Information Algebras in the Theory of Imprecise Probabilities, an Extension

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Abstract

In recent works, we have shown how to construct an information algebra of coherent sets of gambles, considering firstly a particular model to represent questions, called the *multivariate model*, and then generalizing it. Here we further extend the construction made to the highest level of generality, setting up an associated information algebra of *coherent lower previsions*, analyzing the connection of both the information algebras constructed with an instance of *set algebras* and, finally, establishing and inspecting a version of the *marginal problem* in this framework.

Set algebras are particularly important information algebras since they are their *prototypical* structures. They also represent the algebraic counterparts of classical propositional logic. As a consequence, this paper details as well how propositional logic is naturally embedded into the theory of *imprecise probabilities*.

Keywords: Desirability, Information algebras, Order theory, Imprecise probabilities, Coherence.

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1. Introduction and Overview

While analysing the *marginal problem*, i.e., the problem of checking the compatibility of a number of marginal assessments with a global model, in the framework of *desirability*, Miranda and Zaffalon [1] have remarked that their main results could also be obtained using the theory of *information algebras* [2, 3, 4]. This observation has been taken up and deepened in some of our recent works [5, 6, 7]: we have shown that the founding properties of desirability can in fact be abstracted into properties of information algebras. Stated differently, *coherent sets of gambles* induce an information algebra.

Desirability, or the theory of coherent sets of gambles, is a very general theory of uncertainty introduced by Peter Williams in 1975 as a generalization of de Finetti's theory [8, 9, 10]. It provides, in particular, a very general setting for analysing issues of compatibility, since it allows us to work with any possibility space, unrestricted domains and imprecise probabilities. Coherent sets of gambles, indeed, encompass probabilistic models for a possibility space Ω made of lower and upper expectation, respectively called *coherent lower* and *coherent upper previsions*. This, in particular, gives us also the possibility to interpret them as *pieces of information* about Ω. This is the interpretation that allows us to construct *information algebras of coherent sets of gambles* in [5, 6, 7].

Information algebras, in fact, are algebraic structures composed by 'pieces of information' that can be manