\downarrow S})^{\uparrow S\cup T}\cup((D_2\cap\mathcal{L}_T)^{\downarrow T})^{\uparrow S\cup T}), S\cup T).$$

Now, using again properties of Lemma 4, we have

$$\begin{aligned} &h(D_1,S)\cdot h(D_2,T) \\ &\quad := (\mathcal{C}(((D_1\cap\mathcal{L}_S)^{\downarrow S})^{\uparrow S\cup T}\cup((D_2\cap\mathcal{L}_T)^{\downarrow T})^{\uparrow S\cup T}), S\cup T) \\ &\quad = (\mathcal{C}((D_1\cap\mathcal{L}_S)^{\downarrow S\cup T}\cup(D_2\cap\mathcal{L}_T)^{\downarrow S\cup T}), S\cup T) \\ &\quad = (\mathcal{C}(((D_1\cap\mathcal{L}_S)\cup(D_2\cap\mathcal{L}_T))^{\downarrow S\cup T}), S\cup T) = h((D_1,S)\cdot(D_2,T)). \end{aligned}$$

Obviously, $(\mathcal{L}(\Omega), S)$ maps to $(\mathcal{L}(\Omega_S), S)$ and $(\mathcal{L}^+(\Omega), S)$ maps to $(\mathcal{L}^+(\Omega_S), S)$. Then we have, again by definition,

$$\begin{aligned} h(\pi_T(D,S)) &:= h(\epsilon_T(D), T) \\ &:= ((\epsilon_T(D)\cap\mathcal{L}_T)^{\downarrow T}, T) \\ &= ((D\cap\mathcal{L}_T)^{\downarrow T}, T). \end{aligned}$$

Indeed, $D\cap\mathcal{L}_T \subseteq \mathcal{C}(D\cap\mathcal{L}_T)\cap\mathcal{L}_T = \epsilon_T(D)\cap\mathcal{L}_T \subseteq D\cap\mathcal{L}_T$. However, from $T\subseteq S$, it follows $\mathcal{L}_T\subseteq\mathcal{L}_S$. Therefore we have

$$\begin{aligned} h(\pi_T(D,S)) &= ((D\cap\mathcal{L}_T)^{\downarrow T}, T) \\ &= ((D\cap\mathcal{L}_S)\cap\mathcal{L}_T)^{\downarrow T}, T). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \pi_T(h(D,S)) &:= \pi_T((D\cap\mathcal{L}_S)^{\downarrow S}, S) \\ &:= ((\epsilon_T((D\cap\mathcal{L}_S)^{\downarrow S})\cap\mathcal{L}_T(\Omega_S))^{\downarrow T}, T) \\ &= (((D\cap\mathcal{L}_S)^{\downarrow S}\cap\mathcal{L}_T(\Omega_S))^{\downarrow T}, T) \\ &= (((D\cap\mathcal{L}_S)^{\downarrow S}\cap(\mathcal{L}_T(\Omega))^{\downarrow S})^{\downarrow T}, T) \\ &= (((D\cap\mathcal{L}_S)\cap\mathcal{L}_T)^{\downarrow S})^{\downarrow T}, T) \\ &= ((((D\cap\mathcal{L}_S)\cap\mathcal{L}_T)^{\downarrow S})^{\uparrow I})^{\downarrow T}, T) \\ &= ((D\cap\mathcal{L}_S)\cap\mathcal{L}_T)^{\downarrow T}, T) = h(\pi_T(D,S)), \end{aligned}$$

thanks to Lemma 4.

2. Suppose $h(D_1,S) = h(D_2,T)$. Then we have $S=T$ and $(D_1\cap\mathcal{L}_S)^{\downarrow S} = (D_2\cap\mathcal{L}_S)^{\downarrow S}$, from which we derive that $D_1\cap\mathcal{L}_S = D_2\cap\mathcal{L}_S$ and therefore, $D_1 = \mathcal{C}(D_1\cap\mathcal{L}_S) = \mathcal{C}(D_2\cap\mathcal{L}_S) = D_2$. So the map h is injective.

   Moreover, for any $(\tilde{D},S)\in\tilde{\mathbf{\Phi}}$ we have that $(\tilde{D},S) = h(D,S)$ where $(D,S) = (\mathcal{C}(\tilde{D}^{\uparrow I}), S)\in\mathbf{\Phi}$. Indeed:

- $(\mathcal{C}(\tilde{D}^{\uparrow I}), S) \in \mathbf{\Phi}$. In fact, $\epsilon_S(\mathcal{C}(\tilde{D}^{\uparrow I})) \coloneqq \mathcal{C}(\mathcal{C}(\tilde{D}^{\uparrow I}) \cap \mathcal{L}_S)$. Now, $\tilde{D}^{\uparrow I} \subseteq \mathcal{C}(\tilde{D}^{\uparrow I}) \cap \mathcal{L}_S$, therefore $\mathcal{C}(\tilde{D}^{\uparrow I}) \subseteq \mathcal{C}(\mathcal{C}(\tilde{D}^{\uparrow I}) \cap \mathcal{L}_S)$. On the other hand, $\mathcal{C}(\tilde{D}^{\uparrow I}) \cap \mathcal{L}_S \subseteq \mathcal{C}(\tilde{D}^{\uparrow I})$, therefore $\mathcal{C}(\mathcal{C}(\tilde{D}^{\uparrow I}) \cap \mathcal{L}_S) \subseteq \mathcal{C}(\tilde{D}^{\uparrow I})$. Hence, $\epsilon_S(\mathcal{C}(\tilde{D}^{\uparrow I})) \coloneqq \mathcal{C}(\mathcal{C}(\tilde{D}^{\uparrow I}) \cap \mathcal{L}_S) = \mathcal{C}(\tilde{D}^{\uparrow I})$.
- $h(\mathcal{C}(\tilde{D}^{\uparrow I}), S) = (\tilde{D}, S)$. In fact, $h(\mathcal{C}(\tilde{D}^{\uparrow I}), S) \coloneqq ((\epsilon_S(\mathcal{C}(\tilde{D}^{\uparrow I})) \cap \mathcal{L}_S)^{\downarrow S}, S) = ((\mathcal{C}(\tilde{D}^{\uparrow I}) \cap \mathcal{L}_S)^{\downarrow S}, S)$ by previous item. Moreover, $((\mathcal{C}(\tilde{D}^{\uparrow I}) \cap \mathcal{L}_S)^{\downarrow S}, S) = (\tilde{D}, S)$ by item 5 of Lemma 4.

So h is surjective, hence bijective.

This theorem proves that:

- $(\tilde{\mathbf{\Phi}}(\Omega), I, d, \cdot, \{(\mathcal{L}(\Omega_S), S)\}_{S \subseteq I}, \{(\mathcal{L}^+(\Omega_S), S)\}_{S \subseteq I}, \pi)$ is a labeled information algebra, where $d, \cdot$ and π are defined above on $\tilde{\mathbf{\Phi}}$ and $\mathcal{P}(I)$. It can be proven by verifying the axioms for a labeled information algebra.
- h is an isomorphism between the two labeled information algebras $(\mathbf{\Phi}(\Omega), I, d, \cdot, \{(\mathcal{L}(\Omega), S)\}_{S \subseteq I}, \{(\mathcal{L}^+(\Omega), S)\}_{S \subseteq I}, \pi)$ with $d, \cdot, \pi$ defined on $\mathbf{\Phi}$ and $\mathcal{P}(I)$ and $(\tilde{\mathbf{\Phi}}(\Omega), I, d, \cdot, \{(\mathcal{L}(\Omega_S), S)\}_{S \subseteq I}, \{(\mathcal{L}^+(\Omega_S), S)\}_{S \subseteq I}, \pi)$ with $d, \cdot, \pi$ defined on $\tilde{\mathbf{\Phi}}$ and $\mathcal{P}(I)$.

Example 5. Let us return to Example 1. again and let us consider the pairs $(D_1^+, S_1), (D_2^+, S_2), (D_3, S_3), (D_3', S_3) \in \mathbf{\Phi}$. Then consider the map $h : \mathbf{\Phi} \to \tilde{\mathbf{\Phi}}$ defined by $(D, S) \mapsto ((\epsilon_S(D) \cap \mathcal{L}_S)^{\downarrow S}, S) = ((D \cap \mathcal{L}_S)^{\downarrow S}, S)$. We have:

- $(D_1^+ \cap \mathcal{L}_{S_1})^{\downarrow S_1} \coloneqq \{f \in \mathcal{L}(\Omega_{S_1}) : \min\{f(1,0), f(0,1)\} > 0\} \cup \mathcal{L}^+(\Omega_{S_1})$;
- $(D_2^+ \cap \mathcal{L}_{S_2})^{\downarrow S_2} \coloneqq \{f \in \mathcal{L}(\Omega_{S_2}) : f(1) > 0\} \cup \mathcal{L}^+(\Omega_{S_2})$;
- $(D_3 \cap \mathcal{L}_{S_3})^{\downarrow S_3} \coloneqq \{f \in \mathcal{L}(\Omega_{S_3}) : \min\{f(1,0), f(1,1)\} > 0\} \cup \{f \in \mathcal{L}(\Omega_{S_3}) : \min\{f(1,0), f(1,1)\} = 0 < f(0,0)\} \cup \mathcal{L}^+(\Omega_{S_3})$;
- $(D_3' \cap \mathcal{L}_{S_3})^{\downarrow S_3} \coloneqq \{f \in \mathcal{L}(\Omega_{S_3}) : \min\{f(1,0), f(1,1)\} > 0\} \cup \{f \in \mathcal{L}(\Omega_{S_3}) : \min\{f(1,0), f(1,1)\} = 0 < f(0,1)\} \cup \mathcal{L}^+(\Omega_{S_3})$.

Theorem 3 then guarantees that h maintains combination and extraction, therefore

$$h(D_1^+, S_1) \cdot h(D_2^+, S_2) \cdot h(D_3, S_3) \cdot h(D_3', S_3) = h(D_1^+ \cdot D_2^+ \cdot D_3 \cdot D_3', I) = (\mathcal{L}, I).$$

Now, if we define $D \coloneqq D_1^+ \cdot D_2^+ \cdot D_3$ as in Example 2., we have

$$h(D_1^+, S_1) \cdot h(D_2^+, S_2) \cdot h(D_3, S_3) = h(D, I) = (D, I),$$

and

$$\begin{aligned}\pi_{S_3}(h(D_1^+, S_1) \cdot h(D_2^+, S_2) \cdot h(D_3, S_3)) = \pi_{S_3}(h(D, I)) = \\ h(\epsilon_{S_3}(D), S_3) = h(D_3, S_3) = ((D_3 \cap \mathcal{L}_{S_3})^{\downarrow S_3}, S_3).\end{aligned}$$

We remark that also in labeled information algebras, an information order can be defined analogously to the one seen for domain-free ones.

In a computational application of this second labeled version of the information algebra of coherent sets of gambles, one would use the fact that any set $(\tilde{D}, S)$ is determined by gambles defined on the set of possibilities Ω_S, which reduce greatly the efficiency of storage. Observations like this explain why labeled information algebras are better suited for computational purposes.

4.3. Atoms

Maximal coherent sets of gambles $M \in C$ (see Section 2) are atoms in $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ (see Section 3). Indeed, they differ from $\mathcal{L}$ and they have the property that, in information order,

$$M \leq D \text{ for } D \in \Phi \Rightarrow M = D \text{ or } D = \mathcal{L}.$$

This property can alternatively being expressed by combination,

$$M \cdot D = \mathcal{L} \text{ or } M \cdot D = M, \ \forall D \in \Phi.$$

Analogously to Section 3.5, let $At(\Phi)$ denote the set of all atoms (maximal coherent sets) of $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$. Moreover, for any set of gambles $D \in \Phi$, let $At(D)$ denote the subset of $At(\Phi)$ (maximal coherent sets) which contains D,

$$At(D) := \{M \in At(\Phi) : D \leq M\}.$$

In general such sets may be empty. Not so in the case of coherent sets of gambles. Indeed, $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ is completely atomistic (see Section 3).

1. For any set $D \in C$, there is a set $M \in At(\Phi)$ so that in information order $D \leq M$ (i.e., $D \subseteq M$). So $At(D)$, for D coherent, is never empty. Therefore, $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ is atomic.

2. For all sets $D \in C$, we have

$$D = \inf At(D) = \bigcap At(D).$$

Therefore $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ is atomic composed or atomistic.

3. For any, non-empty, subset A of $At(\Phi)$ we have that

$$\inf A = \bigcap A$$

is a coherent set of gambles, i.e., an element of C. Hence, $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ is completely atomistic.

The first two properties are proved in [11], the third follows since coherent sets form a $\bigcap$-structure. Note that, if A is a set of maximal sets of gambles, $A \subseteq At(\bigcap A)$, and in general A is a proper subset of $At(\bigcap A)$.

Regarding the labeled information algebra $(\tilde{\mathbf{\Phi}}(\Omega), I, d, \cdot, \{(\mathcal{L}(\Omega_S), S)\}_{S \subseteq I}, \{(\mathcal{L}^+(\Omega_S), S)\}_{S \subseteq I}, \pi)$ instead, we have that $(\tilde{M}, S)$ where $\tilde{M} \in C(\Omega_S)$ is a maximal set of gambles, are atoms relative to S, for every $S \subseteq I$. Let us indicate with $At_S(\tilde{\mathbf{\Phi}})$ the set of all its atoms relative to S and with $At_S(\tilde{D}, S)$ the subset of $At_S(\tilde{\mathbf{\Phi}})$ dominating $(\tilde{D}, S)$, for every $(\tilde{D}, S) \in \tilde{\mathbf{\Phi}}$.

The properties of the domain-free information algebra $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ of being atomic, atomistic and completely atomistic carry over to this labeled version.

1. *Atomic:* For any element $(\tilde{D}, S) \in \tilde{\mathbf{\Phi}}$, $S \subseteq I$ with $\tilde{D} \in C(\Omega_S)$, there is an atom relative to S, $(\tilde{M}, S) \in At_S(\tilde{\mathbf{\Phi}})$, so that $(\tilde{D}, S) \leq (\tilde{M}, S)$.

2. *Atomistic:* For any element $(\tilde{D}, S) \in \tilde{\mathbf{\Phi}}$, $S \subseteq I$, with $\tilde{D} \in C(\Omega_S)$, $(\tilde{D}, S) = \inf\{(\tilde{M}, S) : (\tilde{M}, S) \in At_S(\tilde{D}, S)\}$.

3. *Completely Atomistic:* For any, non-empty, subset A of $At_S(\tilde{\mathbf{\Phi}})$, $\inf\{(\tilde{M}, S) : (\tilde{M}, S) \in A\}$ exists and belongs to $\tilde{\mathbf{\Phi}}_S$, for every $S \subseteq I$.

It is known that completely atomistic information algebras can be embedded in an information algebra where pieces of information are subsets of their atoms. The latter is a *set algebra*, that is, a prototypical form of information algebra based on the usual set operations [2]. This is an important representation theorem similar to Stone's representation theorem for Boolean algebras, see also [6].

5. Information algebras of lower and upper previsions

In this section we prove that, similarly to coherent sets of gambles, also coherent lower previsions induce a domain-free and two isomorphic labeled information algebras.

We recall that, in what follows, we denote with $\sigma(\mathcal{K})$, for every set $\mathcal{K} \subseteq \mathcal{L}$, the lower prevision $\underline{P}$ constructed from $\mathcal{K}$ through Eq. 1.

5.1. Domain-free version

Let us consider $\underline{\Phi}(\Omega) := \underline{\mathbb{P}}(\Omega) \cup \{\sigma(\mathcal{L}(\Omega))\}$ where we assume for Ω, as in Section 4, a multivariate model, i.e. $\Omega = \bigtimes_{i \in I} \Omega_i$ where I is a not empty index set and Ω_i, for every $i \in I$, is the set of the possible values of a variable X_i. We would like to introduce also here, like in $\Phi(\Omega)$, the operations of combination and extraction.

Let us define for two coherent sets of gambles which are not inconsistent, i.e. such that $D_1 \cdot D_2 \neq 0$, with $\underline{P}_1 := \sigma(D_1)$ and $\underline{P}_2 := \sigma(D_2)$, the lower prevision $\underline{P}' := \max\{\underline{P}_1, \underline{P}_2\}$, which assumes the value:

$$\begin{aligned}\sigma(D_1 \cup D_2)(f) :=& \sup\{\mu \in \mathbb{R} : f - \mu \in D_1 \cup D_2\} \\ =& \max\{\underline{P}_1(f), \underline{P}_2(f)\} =: \underline{P}'(f),\end{aligned}$$

for every gamble f in its domain. Following a reasoning similar to the one considered for coherent sets of gambles, we may take $\underline{E}^*(\underline{P}')$ to define combination of two lower previsions $\underline{P}_1$ and $\underline{P}_2$ in $\underline{\Phi}$. Regarding the extraction, for every $S \subseteq I$, we may take $\underline{E}^*(\underline{P}_S)$, where $\underline{P}_S$ is defined as the restriction of $\underline{P} \in \underline{\Phi}$ to $\mathcal{L}_S$. Thus, in summary, we can define on $\underline{\Phi}$ and $\mathcal{P}(I)$ the following operations.

1. Combination. $\cdot : \underline{\Phi} \times \underline{\Phi} \to \underline{\Phi}$, where $\cdot((\underline{P}_1, \underline{P}_2))$ is denoted by $\underline{P}_1 \cdot \underline{P}_2$ and defined as

$$\underline{P}_1 \cdot \underline{P}_2 := \underline{E}^*(\max\{\underline{P}_1, \underline{P}_2\}), \tag{11}$$

   for every $\underline{P}_1, \underline{P}_2 \in \underline{\Phi}$.

2. Extraction. $\underline{e} : \underline{\Phi} \times \mathcal{P}(I) \to \underline{\Phi}$, where $\underline{e}((\underline{P}, S))$ is denoted by $\underline{e}_S(\underline{P})$ and defined as

$$\underline{e}_S(\underline{P}) := \underline{E}^*(\underline{P}_S),$$

   for every $\underline{P} \in \underline{\Phi}, S \in \mathcal{P}(I)$.

The following theorem permits to conclude that $(\underline{\Phi}(\Omega), I, \cdot, \sigma(\mathcal{L}(\Omega)), \sigma(\mathcal{L}^+(\Omega)), \underline{e})$, where $\cdot, \underline{e}$ are defined above on $\underline{\Phi}$ and $\mathcal{P}(I)$, forms a domain-free information algebra. With the same little abuse of nomenclature introduced before for coherent sets of gambles, we can call it the *domain-free information algebra of coherent lower previsions defined on* $\mathcal{L}(\Omega)$. As usual, we can indicate it also with $(\underline{\Phi}, I, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$ and call it simply *domain-free information algebra of coherent lower previsions* when there is no ambiguity.

Theorem 4. *Let D_1^+, $D_2^+, D^+ \subseteq \mathcal{L}$ be strictly desirable sets of gambles and $S \subseteq I$. Then*

1. $\sigma(\mathcal{L})(f) = \infty$, $\sigma(\mathcal{L}^+)(f) = \inf f$ *for all* $f \in \mathcal{L}$,
2. $\sigma(D_1^+ \cdot D_2^+) = \sigma(D_1^+) \cdot \sigma(D_2^+)$,
3. $\sigma(\epsilon_S(D^+)) = \underline{e}_S(\sigma(D^+))$.

PROOF. 1. It follows from the definition.

2. Assume first that $D_1^+ \cdot D_2^+ = 0$ and let $\underline{P}_1 := \sigma(D_1^+)$, $\underline{P}_2 := \sigma(D_2^+)$. Then there can be no coherent lower prevision $\underline{P}$ dominating both $\underline{P}_1$ and $\underline{P}_2$. Indeed, otherwise we would have $D_1^+ = \tau(\underline{P}_1) \leq \tau(\underline{P})$ and $D_2^+ = \tau(\underline{P}_2) \leq \tau(\underline{P})$, where $\tau(\underline{P})$ is a coherent set of gambles. But this is a contradiction. Vice versa is also true. Therefore, we have $\sigma(D_1^+ \cdot D_2^+)(f) = \infty = (\sigma(D_1^+) \cdot \sigma(D_2^+))(f)$, for all gambles $f \in \mathcal{L}$.

Let then $D_1^+ \cdot D_2^+ \neq 0$. Then $D_1^+ \cdot D_2^+$ as well as $D_1^+ \cup D_2^+$ satisfy the condition of Theorem 11 in Appendix A. Therefore, applying this theorem, we have

$$\begin{aligned}\sigma(D_1^+ \cdot D_2^+) &\coloneqq \sigma(\mathcal{C}(D_1^+ \cup D_2^+)) = \underline{E}(\sigma(D_1^+ \cup D_2^+)) \\ &= \underline{E}(\max\{\sigma(D_1^+), \sigma(D_2^+)\}) =: \sigma(D_1^+) \cdot \sigma(D_2^+).\end{aligned}$$

3. We remark that $D^+ \cap \mathcal{L}_S$ satisfies the conditions of Theorem 11 in Appendix A. Thus we obtain

$$\sigma(\epsilon_S(D^+)) \coloneqq \sigma(\mathcal{C}(D^+ \cap \mathcal{L}_S)) = \underline{E}(\sigma(D^+ \cap \mathcal{L}_S)).$$

Now,

$$\sigma(D^+ \cap \mathcal{L}_S)(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in D^+ \cap \mathcal{L}_S\},\ \forall f \in dom(\sigma(D^+ \cap \mathcal{L}_S)).$$

But $f - \mu \in D^+ \cap \mathcal{L}_S$ if and only if f is S-measurable and $f - \mu \in D^+$. Therefore, we conclude that $\sigma(D^+ \cap \mathcal{L}_S) = \sigma(D^+)_S$. Thus, we have indeed $\sigma(\epsilon_S(D^+)) = \underline{E}(\sigma(D^+)_S) =: \underline{e}_S(\sigma(D^+))$.

This theorem proves firstly that $(\underline{\Phi}, I, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$ is a domain-free information algebra. Moreover, it proves also that it is isomorphic to $(\Phi^+, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$, subalgebra of the information algebra of coherent sets of gambles $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$. There is obviously the connected (isomorphic) information algebra of upper previsions.

The following theorem and corollary instead show that $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ is only *weakly* homomorphic to $(\underline{\Phi}, I, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$.

Consider indeed the map $D \mapsto D^+ \coloneqq \tau(\sigma(D))$, defined from Φ to Φ^+. The next theorem establishes that this map is a *weak* homomorphism between $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ and $(\Phi^+, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$. Weak, because when $D_1 \in \Phi$ and $D_2 \in \Phi$ are mutually inconsistent, that is if $D_1 \cdot D_2 = 0$, then $D_1 \cdot D_2$ can be mapped to something different from $D_1^+ \cdot D_2^+$ (see Example 5. below).

Theorem 5. *Let D_1, D_2 and D be coherent sets of gambles and $S \subseteq I$.*

1. *If $D_1 \cdot D_2 \neq 0$, then $D_1 \cdot D_2 \mapsto (D_1 \cdot D_2)^+ = D_1^+ \cdot D_2^+$,*
2. *$\epsilon_S(D) \mapsto (\epsilon_S(D))^+ = \epsilon_S(D^+)$.*

Proof. 1. Note first that $D_1^+ \subseteq D_1$ and $D_2^+ \subseteq D_2$, thanks to Lemma 7 in Appendix A. So that

$$D_1^+ \cdot D_2^+ = \tau(\sigma(D_1^+ \cdot D_2^+)) \subseteq \tau(\sigma(D_1 \cdot D_2)) =: (D_1 \cdot D_2)^+.$$

Further

$$(D_1 \cdot D_2)^+ \coloneqq \tau(\sigma(D_1 \cdot D_2)) \coloneqq \{f \in \mathcal{L} : \sigma(D_1 \cdot D_2)(f) > 0\} \cup \mathcal{L}^+(\Omega).$$

So, if $f \in (D_1 \cdot D_2)^+$, then either $f \in \mathcal{L}^+(\Omega)$ or

$$\sigma(D_1 \cdot D_2)(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in \mathcal{C}(D_1 \cup D_2)\} > 0. \tag{12}$$

In the first case obviously $f \in D_1^+ \cdot D_2^+$. Let us consider now $f \notin \mathcal{L}^+$, in this case there is a $\delta > 0$ so that $f - \delta \in \mathcal{C}(D_1 \cup D_2)$. This means that $f - \delta = h + \lambda_1 f_1 + \lambda_2 f_2$, where $h \in \mathcal{L}^+(\Omega) \cup \{0\}$, $f_1 \in D_1$, $f_2 \in D_2$ and $\lambda_1, \lambda_2 \geq 0$ and not both equal 0. But then

$$f = h + (\lambda_1 f_1 + \delta/2) + (\lambda_2 f_2 + \delta/2).$$

We have $f_1' \coloneqq \lambda_1 f_1 + \delta/2 \in D_1$ and $f_2' \coloneqq \lambda_2 f_2 + \delta/2 \in D_2$. But this, together with $\lambda_1 f_1 = f_1' - \delta/2 \in D_1$ if $\lambda_1 > 0$ or otherwise $f_1' \in \mathcal{L}^+(\Omega)$, and $\lambda_2 f_2 = f_2' - \delta/2 \in D_2$ if $\lambda_2 > 0$ or otherwise $f_2' \in \mathcal{L}^+(\Omega)$, show according to Lemma 8 in Appendix A that $f_1' \in D_1^+$ and $f_2' \in D_2^+$. So, finally, we have $f \in D_1^+ \cdot D_2^+ = \mathcal{C}(D_1^+ \cup D_2^+)$. This proves that $(D_1 \cdot D_2)^+ = D_1^+ \cdot D_2^+$.

2. Note that $D^+ \subseteq D$, again thanks to Lemma 7 in Appendix A. So that

$$\epsilon_S(D^+) = \tau(\sigma(\epsilon_S(D^+))) \subseteq \tau(\sigma(\epsilon_S(D))) =: (\epsilon_S(D))^+.$$

Further

$$(\epsilon_S(D))^+ := \tau(\sigma(\epsilon_S(D))) := \{f \in \mathcal{L} : \sigma(\epsilon_S(D))(f) > 0\} \cup \mathcal{L}^+(\Omega),$$

where

$$\sigma(\epsilon_S(D))(f) := \sup\{\mu \in \mathbb{R} : f - \mu \in \mathcal{C}(D \cap \mathcal{L}_S)\},$$

for every $f \in \mathcal{L}$. So, if $f \in (\epsilon_S(D))^+$, then either $f \in \mathcal{L}^+(\Omega)$ in which case $f \in \epsilon_S(D^+)$ or there is a $\delta > 0$ so that $f - \delta \in \mathcal{C}(D \cap \mathcal{L}_S) = \mathsf{posi}\{\mathcal{L}^+(\Omega) \cup (D \cap \mathcal{L}_S)\}$. In the second case, if $f \notin \mathcal{L}^+$, $f - \delta = h + g$ where $h \in \mathcal{L}^+(\Omega) \cup \{0\}$ and $g \in D \cap \mathcal{L}_S$. Then we have $f = h + g'$ where $g' := g + \delta$ is still S- measurable and $g' \in D$. But, given the fact that $g = g' - \delta \in D \cap \mathcal{L}_S$, from Lemma 8 in Appendix A, we have $g' \in D^+ \cap \mathcal{L}_S$ and therefore $f \in \epsilon_S(D^+)$. Thus we conclude that $(\epsilon_S(D))^+ = \epsilon_S(D^+)$.

The following corollary permits to conclude that there is a weak homomorphism also between $(\Phi, I, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ and $(\underline{\Phi}, I, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$.

Corollary 1. *Let D_1, D_2 and D be coherent sets of gambles so that $D_1 \cdot D_2 \neq 0$ and $S \subseteq I$. Then*

1. $\sigma(D_1 \cdot D_2) = \sigma(D_1) \cdot \sigma(D_2)$,
2. $\sigma(\epsilon_S(D)) = \underline{e}_S(\sigma(D))$.

Proof. These claims are immediate consequences of Theorems 5 and Theorem 4.

The weak homomorphism, as previously noticed, does not extend to a pair of inconsistent coherent sets of gambles, as the following example shows.

Example 6. As we have previously seen in Example 1., the coherent lower previsions associated with D_1^+, D_2^+, D_3, D_4 are the following.

- $(\forall f \in \mathcal{L})$ $\sigma(D_1^+)(f) = \min_{\{x_3, x_4 \in \{0,1\}\}}\{f(1, 0, x_3, x_4), f(0, 1, x_3, x_4)\}$;
- $(\forall f \in \mathcal{L})$ $\sigma(D_2^+)(f) = \sigma(D_3)(f) = \sigma(D_3')(f) = \min_{\{x_1, x_2, x_4 \in \{0,1\}\}} f(x_1, x_2, 1, x_4)$.

As Example 2. shows moreover, D_3 and D_3' are mutually inconsistent, since we have $D_3 \cdot D_3' = \mathcal{L}$. But, on the other hand,

$$D_3^+ = (D_3')^+ := \tau(\sigma(D_3')) = D_2^+,$$

therefore $D_3 \cdot D_3' = \mathcal{L}$ while $D_3^+ \cdot (D_3')^+ = D_2^+$. Hence $(D_3 \cdot D_3')^+ := \tau(\sigma(D_3 \cdot D_3')) = \mathcal{L} \neq D_3^+ \cdot (D_3')^+ = D_2^+$. Moreover, $\sigma(D_3 \cdot D_3') = \sigma(\mathcal{L}) \neq \sigma(D_3) \cdot \sigma(D_3') = \sigma(D_3^+) \cdot \sigma((D_3')^+) = \sigma(D_2^+) \cdot \sigma(D_2^+) = \sigma(D_2^+)$, thanks to Lemma 7 in Appendix A.

D_1^+, D_2^+, D_3 instead are consistent, therefore Corollary 1 permits to translate operations on lower previsions on the corresponding operations defined on sets of gambles. Denoting with $D := D_1^+ \cdot D_2^+ \cdot D_3 = D_1^+ \cdot D_3$, as in Example 2., we have:

$$\sigma(D_1^+) \cdot \sigma(D_2^+) \cdot \sigma(D_3) = \sigma(D_1^+ \cdot D_2^+ \cdot D_3) = \sigma(D_1^+ \cdot D_3) = \sigma(D),$$

$$\underline{e}_{S_3}(\sigma(D)) = \sigma(\epsilon_{S_3}(D)) = \sigma(D_3).$$

Finally, we claim that extraction distributes over meet (infimum) also in $(\underline{\Phi}, I, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$.

Theorem 6. *Let $\{\underline{P}_j\}_{j \in J}$ be any family of lower previsions in $\underline{\Phi}$ and $S \subseteq I$. Then*

$$\underline{e}_S(\inf\{\underline{P}_j : j \in J\}) = \inf\{\underline{e}_S(\underline{P}_j) : j \in J\}. \tag{13}$$

Proof. We may assume that $\underline{P}_j \in \underline{\mathbb{P}}$ for all $j \in J$ since if some or all $\underline{P}_j = \sigma(\mathcal{L})$, then we may restrict the infimum on both sides over the set $\underline{P}_j \in \underline{\mathbb{P}}$, or the infimum over both sides equals $\sigma(\mathcal{L})$. From Lemma 11 in Appendix A and Corollary 1, it follows that for any family of coherent sets of gambles $\{D_j\}_{j \in J}$,

$$\sigma(\epsilon_S(\bigcap_j D_j)) = \underline{e}_S(\sigma(\bigcap_j D_j)) = \underline{e}_S(\inf\{\sigma(D_j) : j \in J\})$$

and

$$\sigma(\bigcap_j \epsilon_S(D_j)) = \inf\{\sigma(\epsilon_S(D_j)) : j \in J\} = \inf\{\underline{e}_S(\sigma(D_j)) : j \in J\}.$$

The result then follows from Eq. (10).

Also here then, analogously to Section 4, we can define an information order on lower previsions in $\underline{\Phi}$, defined as $\underline{P}_1 \leq \underline{P}_2$ if and only if $\underline{P}_1 \cdot \underline{P}_2 = \underline{P}_2$. Also in this case, it is easy to verify that it coincides with the usual partial order on lower previsions introduced in Section 2, restricted to $\underline{\Phi}$, see [4, Theorem 3.1.2].

5.2. Labeled versions

Let us introduce the concept of support also for lower previsions.

Definition 21 (Support for lower previsions). A subset S of I is called *support* of a lower prevision $\underline{P} \in \underline{\Phi}$, if $\underline{e}_S(\underline{P}) = \underline{P}$.

There is a clear connection between supports of coherent sets of gambles and supports of the associated lower previsions.

Corollary 2. *Consider $D \in \Phi$. If $S \subseteq I$ is a support of D, then it is a support also of $\sigma(D)$. Vice versa, starting from $\underline{P} \in \underline{\Phi}$, if $S \subseteq I$ is a support of $\underline{P}$ then it is also a support of $\tau(\underline{P})$.*

Proof. The first result derives directly from item 2. of Corollary 1. Regarding the second one, $\underline{e}_S(\underline{P}) = \underline{P}$ implies, again thanks to item 2. of Corollary 1, $\sigma(\epsilon_S(\tau(\underline{P}))) = \sigma(\tau(\underline{P}))$. Applying then τ to both the terms of the equivalence we have the result.

Now, we can construct the first version of the labeled information algebra of coherent lower previsions using the same standard procedure described in Section 3.4 and used in Section 4.2 for coherent sets of gambles. So, let us consider as before a multivariate model for the possibility set Ω, i.e. let us suppose that $\Omega = \times_{i \in I} \Omega_i$ for some non-empty index set I. Then, let us define $\underline{\mathbf{\Phi}}_S(\Omega) \coloneqq \{(\underline{P}, S) : S \text{ is a support of } \underline{P} \in \underline{\Phi}(\Omega)\}$ for every $S \subseteq I$ and $\underline{\mathbf{\Phi}}(\Omega) \coloneqq \bigcup_{S \subseteq I} \underline{\mathbf{\Phi}}_S(\Omega)$. As usual, we can refer to them also with $\underline{\mathbf{\Phi}}_S$ and $\underline{\mathbf{\Phi}}$ respectively, when there is no possible ambiguity. It is possible to define on $\underline{\mathbf{\Phi}}$ and $\mathcal{P}(I)$, analogously to the case of coherent sets of gambles, the following operations in terms of the ones defined on $(\underline{\Phi}, I, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$.

1. Labeling. $d : \underline{\mathbf{\Phi}} \to \mathcal{P}(I)$, defined as
$$d(\underline{P}, S) \coloneqq S,$$
for every $(\underline{P}, S) \in \underline{\mathbf{\Phi}}$.
2. Combination. $\cdot : \underline{\mathbf{\Phi}} \times \underline{\mathbf{\Phi}} \to \underline{\mathbf{\Phi}}$, where $\cdot(((\underline{P}_1, S), (\underline{P}_2, T)))$ is denoted by $(\underline{P}_1, S) \cdot (\underline{P}_2, T)$ and defined as
$$(\underline{P}_1, S) \cdot (\underline{P}_2, T) \coloneqq (\underline{P}_1 \cdot \underline{P}_2, S \cup T),$$
for every $(\underline{P}_1, S), (\underline{P}_2, T) \in \underline{\mathbf{\Phi}}$.
3. Marginalisation. $\underline{\pi} : (dom(\underline{\pi}) \subseteq \underline{\mathbf{\Phi}} \times \mathcal{P}(I)) \to \underline{\mathbf{\Phi}}$, where $\underline{\pi}(((\underline{P}, S), T))$ is denoted by $\underline{\pi}_T(\underline{P}, S)$ and defined as
$$\underline{\pi}_T(\underline{P}, S) \coloneqq (\underline{e}_T(\underline{P}), T),$$
for every $(\underline{P}, S) \in \underline{\mathbf{\Phi}}$, $T \subseteq S \subseteq I$.

Therefore, $(\underline{\mathbf{\Phi}}(\Omega), I, d, \cdot, \{(\sigma(\mathcal{L}(\Omega)), S)\}_{S\subseteq I}, \{(\sigma(\mathcal{L}^+(\Omega)), S)\}_{S\subseteq I}, \underline{\pi})$, or more simply $(\underline{\mathbf{\Phi}}, I, d, \cdot, \{(\sigma(\mathcal{L}), S)\}_{S\subseteq I}, \{(\sigma(\mathcal{L}^+), S)\}_{S\subseteq I}, \underline{\pi})$, where $d, \cdot, \underline{\pi}$ are the operations defined above on $\underline{\mathbf{\Phi}}$ and $\mathcal{P}(I)$, forms a labeled information algebra.

Now, we can proceed by constructing the second version of this labeled information algebra, again following the same reasoning used for coherent sets of gambles. So, for any subset S of I let

$$\tilde{\underline{\mathbf{\Phi}}}_S(\Omega) := \{(\tilde{\underline{P}}, S) : \tilde{\underline{P}} \in \underline{\Phi}(\Omega_S)\},$$

where $\underline{\Phi}(\Omega_S) := \underline{\mathbb{P}}(\Omega_S) \cup \{\sigma(\mathcal{L}(\Omega_S))\}$. Further let

$$\tilde{\underline{\mathbf{\Phi}}}(\Omega) := \bigcup_{S\subseteq I} \tilde{\underline{\mathbf{\Phi}}}_S(\Omega).$$

As usual, we can refer to $\tilde{\underline{\mathbf{\Phi}}}_S(\Omega)$ and $\tilde{\underline{\mathbf{\Phi}}}(\Omega)$, also with $\tilde{\underline{\mathbf{\Phi}}}_S$ and $\tilde{\underline{\mathbf{\Phi}}}$ respectively, when there is no possible ambiguity.

On $\tilde{\underline{\mathbf{\Phi}}}$ and $\mathcal{P}(I)$ we define the following operations.

1. Labeling: $d : \tilde{\underline{\mathbf{\Phi}}} \to \mathcal{P}(I)$, defined as
$$d(\tilde{\underline{P}}, S) := S,$$
for every $(\tilde{\underline{P}}, S) \in \tilde{\underline{\mathbf{\Phi}}}$.
2. Combination. $\cdot : \tilde{\underline{\mathbf{\Phi}}} \times \tilde{\underline{\mathbf{\Phi}}} \to \tilde{\underline{\mathbf{\Phi}}}$, where $\cdot(((\tilde{\underline{P}}_1, S), (\tilde{\underline{P}}_2, T)))$ is denoted by $(\tilde{\underline{P}}_1, S) \cdot (\tilde{\underline{P}}_2, T)$ and defined as
$$(\tilde{\underline{P}}_1, S) \cdot (\tilde{\underline{P}}_2, T) := (\underline{E}^*(\tilde{\underline{P}}_1^{\uparrow S\cup T}) \cdot \underline{E}^*(\tilde{\underline{P}}_2^{\uparrow S\cup T}), S\cup T),$$
for every $(\tilde{\underline{P}}_1, S), (\tilde{\underline{P}}_2, T) \in \tilde{\underline{\mathbf{\Phi}}}$, where given a lower prevision $\underline{P}$ with $dom(\underline{P}) \subseteq \mathcal{L}(\Omega_Z)$ with $Z \subseteq S\cup T$, $\underline{P}^{\uparrow S\cup T}(f) := \underline{P}(f^{\downarrow Z})$, for every $f \in \mathcal{L}_Z(\Omega_{S\cup T})$ such that $f^{\downarrow Z} \in dom(\underline{P})$.
3. Marginalisation. $\underline{\pi} : (dom(\underline{\pi}) \subseteq \tilde{\underline{\mathbf{\Phi}}} \times \mathcal{P}(I)) \to \tilde{\underline{\mathbf{\Phi}}}$, where $\underline{\pi}(((\tilde{\underline{P}}, S), T))$ is denoted by $\underline{\pi}_T(\tilde{\underline{P}}, S)$ and defined as
$$\underline{\pi}_T(\tilde{\underline{P}}, S) := (\underline{e}_T(\tilde{\underline{P}})_T^{\downarrow T}, T),$$
for every $T \subseteq S \subseteq I$, where given a lower prevision $\underline{P}$ with $dom(\underline{P}) \subseteq \mathcal{L}(\Omega_Z)$ and $T \subseteq Z$, $\underline{P}_T^{\downarrow T}(f) := \underline{P}_T(f^{\uparrow Z}) = \underline{P}(f^{\uparrow Z})$ for every $f \in \mathcal{L}(\Omega_T)$ such that $f^{\uparrow Z} \in dom(\underline{P})$.

We can show that the map $\underline{h} : \underline{\mathbf{\Phi}} \to \tilde{\underline{\mathbf{\Phi}}}$, where $\underline{h}((\underline{P}, S))$ is denoted by $\underline{h}(\underline{P}, S)$ and defined as

$$\underline{h}(\underline{P}, S) := (\underline{e}_S(\underline{P})_S^{\downarrow S}, S) = (\underline{P}_S^{\downarrow S}, S),$$

for every $(\underline{P}, S) \in \underline{\mathbf{\Phi}}$, is bijective and mantains operations.

We claim that:

$$\underline{P}_S^{\downarrow S} = \sigma((D^+ \cap \mathcal{L}_S)^{\downarrow S}), \tag{14}$$

where $D^+ = \tau(\underline{P})$, for every $\underline{P} \in \underline{\Phi}$, $S \subseteq I$. In fact, for every $f \in \mathcal{L}(\Omega_S)$ both $\underline{P}_S^{\downarrow S}(f)$ and $\sigma((D^+\cap\mathcal{L}_S)^{\downarrow S})(f)$ are defined and

$$\begin{aligned}\sigma((D^+ \cap \mathcal{L}_S)^{\downarrow S})(f) &:= \sup\{\mu \in \mathbb{R} : f - \mu \in (D^+ \cap \mathcal{L}_S)^{\downarrow S}\}\\ &= \sup\{\mu \in \mathbb{R} : f^{\uparrow I} - \mu \in D^+ \cap \mathcal{L}_S\} =: \sigma(D^+)_S^{\downarrow S}(f).\end{aligned}$$

Finally, $(D^+ \cap \mathcal{L}_S)^{\downarrow S} \in \Phi(\Omega_S)$, hence $\underline{P}_S^{\downarrow S} \in \underline{\Phi}(\Omega_S)$. So that the map $\underline{h}$ is well defined.

These considerations suggest defining two new maps: $\boldsymbol{\sigma} : \mathbf{\Phi} \to \underline{\mathbf{\Phi}}$ defined as $(D, S) \mapsto (\sigma(D), S)$ and $\tilde{\boldsymbol{\sigma}} : \tilde{\mathbf{\Phi}} \to \tilde{\underline{\mathbf{\Phi}}}$ defined as $(\tilde{D}, S) \mapsto (\sigma(\tilde{D}), S)$.[8] Now the previous equivalence $\underline{P}_S^{\downarrow S} = \sigma((\tau(\underline{P}) \cap \mathcal{L}_S)^{\downarrow S})$ proves that:

$$\underline{h}(\underline{P}, S) = \underline{h}(\boldsymbol{\sigma}(D^+, S)) = \tilde{\boldsymbol{\sigma}}(h(D^+, S)), \tag{15}$$

for every $(\underline{P}, S) \in \underline{\mathbf{\Phi}}$, with $D^+ = \tau(\underline{P})$.

Now, we can prove the main result.

[8] Notice that, if $\tilde{D} \in \Phi(\Omega_S)$, then $\sigma(\tilde{D}) \in \underline{\Phi}(\Omega_S)$. So that $\tilde{\boldsymbol{\sigma}}$ is well defined.

Theorem 7. *The map $\underline{h}$ has the following properties.*

1. *Let $(\underline{P}, S)$, $(\underline{P}_1, S)$, $(\underline{P}_2, T) \in \underline{\boldsymbol{\Phi}}$:*

$$\begin{aligned}
\underline{h}((\underline{P}_1, S) \cdot (\underline{P}_2, T)) &= \underline{h}(\underline{P}_1, S) \cdot \underline{h}(\underline{P}_2, T), \\
\underline{h}(\sigma(\mathcal{L}), S) &= (\sigma(\mathcal{L}(\Omega_S)), S), \ \forall S \subseteq I, \\
\underline{h}(\sigma(\mathcal{L}^+), S) &= (\sigma(\mathcal{L}^+(\Omega_S)), S), \ \forall S \subseteq I, \\
\underline{h}(\underline{\pi}_T(\underline{P}, S)) &= \underline{\pi}_T(\underline{h}(\underline{P}, S)), \ \text{if } T \subseteq S.
\end{aligned}$$

2. *$\underline{h}$ is bijective.*

Proof. 1. We have, by definition,

$$\begin{aligned}
\underline{h}((\underline{P}_1, S) \cdot (\underline{P}_2, T)) &\coloneqq \underline{h}(\underline{P}_1 \cdot \underline{P}_2, S \cup T) \\
&= \underline{h}(\sigma(D_1^+) \cdot \sigma(D_2^+), S \cup T),
\end{aligned}$$

if we define $D_1^+ \coloneqq \tau(\underline{P}_1)$ and $D_2^+ \coloneqq \tau(\underline{P}_2)$. Now, thanks to Theorem 4, we have

$$\begin{aligned}
\underline{h}((\underline{P}_1, S) \cdot (\underline{P}_2, T)) &= \underline{h}(\sigma(D_1^+) \cdot \sigma(D_2^+), S \cup T) \\
&= \underline{h}(\sigma(D_1^+ \cdot D_2^+), S \cup T) \\
&=: \underline{h}(\boldsymbol{\sigma}(D_1^+ \cdot D_2^+, S \cup T)).
\end{aligned}$$

By Eq.(15), $\underline{h}(\boldsymbol{\sigma}(D^+, S)) = \tilde{\boldsymbol{\sigma}}(h(D^+, S))$, for every $(D^+, S) \in \boldsymbol{\Phi}$ such that $D^+ \in \Phi^+$. Therefore,

$$\begin{aligned}
\underline{h}((\underline{P}_1, S) \cdot (\underline{P}_2, T)) &= \underline{h}(\boldsymbol{\sigma}(D_1^+ \cdot D_2^+, S \cup T)) \\
&= \tilde{\boldsymbol{\sigma}}(h(D_1^+ \cdot D_2^+, S \cup T)) \\
&=: \tilde{\boldsymbol{\sigma}}(h((D_1^+, S) \cdot (D_2^+, T))) \\
&= \tilde{\boldsymbol{\sigma}}(h(D_1^+, S) \cdot h(D_2^+, T)),
\end{aligned}$$

thanks to Theorem 3. Now, we claim that

$$\tilde{\boldsymbol{\sigma}}(h(D_1^+, S) \cdot h(D_2^+, T)) = \tilde{\boldsymbol{\sigma}}(h(D_1^+, S)) \cdot \tilde{\boldsymbol{\sigma}}(h(D_2^+, T)).$$

Indeed, on the one hand, we have

$$\begin{aligned}
&\tilde{\boldsymbol{\sigma}}(h(D_1^+, S) \cdot h(D_2^+, T)) \\
&\quad \coloneqq \tilde{\boldsymbol{\sigma}}(((D_1^+ \cap \mathcal{L}_S)^{\downarrow S}, S) \cdot ((D_2^+ \cap \mathcal{L}_T)^{\downarrow T}, T)) \\
&\quad \coloneqq (\sigma(\mathcal{C}(((D_1^+ \cap \mathcal{L}_S)^{\downarrow S})^{\uparrow S \cup T}) \cdot \mathcal{C}(((D_2^+ \cap \mathcal{L}_T)^{\downarrow T})^{\uparrow S \cup T})), S \cup T) \\
&\quad \coloneqq (\sigma(\mathcal{C}(D_{1,S\cup T}^+) \cdot \mathcal{C}(D_{2,S\cup T}^+)), S \cup T),
\end{aligned}$$

where $D_{1,S\cup T}^+ \coloneqq ((D_1^+ \cap \mathcal{L}_S)^{\downarrow S})^{\uparrow S \cup T}$ and $D_{2,S\cup T}^+ \coloneqq ((D_2^+ \cap \mathcal{L}_T)^{\downarrow T})^{\uparrow S \cup T}$. On the other hand instead, we have

$$\begin{aligned}
&\tilde{\boldsymbol{\sigma}}(h(D_1^+, S)) \cdot \tilde{\boldsymbol{\sigma}}(h(D_2^+, T)) \\
&\quad \coloneqq \tilde{\boldsymbol{\sigma}}((D_1^+ \cap \mathcal{L}_S)^{\downarrow S}, S) \cdot \tilde{\boldsymbol{\sigma}}((D_2^+ \cap \mathcal{L}_T)^{\downarrow T}, T) \\
&\quad \coloneqq (\sigma((D_1^+ \cap \mathcal{L}_S)^{\downarrow S}), S) \cdot (\sigma((D_2^+ \cap \mathcal{L}_T)^{\downarrow T}), T) \\
&\quad \coloneqq (\underline{E}^*((\underline{P}_{1,S}^{\downarrow S})^{\uparrow S \cup T}) \cdot \underline{E}^*((\underline{P}_{2,T}^{\downarrow T})^{\uparrow S \cup T}), S \cup T).
\end{aligned}$$

Now, we can show that $(\underline{P}_{1,S}^{\downarrow S})^{\uparrow S \cup T} = \sigma(D_{1,S\cup T}^+) = \sigma(((D_1^+ \cap \mathcal{L}_S)^{\downarrow S})^{\uparrow S \cup T})$. Indeed,

$$\begin{aligned}
\sigma(((D_1^+ \cap \mathcal{L}_S)^{\downarrow S})^{\uparrow S \cup T})(f) &\coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in ((D_1^+ \cap \mathcal{L}_S)^{\downarrow S})^{\uparrow S \cup T}\} \\
&= \sup\{\mu \in \mathbb{R} : f^{\downarrow S} - \mu \in (D_1^+ \cap \mathcal{L}_S)^{\downarrow S}\} =: (\underline{P}_{1,S}^{\downarrow S})^{\uparrow S \cup T}(f),
\end{aligned}$$

for every $f \in \mathcal{L}_S(\Omega_{S\cup T})$. Analogously, we can show that $(\underline{P}_{2,T}^{\downarrow T})^{\uparrow S\cup T} = \sigma(D^+_{2,S\cup T}) = \sigma(((D_2^+ \cap \mathcal{L}_T)^{\downarrow T})^{\uparrow S\cup T})$. So, we have:

$$
\begin{aligned}
&(\underline{E}^*((\underline{P}_{1,S}^{\downarrow S})^{\uparrow S\cup T}) \cdot \underline{E}^*((\underline{P}_{2,T}^{\downarrow T})^{\uparrow S\cup T}), S\cup T) \\
&\quad = (\underline{E}^*(\sigma(D^+_{1,S\cup T})) \cdot \underline{E}^*(\sigma(D^+_{2,S\cup T})), S\cup T) = \\
&\quad = (\sigma(\mathcal{C}(D^+_{1,S\cup T})) \cdot \sigma(\mathcal{C}(D^+_{2,S\cup T})), S\cup T).
\end{aligned}
$$

In fact, if $D_1^+ = \mathcal{L}$, then $\underline{E}^*(\sigma(D^+_{1,S\cup T})) = \sigma(\mathcal{C}(D^+_{1,S\cup T}))$. Otherwise, $D^+_{1,S\cup T}$ satisfies the hypotheses of Theorem 11 in Appendix A:

- $0 \notin \mathcal{E}(D^+_{1,S\cup T})$. Otherwise, $0 \in \mathcal{E}((D_1^+ \cap \mathcal{L}_S)^{\downarrow S\cup T})$ by Lemma 4. Again by Lemma 4, this means $(\mathcal{C}(D_1^+ \cap \mathcal{L}_S) \cap \mathcal{L}_{S\cup T})^{\downarrow S\cup T} = 0$ that implies $0 \in \mathcal{E}(D_1^+ \cap \mathcal{L}_S) \subseteq D_1^+$, which is a contradiction;
- if $f \in D^+_{1,S\cup T} \setminus \mathcal{L}^+(\Omega_{S\cup T})$, then $f \in (D_1^+ \cap \mathcal{L}_S)^{\downarrow S\cup T} \setminus \mathcal{L}^+(\Omega_{S\cup T})$ thanks to Lemma 4. Hence $f^{\uparrow S\cup T} \in D_1^+ \cap \mathcal{L}_S \setminus \mathcal{L}_S^+$ and then, there exists $\delta > 0$ such that $f^{\uparrow S\cup T} - \delta = (f-\delta)^{\uparrow S\cup T} \in D_1^+ \cap \mathcal{L}_S$ that means $f - \delta \in (D_1^+ \cap \mathcal{L}_S)^{\downarrow S\cup T} = D^+_{1,S\cup T}$.

Therefore, thanks to Theorem 11 in Appendix A, we have: $\underline{E}^*(\sigma(D^+_{1,S\cup T})) = \sigma(\mathcal{C}(D^+_{1,S\cup T}))$. Analogously, we can show that $\underline{E}^*(\sigma(D^+_{2,S\cup T})) = \sigma(\mathcal{C}(D^+_{2,S\cup T}))$.

Moreover, given the fact that $\mathcal{C}(D^+_{1,S\cup T}), \mathcal{C}(D^+_{1,S\cup T}) \in \Phi^+(\Omega_{S\cup T})$ and thanks again to Theorem 4, we have

$$
\begin{aligned}
&\tilde{\boldsymbol{\sigma}}(h(D_1^+, S)) \cdot \tilde{\boldsymbol{\sigma}}(h(D_2^+, T)) = (\sigma(\mathcal{C}(D^+_{1,S\cup T})) \cdot \sigma(\mathcal{C}(D^+_{2,S\cup T})), S\cup T) = \\
&\quad = (\sigma(\mathcal{C}(D^+_{1,S\cup T}) \cdot \mathcal{C}(D^+_{2,S\cup T})), S\cup T) = \tilde{\boldsymbol{\sigma}}(h(D_1^+, S) \cdot h(D_2^+, T)).
\end{aligned}
$$

Hence, we have finally:

$$
\begin{aligned}
\underline{h}((\underline{P}_1, S) \cdot (\underline{P}_2, T)) &= \tilde{\boldsymbol{\sigma}}(h(D_1^+, S) \cdot h(D_2^+, T)) \\
&= \tilde{\boldsymbol{\sigma}}(h(D_1^+, S)) \cdot \tilde{\boldsymbol{\sigma}}(h(D_2^+, T)) \\
&= \underline{h}(\boldsymbol{\sigma}(D_1^+, S)) \cdot \underline{h}(\boldsymbol{\sigma}(D_2^+, T)) = \\
&= \underline{h}(\underline{P}_1, S) \cdot \underline{h}(\underline{P}_2, T).
\end{aligned}
$$

Obviously, $\underline{h}(\sigma(\mathcal{L}), S) = (\sigma(\mathcal{L}(\Omega_S)), S)$ and $\underline{h}(\sigma(\mathcal{L}^+), S) = (\sigma(\mathcal{L}^+(\Omega_S)), S)$.

Next, we have

$$
\underline{h}(\underline{\pi}_T(\underline{P}, S)) \coloneqq \underline{h}(\underline{e}_T(\underline{P}), T) = \underline{h}(\underline{e}_T(\sigma(D^+)), T),
$$

if $D^+ = \tau(\underline{P})$. Therefore using Theorem 3, Theorem 4 and Eq. (15), we have

$$
\begin{aligned}
\underline{h}(\underline{\pi}_T(\underline{P}, S)) &\coloneqq \underline{h}(\underline{e}_T(\sigma(D^+)), T) \\
&= \underline{h}(\sigma(\epsilon_T(D^+)), T) \\
&=: \underline{h}(\boldsymbol{\sigma}(\pi_T(D^+, S))) \\
&= \tilde{\boldsymbol{\sigma}}(h(\pi_T(D^+, S))) \\
&= \tilde{\boldsymbol{\sigma}}(\pi_T(h(D^+, S))) \\
&\coloneqq \tilde{\boldsymbol{\sigma}}(\pi_T((D^+ \cap \mathcal{L}_S)^{\downarrow S}, S)) \\
&\coloneqq \tilde{\boldsymbol{\sigma}}((\epsilon_T((D^+ \cap \mathcal{L}_S)^{\downarrow S}) \cap \mathcal{L}_T(\Omega_S))^{\downarrow T}, T)) \\
&\coloneqq (\sigma(((D^+ \cap \mathcal{L}_S)^{\downarrow S} \cap \mathcal{L}_T(\Omega_S))^{\downarrow T}), T).
\end{aligned}
$$

At the end we use the fact that $D \cap \mathcal{L}_T(\Omega_S) \subseteq \epsilon_T(D) \cap \mathcal{L}_T(\Omega_S) \subseteq D \cap \mathcal{L}_T(\Omega_S)$ for every $D \in \Phi(\Omega_S)$ and $T \subseteq S$, similarly to what observed in the proof of Theorem 3. Now, we have

$$
\begin{aligned}
&\underline{h}(\underline{\pi}_T(\underline{P}, S)) = (\sigma(((D^+ \cap \mathcal{L}_S)^{\downarrow S} \cap \mathcal{L}_T(\Omega_S))^{\downarrow T}), T) \\
&= ((\underline{P}_S^{\downarrow S})_T^{\downarrow T}, T) \\
&= \underline{\pi}_T(\underline{h}(\underline{P}, S)).
\end{aligned}
$$

Indeed, by a reasoning similar to the one that leads to Eq.(14), we can show that $\sigma((\tau(\underline{P}_S^{\downarrow S}) \cap \mathcal{L}_T)^{\downarrow T}) = (\underline{P}_S^{\downarrow S})_T^{\downarrow T}$. Moreover, as observed in Section 2, we have $\underline{e}_T(\underline{P}_S^{\downarrow S})_T^{\downarrow T} \coloneqq \underline{E}^*((\underline{P}_S^{\downarrow S})_T)_T^{\downarrow T} = (\underline{P}_S^{\downarrow S})_T^{\downarrow T}$. So we have the result.

2. Suppose $\underline{h}(\underline{P}_1, S) = \underline{h}(\underline{P}_2, T)$. Then we have $S = T$ and $\underline{P}_{1,S}^{\downarrow S} = \underline{P}_{2,S}^{\downarrow S}$, from which we derive that $\underline{P}_{1,S} = \underline{P}_{2,S}$ and therefore, $\underline{P}_1 = \underline{e}_S(\underline{P}_1) = \underline{E}^*(\underline{P}_{1,S}) = \underline{E}^*(\underline{P}_{2,S}) = \underline{e}_S(\underline{P}_2) = \underline{P}_2$. So the map $\underline{h}$ is injective.

   Moreover, for any $(\tilde{\underline{P}}, S) \in \tilde{\underline{\mathbf{\Phi}}}$, if we call $D^+ \coloneqq \tau(\underline{P})$, we claim that $(\tilde{\underline{P}}, S) = \underline{h}(\boldsymbol{\sigma}(\mathcal{C}((\tilde{D}^+)^{\uparrow I}), S)) = \underline{h}(\underline{E}^*(\tilde{\underline{P}}^{\uparrow I}), S)$. Since $(\mathcal{C}((\tilde{D}^+)^{\uparrow I}), S) \in \mathbf{\Phi}$, see the proof of item 2 of Theorem 3, we have that $(\underline{E}^*(\tilde{\underline{P}}^{\uparrow I}), S) \in \underline{\mathbf{\Phi}}$ and hence $\underline{h}$ is surjective.

   Indeed $h((\mathcal{C}((\tilde{D}^+)^{\uparrow I}), S)) = (\tilde{D}^+, S)$ again from the proof of item 2 of Theorem 3, therefore:

$$
\begin{aligned}
\underline{h}(\boldsymbol{\sigma}(\mathcal{C}((\tilde{D}^+)^{\uparrow I}), S)) &= \tilde{\boldsymbol{\sigma}}(h(\mathcal{C}((\tilde{D}^+)^{\uparrow I}), S))) \\
&= \tilde{\boldsymbol{\sigma}}(\tilde{D}^+, S) = (\sigma(\tilde{D}^+), S) = (\tilde{\underline{P}}, S).
\end{aligned}
$$

   Moreover, we have $\sigma((\tilde{D}^+)^{\uparrow I}) = \tilde{\underline{P}}^{\uparrow I}$. This follows from:

$$
\begin{aligned}
\sigma((\tilde{D}^+)^{\uparrow I})(f) &\coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in (\tilde{D}^+)^{\uparrow I}\} = \\
&\sup\{\mu \in \mathbb{R} : f^{\downarrow S} - \mu \in \tilde{D}^+\} \eqqcolon \tilde{\underline{P}}^{\uparrow I}(f)
\end{aligned}
$$

   for every $f \in \mathcal{L}_S(\Omega_I)$.

   Finally, if $\underline{P} = \sigma(\mathcal{L}(\Omega_S))$, we already have $\sigma(\mathcal{C}((\tilde{D}^+)^{\uparrow I})) = \underline{E}^*(\sigma((\tilde{D}^+)^{\uparrow I})) = E^*(\tilde{\underline{P}}^{\uparrow I})$, otherwise, to obtain this equivalence, we use Theorem 11 of Appendix A (that can be applied on $(\tilde{D}^+)^{\uparrow I}$).

   So h is surjective, hence bijective.

This theorem proves that $(\tilde{\underline{\mathbf{\Phi}}}(\Omega), I, d, \cdot, \{(\sigma(\mathcal{L}(\Omega_S)),S)\}_{S\subseteq I}, \{(\sigma(\mathcal{L}^+(\Omega_S)), S)\}_{S\subseteq I}, \underline{\pi})$, where $d, \cdot, \underline{\pi}$ are defined on $\tilde{\underline{\mathbf{\Phi}}}$ and $\mathcal{P}(I)$, is a labeled information algebra. Moreover, it proves that there is an isomorphism between the labeled information algebra $(\underline{\mathbf{\Phi}}(\Omega), I, d, \cdot, \{(\sigma(\mathcal{L}(\Omega)), S)\}_{S\subseteq I}, \{(\sigma(\mathcal{L}^+(\Omega)), S)\}_{S\subseteq I}, \underline{\pi})$, where $d, \cdot, \underline{\pi}$ are defined on $\underline{\mathbf{\Phi}}$ and $\mathcal{P}(I)$, and the labeled information algebra $(\tilde{\underline{\mathbf{\Phi}}}(\Omega), I, d, \cdot, \{(\sigma(\mathcal{L}(\Omega_S)), S)\}_{S\subseteq I}, \{(\sigma(\mathcal{L}^+(\Omega_S)), S)\}_{S\subseteq I}, \underline{\pi})$ where $d, \cdot, \underline{\pi}$ are defined on $\tilde{\underline{\mathbf{\Phi}}}$ and $\mathcal{P}(I)$.

Example 7. Let us return to the previous examples and consider the pairs $(\sigma(D_1^+), S_1), (\sigma(D_2^+), S_2), (\sigma(D_3), S_3), (\sigma(D'_3), S$ $\underline{\mathbf{\Phi}}$. Then consider the map $\underline{h} : \underline{\mathbf{\Phi}} \to \tilde{\underline{\mathbf{\Phi}}}$ defined by $(\underline{P}, S) \mapsto (\underline{e}_S(\underline{P})_S^{\downarrow S}, S) = (\underline{P}_S^{\downarrow S}, S)$. Then we have:

- $\underline{h}(\sigma(D_1^+), S_1) = \underline{h}(\boldsymbol{\sigma}(D_1^+, S_1)) = \tilde{\boldsymbol{\sigma}}(h(D_1^+, S_1)) = (\sigma((D_1^+ \cap \mathcal{L}_{S_1})^{\downarrow S_1}), S_1)$, with $\sigma((D_1^+ \cap \mathcal{L}_{S_1})^{\downarrow S_1})(f) = \min\{f(0,1), f(1,0)\}$, for every $f \in \mathcal{L}(\Omega_{S_1})$;
- $\underline{h}(\sigma(D_2^+), S_2) = \underline{h}(\boldsymbol{\sigma}(D_2^+, S_2)) = \tilde{\boldsymbol{\sigma}}(h(D_2^+, S_2)) = (\sigma((D_2^+ \cap \mathcal{L}_{S_2})^{\downarrow S_2}), S_2)$ with $\sigma((D_2^+ \cap \mathcal{L}_{S_2})^{\downarrow S_2})(f) = f(1)$, for every $f \in \mathcal{L}(\Omega_{S_2})$;
- $\underline{h}(\sigma(D_3), S_3) = \underline{h}(\sigma(D'_3), S_3) = \underline{h}(\sigma(D_2^+), S_3) = \underline{h}(\boldsymbol{\sigma}(D_2^+, S_3)) = \tilde{\boldsymbol{\sigma}}(h(D_2^+, S_3)) = (\sigma((D_2^+ \cap \mathcal{L}_{S_3})^{\downarrow S_3}), S_3)$ with $\sigma((D_2^+ \cap \mathcal{L}_{S_3})^{\downarrow S_3})(f) = \min\{f(1,0), f(1,1)\}$, for every $f \in \mathcal{L}(\Omega_{S_3})$.

Moreover, Theorem 7 guarantees that $\underline{h}$ maintains combination and extraction, therefore

$$\begin{aligned}&\underline{h}(\sigma(D_1^+), S_1) \cdot \underline{h}(\sigma(D_2^+), S_2) \cdot \underline{h}(\sigma(D_3), S_3) \cdot = \underline{h}(\sigma(D_1^+) \cdot \sigma(D_2^+) \cdot \sigma(D_3), I) = \\ &\underline{h}(\sigma(D_1^+ \cdot D_2^+ \cdot D_3), I) = \underline{h}(\sigma(D), I) = \tilde{\boldsymbol{\sigma}}(h(D, I)),\end{aligned}$$

thanks to Corollary 1. And

$$\begin{aligned}&\underline{\pi}_{S_3}(\underline{h}(\sigma(D_1^+), S_1) \cdot \underline{h}(\sigma(D_2^+), S_2) \cdot \underline{h}(\sigma(D_3), S_3)) = \underline{\pi}_{S_3}(\underline{h}(\sigma(D), I)) = \\ &\underline{h}(\underline{\pi}_{S_3}(\sigma(D), I)) = \underline{h}(\underline{e}_{S_3}(\sigma(D)), S_3) = \underline{h}(\sigma(\epsilon_{S_3}(D)), S_3) = \\ &\tilde{\boldsymbol{\sigma}}(h(\epsilon_{S_3}(D), S_3)) = \tilde{\boldsymbol{\sigma}}(h(D_3, S_3)),\end{aligned}$$

thanks again to Corollary 1.

We remark that also in this case, an information order can be defined analogously to the one seen for the domain-free case.

The results of this section show that the domain-free information algebra of coherent lower (and upper) previsions is closely related to the domain-free information algebra of coherent sets of gambles. This relationship carries over to the labeled versions of the information algebras involved. Moreover, we have shown that the information algebra of coherent sets of gambles is completely atomistic. In the next section we discuss what this means for the information algebras of coherent lower previsions.

5.3. Atoms

Linear previsions have an important role in the theory of imprecise probabilities. Therefore, in this section, they will be examined from the point of view of information algebras.

From Lemma 9 in Appendix A, we may deduce that linear previsions are atoms in the domain-free information algebra of coherent lower previsions. Indeed we have $\underline{P} \cdot P = P$ or $\underline{P} \cdot P = 0$, where here 0 is the null element $\underline{P}(f) = \infty$, for all $f \in \mathcal{L}(\Omega)$. The information algebra of coherent sets of gambles is completely atomistic. It is to be expected that the same holds for the information algebra of coherent lower previsions. Let $At(\underline{\Phi}) \coloneqq \mathbb{P}$ be the set of all linear previsions (atoms) and $At(\underline{P})$ the set of all linear previsions (atoms) dominating $\underline{P} \in \Phi$,

$$At(\underline{P}) \coloneqq \{P \in At(\underline{\Phi}) : \underline{P} \leq P\}.$$

Then the following theorem shows that the information algebra $(\underline{\Phi}, I, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$ is completely atomistic.

Theorem 8. *Consider the set of lower previsions $\underline{\Phi}$. The following holds.*

1. *If $\underline{P} \in \underline{\mathbb{P}}$, then $At(\underline{P}) \neq \emptyset$ and*

$$\underline{P} = \inf At(\underline{P}).$$

2. *If A is any non-empty subset of $At(\underline{\Phi})$, then*

$$\underline{P} \coloneqq \inf A$$

exists and it belongs to $\underline{\mathbb{P}}$.

For the proof of this theorem, see Theorem 2.6.3, Corollary 2.8.6 and Theorem 3.3.3 in [4].

According to this theorem, if A is any non-empty family of linear previsions, then $\inf A$ exists and it is a coherent lower prevision $\underline{P}$. Then we have $A \subseteq At(\underline{P})$ and it follows

$$\underline{P} \coloneqq \inf A = \inf At(\underline{P}).$$

So, the coherent lower prevision $\underline{P}$ is the lower envelope of the linear previsions (atoms) which dominate it.

We now examine linear previsions in the labeled view of the information algebra of coherent lower previsions. The elements $(\tilde{P}, S)$, where $\tilde{P} \in \mathbb{P}(\Omega_S)$ and $S \subseteq I$, are atoms relative to S in $(\tilde{\underline{\mathbf{\Phi}}}(\Omega), I, d, \cdot, \{(\sigma(\mathcal{L}(\Omega_S)), S)\}_{S\subseteq I}, \{(\sigma(\mathcal{L}^+(\Omega_S)), S)\}_{S\subseteq I}, \underline{\pi})$, i.e., if $(\tilde{P}, S) \leq (\tilde{\underline{P}}, S)$, then either $(\tilde{\underline{P}}, S) = (\tilde{P}, S)$ or $(\tilde{\underline{P}}, S) = (\sigma(\mathcal{L}(\Omega_S)), S)$, which is the null element for label S. This follows from Lemma 9 in Appendix A.

Also here, the properties of the domain-free information algebra $(\underline{\mathbf{\Phi}}, I, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$ of being atomic, atomistic and completely atomistic carry over to this labeled version. Let $At_S(\tilde{\underline{\mathbf{\Phi}}})$ be the set of atoms $(\tilde{P}, S)$ relative to S of this labeled information algebra, and let $At_S(\tilde{\underline{P}}, S)$ the subset of $At_S(\tilde{\underline{\mathbf{\Phi}}})$ dominating $(\tilde{\underline{P}}, S) \in \tilde{\underline{\mathbf{\Phi}}}$.

- *Atomic:* For any element $(\tilde{\underline{P}}, S) \in \tilde{\underline{\mathbf{\Phi}}}, S \subseteq I$ with $\tilde{\underline{P}} \in \underline{\mathbb{P}}(\Omega_S)$, there is an atom relative to S, $(\tilde{P}, S) \in At_S(\tilde{\underline{\mathbf{\Phi}}})$, so that $(\tilde{\underline{P}}, S) \leq (\tilde{P}, S)$. That is, $At_S(\tilde{\underline{P}}, S)$ is not empty.
- *Atomistic*: For any element, $(\tilde{\underline{P}}, S) \in \tilde{\underline{\mathbf{\Phi}}}, S \subseteq I$, with $\tilde{\underline{P}} \in \underline{\mathbb{P}}(\Omega_S)$, we have $(\tilde{\underline{P}}, S) = \inf At_S(\tilde{\underline{P}}, S)$.
- *Completely Atomistic:* For any, not empty, subset A of $At_S(\tilde{\underline{\mathbf{\Phi}}})$, $\inf A$ exists and belongs to $\tilde{\underline{\mathbf{\Phi}}}_S$, for every $S \subseteq I$.

6. The marginal problem

Compatibility is the problem of checking whether some given probabilistic assessments have a common joint probabilistic model.

Here, as an application of the results found in this paper, we treat the compatibility problem with respect to coherent sets of gambles and coherent lower previsions, already analysed in [1], in a more natural and easy way using results of information algebras.

Two or more pieces of information can be considered as *consistent*, if their combination is not the null element. This translates for coherent sets of gambles in the following way.

Definition 22 (Consistency for coherent sets of gambles). A finite family of coherent sets of gambles $D_1, \ldots, D_n$ is *consistent*, or $D_1, \ldots, D_n$ are *consistent*, if and only if $0 \neq D_1 \cdot \ldots \cdot D_n$.

This is called "avoiding partial loss" in desirability [1]. Otherwise, the family is, or the sets of gambles are, called *inconsistent.* There is, however, a more restrictive concept of consistency called *compatibility* [1]. We translate it using the language of information algebras.

Definition 23 (Compatibility for coherent sets of gambles). A finite family of coherent sets of gambles $D_1, \ldots, D_n$, where D_i has support S_i for every $i = 1, \ldots, n$ respectively, is called *compatible*, or $D_1, \ldots, D_n$ are called *compatible*, if and only if there is a coherent set of gambles D such that $\epsilon_{S_i}(D) = D_i$ for $i = 1, \ldots, n$.

To decide whether a family of D_i is compatible in this sense is also called *the marginal problem*, since extractions are (in the labeled view) projections or marginals.

In [1] a definition of *pairwise compatibility* for coherent sets of gambles is also given. We can reformulate it as follows.

Definition 24 (Pairwise compatibility for coherent sets of gambles). Two coherent sets D_i and D_j, where D_i has support S_i and D_j support S_j, are called *pairwise compatible* if and only if

$$\epsilon_{S_j}(D_i) = \epsilon_{S_i}(D_j). \tag{16}$$

Analogously, a finite family of coherent sets of gambles $D_1, ..., D_n$, where D_i has support S_i for every $i = 1, ..., n$ respectively, is pairwise compatible, or again $D_1, ..., D_n$ are pairwise compatible, if and only if pairs D_i, D_j are pairwise compatible for every $i, j \in \{1, ...n\}$.

From this definition moreover, it follows that

$$\epsilon_{S_i\cap S_j}(D_i) = \epsilon_{S_i\cap S_j}(\epsilon_{S_j}(D_i)) = \epsilon_{S_i\cap S_j}(\epsilon_{S_i}(D_j)) = \epsilon_{S_i\cap S_j}(D_j).$$

In an information algebra in general we could take also this as a definition of pairwise compatibility. Indeed, from this we may recover Eq. (16), since by item 5 of the list of properties of support (Section 4.2), if S_i is a support of D_i and S_j of D_j, we have $\epsilon_{S_j}(D_i) = \epsilon_{S_i\cap S_j}(D_i)$ and $\epsilon_{S_i}(D_j) = \epsilon_{S_i\cap S_j}(D_j)$.

Now let us consider D_i, D_j consistent and suppose S_i is a support of D_i and S_j is a support of D_j. Let us also define $D \coloneqq D_i \cdot D_j$. If D_i and D_j are pairwise compatible, then $\epsilon_{S_i}(D) = D_i \cdot \epsilon_{S_i}(D_j) = D_i \cdot \epsilon_{S_j}(D_i) = D_i$ and also $\epsilon_{S_j}(D) = D_j$. So, a pairwise compatible pair of pieces of information are compatible. And, conversely, if D_i and D_j are compatible, then there exists a coherent set D, such that $\epsilon_{S_i\cap S_j}(D) = \epsilon_{S_i\cap S_j}(\epsilon_{S_i}(D)) = \epsilon_{S_i\cap S_j}(D_i)$ and similarly $\epsilon_{S_i\cap S_j}(D) = \epsilon_{S_i\cap S_j}(D_j)$. Therefore the two elements are pairwise compatible.

It is well-known that pairwise compatibility among a family of $D_1, \ldots, D_n$ of pieces of information is not sufficient for the family to be compatible (the inverse however is true: a compatible family of coherent sets of gambles is always pairwise compatible). A well-known sufficient condition to obtain compatibility from pairwise compatibility is that the family of supports $S_1, \ldots, S_n$ of $D_1, ..., D_n$ satisfies the running intersection property (RIP).

RIP For $i = 1$ to $n-1$ there is an index $p(i)$, $i+1 \leq p(i) \leq n$ such that

$$S_i \cap S_{p(i)} = S_i \cap (\cup_{j=i+1}^{n} S_j).$$

Then we have the following theorem, translation of Theorem 2 and Proposition 1 in [1], a theorem that in fact is a theorem of information algebras in general.

Theorem 9. *Consider a finite family of consistent coherent sets of gambles $D_1, \ldots, D_n$ with $n > 1$ where D_i has support S_i for every $i = 1, ..., n$ respectively. If $S_1, \ldots, S_n$ satisfy RIP and $D_1, \ldots, D_n$ are pairwise compatible, then they are compatible and $\epsilon_{S_i}(D_1 \cdot \ldots \cdot D_n) = D_i$ for $i = 1, \ldots, n$.*

Proof. We give a proof in the framework of general domain-free information algebras. Let $Y_i \coloneqq S_{i+1} \cup \ldots \cup S_n$ for $i = 1, \ldots, n-1$ and $D \coloneqq D_1 \cdot \ldots \cdot D_n$. Then by RIP

$$\begin{aligned}\epsilon_{Y_1}(D) &= \epsilon_{Y_1}(D_1) \cdot D_2 \cdot \ldots \cdot D_n = \epsilon_{Y_1}(\epsilon_{S_1}(D_1)) \cdot D_2 \cdot \ldots \cdot D_n \\ &= \epsilon_{S_1\cap Y_1}(D_1) \cdot D_2 \cdot \ldots \cdot D_n = \epsilon_{S_1\cap S_{p(1)}}(D_1) \cdot D_2 \cdot \ldots \cdot D_n.\end{aligned}$$

But by pairwise compatibility $\epsilon_{S_1\cap S_{p(1)}}(D_1) = \epsilon_{S_1\cap S_{p(1)}}(D_{p(1)})$, hence by Idempotency

$$\epsilon_{Y_1}(D) = D_2 \cdot \ldots \cdot D_n.$$

By induction on i, one shows exactly in the same way that

$$\epsilon_{Y_i}(D) = D_{i+1} \cdot \ldots \cdot D_n,\ \forall i = 1, .., n-1.$$

So, we obtain $\epsilon_{S_n}(D) = \epsilon_{Y_{n-1}}(D) = D_n$. Now, we claim that $\epsilon_{S_i}(D) = \epsilon_{S_i\cap S_{p(i)}}(D) \cdot D_i$ for every $i = 1, ..., n-1$. Since $S_{p(i)} \subseteq Y_i$, we have by RIP:

$$\begin{aligned}&D_i \cdot \epsilon_{S_i\cap S_{p(i)}}(D) = D_i \cdot \epsilon_{S_i\cap S_{p(i)}}(\epsilon_{Y_i}(D)) = D_i \cdot \epsilon_{S_i\cap Y_i}(\epsilon_{Y_i}(D)) = \\ &D_i \cdot \epsilon_{S_i}(\epsilon_{Y_i}(D)) = D_i \cdot \epsilon_{S_i}(D_{i+1} \cdot \ldots \cdot D_n) = \epsilon_{S_i}(D_i \cdot D_{i+1} \cdot \ldots \cdot D_n).\end{aligned}$$

Now, if $i = 1$ we have the result, otherwise we have

$$D_i \cdot \epsilon_{S_i\cap S_{p(i)}}(D) = \epsilon_{S_i}(D_i \cdot D_{i+1} \cdot \ldots \cdot D_n) = \epsilon_{S_i}(\epsilon_{Y_{i-1}}(D)) = \epsilon_{S_i}(D).$$

Then, by backward induction, based on the induction assumption $\epsilon_{S_j}(D) = D_j$ for $j > i$, and rooted in $\epsilon_{S_n}(D) = D_n$, for $i = n-1, \ldots 1$, we have by pairwise compatibility

$$\begin{aligned}\epsilon_{S_i}(D) &= \epsilon_{S_i \cap S_{p(i)}}(D) \cdot D_i = \epsilon_{S_i \cap S_{p(i)}}(\epsilon_{S_{p(i)}}(D)) \cdot D_i \\ &= \epsilon_{S_i \cap S_{p(i)}}(D_{p(i)}) \cdot D_i = \epsilon_{S_i \cap S_{p(i)}}(D_i) \cdot D_i = D_i.\end{aligned}$$

This concludes the proof.

Note that the condition $\epsilon_{S_i}(D_1 \cdot \ldots \cdot D_n) = D_i$ for every i, implies that the family $D_1, \ldots, D_n$ is compatible. So, this is a sufficient condition for compatibility. This theorem is a theorem of information algebras, it holds not only for coherent sets of gambles but for any domain-free information algebra, in particular for the domain-free information algebra of coherent lower previsions for instance.

Example 8. Consider again the framework of the previous examples. The sets of variables S_1, S_2, S_3 satisfy the running intersection property. Let us concentrate ourselves on consistent sets D_1^+, D_2^+, D_3. Theorem 9 tells us that D_1^+, D_2^+, D_3 will be compatible if they are pairwise compatible.

Hence, it is sufficient to check pairwise compatibility of D_1^+, D_2^+, D_3:

$$\epsilon_{S_1 \cap S_2}(D_1^+) = \mathcal{L}^+ = \epsilon_{S_1 \cap S_2}(D_2^+),$$

$$\epsilon_{S_1 \cap S_3}(D_1^+) = \mathcal{L}^+ = \epsilon_{S_1 \cap S_3}(D_3),$$

$$\epsilon_{S_2 \cap S_3}(D_2^+) = D_2^+ = \epsilon_{S_2 \cap S_3}(D_3).$$

Since they are pairwise compatible, they are compatible.

The definition of compatibility and pairwise compatibility depend on the supports of the elements D_i. But D_i may have different supports. How does this influence compatibility? Assume D_i and D_j are coherent and pairwise compatible according to their supports S_i and S_j, that is $\epsilon_{S_i \cap S_j}(D_i) = \epsilon_{S_i \cap S_j}(D_j)$. It may be that a set $S_i' \subseteq S_i$ is still a support of D_i and a subset $S_j' \subseteq S_j$ a support of D_j. Then

$$\epsilon_{S_i' \cap S_j'}(D_i) = \epsilon_{S_i' \cap S_j'}(\epsilon_{S_i \cap S_j}(D_i)) = \epsilon_{S_i' \cap S_j'}(\epsilon_{S_i \cap S_j}(D_j)) = \epsilon_{S_i' \cap S_j'}(D_j).$$

So, D_i and D_j are also pairwise compatible relative to the smaller supports S_i' and S_j'. The finite supports of a coherent set of gambles D_i, have a least support

$$d_i := \bigcap \{S : S \text{ support of } D_i\}.$$

This is called the *dimension* of D_i, it is itself a support of D_i (see item 4 on the list of properties of supports). So, if D_i and D_j are pairwise compatible relative to two of their respective supports S_i and S_j, they are pairwise compatible relative to their dimensions d_i and d_j. This makes pairwise compatibility independent of an ad hoc selection of supports.

But what about compatibility? Assume that the family of coherent sets $D_1, \ldots, D_n$ is compatible relative to the supports S_i of D_i, that is, there exists a coherent set D such that $\epsilon_{S_i}(D) = D_i$ for every $i = 1, \ldots, n$. Then we have

$$\epsilon_{d_i}(D) = \epsilon_{d_i}(\epsilon_{S_i}(D)) = \epsilon_{d_i}(D_i) = D_i.$$

So, the family $D_1, \ldots, D_n$ is also compatible with respect to the system of their dimensions. Again, this makes the definition of compatibility independent of a particular selection of supports. We remark that the set $\{S : S \text{ support of } D_i\}$ is an upset, that is with any element S in the set an element $S' \supseteq S$ belongs also to the set (item 6 on the list of properties of supports). In fact,

$$\{S : S \text{ support of } D_i\} = \uparrow d_i$$

is the set of all supersets of the dimension. Now, assume that the finite family of coherent sets $D_1, \ldots, D_n$ with $n > 1$, is consistent and pairwise compatible. The dimensions d_i may not satisfy RIP, but some sets $S_i \supseteq d_i$ may. Then by Theorem 9 and this discussion, $D_1, \ldots, D_n$ are compatible.

From a point of view of information, compatibility of pieces of information $D_1, \ldots, D_n$ is not always desirable. It is a kind of irrelevance or (conditional) independence condition. In fact, if the members of the family of coherent sets $D_1, \ldots, D_n$ with $n > 1$ are consistent, pairwise compatible, and their supports S_i satisfy RIP, then $D_i = \epsilon_{S_i}(D_1 \cdot \ldots \cdot D_n)$ means that, the pieces of information D_j for $j \neq i$ give no new information relative to variables in S_i. If, on the other hand, the family $D_1, \ldots, D_n$ is not compatible, but consistent in the sense that $D := D_1 \cdot \ldots \cdot D_n \neq 0$, then, if S_1 to S_n satisfy RIP, we have that the family $\epsilon_{S_i}(D) \geq D_i$ (in the information order). Indeed $\epsilon_{S_i}(D) \cdot D_i = \epsilon_{S_i}(D_1 \cdot \ldots \cdot D_{i-1} \cdot D_{i+1} \cdot D_n) \cdot D_i \cdot D_i = \epsilon_{S_i}(D_1 \cdot \ldots \cdot D_{i-1} \cdot D_{i+1} \cdot D_n) \cdot D_i = \epsilon_{S_i}(D)$. This means that D_j may provide additional information on the variables in S_i for $i \neq j$. By formula (4.33), p. 119 in [2], we have

$$D = \epsilon_{S_1}(D) \cdot \ldots \cdot \epsilon_{S_n}(D).$$

Obviously the $D'_i := \epsilon_{S_i}(D)$ are pairwise compatible and by definition compatible.

To conclude, note that most of this discussion of compatibility (in particular Theorem 9) depends strongly on idempotency E2 of the information algebra. For instance the valuation algebra corresponding to Bayesian networks is not idempotent, as well as many other semiring-valuation algebras [15]. So Theorem 9 does not apply.

We have remarked that compatibility is essentially an issue of information algebras. So, we may expect that concepts and results on compatibility of coherent sets of gambles carry over to coherent lower and upper previsions.

Definition 25 (Consistency for coherent lower previsions). A finite family of coherent lower previsions $\underline{P}_1, \ldots, \underline{P}_n$, is *consistent*, or $\underline{P}_1, \ldots, \underline{P}_n$ are *consistent*, if and only if $0 \neq \underline{P}_1 \cdot \ldots \cdot \underline{P}_n$.

Definition 26 (Compatibility for coherent lower previsions). A finite family of coherent lower previsions $\underline{P}_1, \ldots, \underline{P}_n$, where $\underline{P}_i$ has support S_i for every $i = 1, \ldots, n$ respectively, is called *compatible*, or $\underline{P}_1, \ldots, \underline{P}_n$ are called *compatible*, if and only if there is a coherent lower prevision $\underline{P}$ such that $\underline{e}_{S_i}(\underline{P}) = \underline{P}_i$ for $i = 1, \ldots, n$.

Definition 27 (Pairwise compatibility for coherent lower previsions). Two coherent lower previsions $\underline{P}_i$ and $\underline{P}_j$, where $\underline{P}_i$ has support S_i and $\underline{P}_j$ support S_j, are called *pairwise compatible*, if and only if

$$\underline{e}_{S_i \cap S_j}(\underline{P}_j) = \underline{e}_{S_i \cap S_j}(\underline{P}_i).$$

Analogously, a finite family of coherent lower previsions $\underline{P}_i, ..., \underline{P}_n$, where $\underline{P}_i$ has support S_i for every $i = 1, ..., n$ respectively, is pairwise compatible, or again $\underline{P}_i, ..., \underline{P}_n$ are pairwise compatible, if and only if pairs $\underline{P}_i, \underline{P}_j$ are pairwise compatible for every $i, j \in \{1, ...n\}$.

Theorem 9 carries over, since it is in fact a theorem of information algebras.

Theorem 10. *Consider a finite family of consistent coherent lower previsions $\underline{P}_1, \ldots, \underline{P}_n$ with $n > 1$, where $\underline{P}_i$ has support S_i for every $i = 1, ..., n$ respectively. If $S_1, \ldots, S_n$ satisfy RIP and $\underline{P}_1, \ldots, \underline{P}_n$ are pairwise compatible, then they are compatible and $\underline{e}_{S_i}(\underline{P}_1 \cdot \ldots \cdot \underline{P}_n) = \underline{P}_i$ for $i = 1, \ldots, n$.*

Of course, there are close relations between compatibility of coherent sets of gambles and coherent lower previsions by the homomorphism between the related algebras. If a family of coherent sets $D_1, \ldots, D_n$ with supports $S_1, \ldots, S_n$ respectively, is compatible, then the associated family of coherent lower previsions $\sigma(D_1), \ldots, \sigma(D_n)$ is compatible too, since $\underline{e}_{S_i}(\sigma(D)) = \sigma(\epsilon_{S_i}(D)) = \sigma(D_i)$. Conversely, if $\underline{P}_1, \ldots, \underline{P}_n$ is a compatible family of coherent lower previsions with support $S_1, \ldots, S_n$ respectively, then there is a compatible family of strictly desirable sets of gambles $D_i^+ := \tau(\underline{P}_i)$ for every $i = 1, ..., n$.

7. Outlook

This paper presents a first approach to information algebras related to coherent sets of gambles and coherent lower and upper previsions. This leads us the possibility to abstract away properties of desirability that can be regarded as properties of the more general algebraic structure of information algebras rather than special ones of desirability.

De Cooman, in [10], pursued a similar purpose. He showed indeed that there is a common order-theoretic structure that he calls *belief structure*, underlying many of the models for representing beliefs in the literature such as, for example, classical propositional logic, almost desirable sets of gambles or lower and upper previsions.

There are surely important and interesting connections between de Cooman's belief structures [10] and information algebras. In particular between belief structures and information algebras based on closure operations, linked with information systems [2]. This hints at a profound connection between the two theories, which certainly deserves careful study. However, this is a subject which has yet to be worked out and may advance both information algebra theory as well as belief structures.

There are also other aspects and issues which are not addressed here. In particular, we limit our work to multivariate models, where coherent sets of gambles and coherent lower previsions represent pieces of information or belief relative to sets of variables. However, more general possibility spaces can be considered. In the view of information algebras, this translates in considering coherent sets of gambles and coherent lower previsions as pieces of information regarding more general partitions of the set of possibilities. This case has been analysed in more detail in [16] and in [17]. Moreover, another important issue which is not been addressed here is the issue of conditioning. It should be analysed both for the multivariate and for the more general cases of possibility spaces. This would also serve to analyse the issue of conditional independence, which seems to be fundamental for any theory of information.

Appendix A. Technical preliminaries

In this appendix are presented some new results on consequence operators, coherent and almost desirable sets of gambles and lower previsions that are preliminary to the rest of the work.

Lemma 5. *If $\mathcal{C}$ is a consequence operator on $\mathcal{P}(\mathcal{L})$ then, for any $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{L}$:*

$$\mathcal{C}(\mathcal{K}_1 \cup \mathcal{K}_2) = \mathcal{C}(\mathcal{C}(\mathcal{K}_1) \cup \mathcal{K}_2).$$

Proof. Obviously, $\mathcal{C}(\mathcal{K}_1 \cup \mathcal{K}_2) \subseteq \mathcal{C}(\mathcal{C}(\mathcal{K}_1) \cup \mathcal{K}_2)$. On the other hand $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{K}_1 \cup \mathcal{K}_2$, hence $\mathcal{C}(\mathcal{K}_1) \subseteq \mathcal{C}(\mathcal{K}_1 \cup \mathcal{K}_2)$ and $\mathcal{K}_2 \subseteq \mathcal{C}(\mathcal{K}_1 \cup \mathcal{K}_2)$. This implies $\mathcal{C}(\mathcal{C}(\mathcal{K}_1) \cup \mathcal{K}_2) \subseteq \mathcal{C}(\mathcal{C}(\mathcal{K}_1 \cup \mathcal{K}_2)) = \mathcal{C}(\mathcal{K}_1 \cup \mathcal{K}_2)$.

The following lemma is about the domain of lower previsions and it is particularly important in Section 5.

Lemma 6. *Given a non-empty set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$, we have*

1. *$\mathcal{K} \subseteq dom(\sigma(\mathcal{K}))$.*
2. *If $0 \notin \mathcal{E}(\mathcal{K})$, then $\sigma(\mathcal{K})(f) \in \mathbb{R}$ for every $f \in \mathcal{K}$.*
3. *If $\mathcal{K} \in C(\Omega)$, then $dom(\sigma(\mathcal{K})) = \mathcal{L}(\Omega)$ and $\sigma(\mathcal{K})(f) \in \mathbb{R}$ for every $f \in \mathcal{L}(\Omega)$.*
4. *If $\mathcal{K} \in \overline{C}(\Omega)$, then $dom(\sigma(\mathcal{K})) = \mathcal{L}(\Omega)$ and $\sigma(\mathcal{K})(f) \in \mathbb{R}$ for every $f \in \mathcal{L}(\Omega)$.*

Proof.
1. Consider $f \in \mathcal{K}$. Then the set $\{\mu \in \mathbb{R} : f - \mu \in \mathcal{K}\}$ is not empty, since it contains at least 0.
2. Assume $f - \mu \in \mathcal{K}$. If $\mu \geq \sup f$, then $f(\omega) - \mu \leq 0$ for all ω, but then $0 \in \mathcal{E}(\mathcal{K})$, contrary to assumption. So, the set $\{\mu \in \mathbb{R} : f - \mu \in \mathcal{K}\}$ is not empty and bounded from above for every $f \in \mathcal{K}$.

3. If $\mathcal{K}$ is a coherent set of gambles, then $0 \notin \mathcal{E}(\mathcal{K}) = \mathcal{C}(\mathcal{K}) = \mathcal{K}$ so that $\mathcal{K} \subseteq dom(\sigma(\mathcal{K}))$ and $\sigma(\mathcal{K})(f) \in \mathbb{R}$, for every $f \in \mathcal{K}$. Consider therefore $f \in \mathcal{L}(\Omega) \setminus \mathcal{K}$. If there would be a $\mu \geq 0$ so that $f - \mu \in \mathcal{K}$, then $f - \mu \leq f \in \mathcal{K}$, which contradicts the assumption. Now, if $\mu \leq \inf f < 0$, then $f - \mu \in \mathcal{L}^+(\Omega) \subseteq \mathcal{K}$, so it follows $\inf f \leq \sigma(\mathcal{K})(f) < 0$ and $dom(\sigma(\mathcal{K})) = \mathcal{L}(\Omega)$.

4. From item 1., we have that $dom(\sigma(\mathcal{K})) \supseteq \mathcal{K}$. Moreover, if $\mathcal{K} \in \overline{C}(\Omega)$, then $-1 \notin \mathcal{K}$. Therefore, for every $f \in \mathcal{K}$, if $\mu \geq \sup f + 1$ then $f - \mu \leq -1 \notin \mathcal{K}$. So, $\{\mu \in \mathbb{R} : \; f - \mu \in \mathcal{K}\}$ is not empty and bounded from above for every $f \in \mathcal{K}$.

   For $f \in \mathcal{L}(\Omega) \setminus \mathcal{K}$, we can repeat the procedure of item 3.

The following lemma establishes how coherent, strictly desirable and almost desirable sets are linked relative to the coherent lower previsions they induce. This result follows also from the fact that, in the sup-norm topology, given a coherent set D its relative interior plus the non-negative, non-zero gambles D^+ is a strictly desirable set of gambles and $\overline{D}$, the relative closure of D, is an almost desirable set of gambles [4].

Lemma 7. *Let $D \subseteq \mathcal{L}(\Omega)$ be a coherent set of gambles. Then*

$$D^+ \coloneqq \tau(\sigma(D)) \subseteq D \subseteq \overline{\tau}(\sigma(D)) \coloneqq \overline{D}$$

and $\sigma(D^+) = \sigma(D) = \sigma(\overline{D})$.

Proof. Let $\underline{P} \coloneqq \sigma(D)$. Then $f \in D^+$ means that $0 < \underline{P}(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in D\}$, or $f \in \mathcal{L}^+(\Omega)$. If $f \in \mathcal{L}^+(\Omega)$ then $f \in D$. Otherwise, there is a δ such that $0 < \delta < \underline{P}(f)$ and $f - \delta \in D$. Therefore $f \in D$ and $D^+ \subseteq D$. Further, consider $f \in D$. Then we must have $\underline{P}(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in D\} \geq 0$, hence $f \in \overline{D}$. The second part follows since τ and $\overline{\tau}$ are the inverse maps of σ on strictly desirable and almost desirable sets of gambles.

We can give also a further characterization of the elements of a strictly desirable set of gambles, particularly useful in Section 5.

Lemma 8. *Let D be a coherent set of gambles and $D^+ \coloneqq \tau(\sigma(D))$, if $f \in D \setminus \mathcal{L}^+(\Omega)$ then $f \in D^+$ if and only if there is a $\delta > 0$ so that $f - \delta \in D$.*

Proof. One part is by definition: if $f \in D^+$ and $f \notin \mathcal{L}^+(\Omega)$, then there is a $\delta > 0$ so that $f - \delta \in D^+ \subseteq D$. Conversely, consider $f - \delta \in D$ for some $\delta > 0$ with $f \notin \mathcal{L}^+(\Omega)$ and note that $D^+ \coloneqq \{g \in \mathcal{L} : \sigma(D)(g) > 0\} \cup \mathcal{L}^+(\Omega)$ and $\sigma(D)(g) \coloneqq \sup\{\mu \in \mathbb{R} : g - \mu \in D\}$ for every $g \in \mathcal{L}$. From $f - \delta \in D$ we deduce that $\sigma(D)(f) > 0$, hence $f \in D^+$.

Now, let us concentrate ourselves on the particular case of strictly desirable sets of gambles associated to a linear prevision P,

$$\tau(P) \coloneqq \{f \in \mathcal{L} : P(f) > 0\} \cup \mathcal{L}^+(\Omega) = \{f \in \mathcal{L} : -P(-f) > 0\} \cup \mathcal{L}^+(\Omega).$$

We call these sets *maximal strictly desirable* sets of gambles and we indicate them with M^+. We can prove then the following results.

Lemma 9. *Let $\underline{P}$ be an element of $\underline{\Phi}$ and P a linear prevision. Then $P \leq \underline{P}$ implies either $\underline{P} = P$ or $\underline{P}(f) = +\infty$ for all $f \in \mathcal{L}$.*

Proof. Clearly $\underline{P}(f) = +\infty$ for all $f \in \mathcal{L}$ is a possible solution. Consider instead the case in which $\underline{P}$ is coherent.

From [4], we know that $\underline{P}(f) \leq \overline{P}(f)$, for all $f \in \mathcal{L}(\Omega)$. Then, we have:

$$\underline{P}(f) \leq \overline{P}(f) \coloneqq -\underline{P}(-f) \leq -P(-f) = P(f), \; \forall f \in \mathcal{L}(\Omega). \tag{A.1}$$

Given the fact that, by hypothesis, we have also $\underline{P}(f) \geq P(f)$, for all $f \in \mathcal{L}(\Omega)$, we have the result.

Lemma 10. *Let D^+ be an element of Φ^+ and M^+ a maximal strictly desirable set of gambles. Then $M^+ \leq D^+$ implies either $D^+ = M^+$ or $D^+ = \mathcal{L}(\Omega)$.*

Proof. Notice that

$$M^+ \leq D^+ \Rightarrow \sigma(M^+) \leq \sigma(D^+). \tag{A.2}$$

Therefore, from Lemma 9 in Appendix A, we have $\sigma(D^+) = \sigma(M^+)$ or $\sigma(D^+) = \sigma(\mathcal{L})$, from which we derive $D^+ = \tau(\sigma(D^+)) = \tau(\sigma(M^+)) = M^+$ or $D^+ = \tau(\sigma(D^+)) = \tau(\sigma(\mathcal{L})) = \mathcal{L}$.

Now, we establishes and prove a fundamental result for Section 5: the map σ commutes with natural extension under certain conditions.

Before stating it, we need the following lemma. It states that the map σ restricted to coherent sets of gambles preserves infima. Here we define the functional $\inf\{\underline{P}_j : j \in J\}$ by $\inf\{\underline{P}_j(f) : j \in J\}$ for all $f \in \mathcal{L}(\Omega)$.

Lemma 11. *Let $\{D_j\}_{j\in J}$ be any family of coherent sets. Then we have*

$$\sigma(\bigcap_{j\in J} D_j) = \inf\{\sigma(D_j) : j \in J\}.$$

Proof. Note that the intersection of the coherent sets D_j equals a coherent set D. Moreover, $\inf\{\sigma(D_j) : j \in J\}$ is coherent [4]. We have $\sigma(\bigcap_{j\in J} D_j) := \sigma(D) =: \underline{P} \leq \sigma(D_j)$, for all $j \in J$. So $\underline{P}(f) \leq \sigma(D_j)(f)$ for all $f \in \mathcal{L}$ and $j \in J$, therefore $\underline{P} \leq \inf\{\sigma(D_j) : j \in J\}$. However, given the fact that $\inf\{\sigma(D_j) : j \in J\}$ is coherent, we have also $\tau(\inf\{\sigma(D_j) : j \in J\}) \subseteq \tau(\sigma(D_j)) \subseteq D_j$ for all $j \in J$, by definition of $\inf\{\sigma(D_j) : j \in J\}$. Hence $\tau(\inf\{\sigma(D_j) : j \in J\}) \subseteq \bigcap_j D_j =: D$. But this implies $\inf\{\sigma(D_j) : j \in J\} \leq \sigma(D) =: \underline{P}$. This concludes the proof.

Now we can state the main result.

Theorem 11. *Let $\mathcal{K} \subseteq \mathcal{L}(\Omega)$ be a non-empty set of gambles which satisfies the following two conditions:*

1. *$0 \notin \mathcal{E}(\mathcal{K})$,*
2. *for all $f \in \mathcal{K} \setminus \mathcal{L}^+(\Omega)$ there exists a $\delta > 0$ such that $f - \delta \in \mathcal{K}$.*

Then we have

$$\sigma(\mathcal{C}(\mathcal{K})) = \underline{E}^*(\sigma(\mathcal{K})) = \underline{E}(\sigma(\mathcal{K})).$$

Proof. If $\mathcal{K} = \mathcal{L}^+(\Omega)$, then $\mathcal{K} \in C(\Omega)$ and $\sigma(\mathcal{C}(\mathcal{K})) = \underline{E}^*(\sigma(\mathcal{K})) = \underline{E}(\sigma(\mathcal{K}))$ because the lower prevision associated with $\mathcal{L}^+(\Omega)$ is already coherent. So, assume that $\mathcal{K} \neq \mathcal{L}^+(\Omega)$. We have then $\mathcal{E}(\mathcal{K}) = \mathcal{C}(\mathcal{K}) \in C(\Omega)$, so that

$$\mathcal{C}(\mathcal{K}) = \bigcap\{D \in C(\Omega) : \mathcal{K} \subseteq D\}.$$

Let $\underline{P} := \sigma(\mathcal{K})$, then $\sigma(\mathcal{C}(\mathcal{K})) \geq \underline{P}$ and moreover $\sigma(\mathcal{C}(\mathcal{K}))$ is coherent, hence $\sigma(\mathcal{C}(\mathcal{K})) \geq \underline{E}^*(\underline{P}) = \underline{E}(\underline{P})$.

Now, $\mathcal{E}(\mathcal{K})$ is a strictly desirable set of gambles such that $\mathcal{E}(\mathcal{K}) \supseteq \mathcal{K}$. So, there exists at least a strictly desirable set of gambles containing $\mathcal{K}$. Therefore we have:

$$\sigma(\mathcal{C}(\mathcal{K})) = \sigma(\bigcap\{D \in C(\Omega) : \mathcal{K} \subseteq D\}) \leq \sigma(\bigcap\{D^+ \in C^+(\Omega) : \mathcal{K} \subseteq D^+\}).$$

Clearly, if $\mathcal{K} \subseteq D^+$, then $\underline{P} \leq \sigma(D^+)$, where $\sigma(D^+)$ is a coherent lower prevision. We claim that the converse is also valid. Indeed, let us consider a coherent lower prevision $\underline{P}'$ such that $\underline{P} \leq \underline{P}'$ and its associated strictly desirable set of gambles $D^+ := \tau(\underline{P}')$. If $f \in \mathcal{K}$, then $\underline{P}(f) \geq 0$. If $f \in \mathcal{L}^+(\Omega)$, then $f \in \tau(\underline{P}')$, otherwise, if $f \in \mathcal{K} \setminus \mathcal{L}^+(\Omega)$, then there is by assumption a $\delta > 0$ such that $f - \delta \in \mathcal{K}$, hence $0 < \underline{P}(f) \leq \underline{P}'(f)$. But this means again that $f \in \tau(\underline{P}')$. So, thanks to Lemma 11 we have:

$$\sigma(\bigcap\{D^+ \in C^+(\Omega) : \mathcal{K} \subseteq D^+\}) = \inf\{\underline{P}' \in \underline{\mathbb{P}}(\Omega) : \underline{P} \leq \underline{P}'\} = \underline{E}^*(\underline{P}) = \underline{E}(\underline{P})$$

so that $\sigma(\mathcal{C}(\mathcal{K})) = \underline{E}(\sigma(\mathcal{K}))$, concluding the proof.

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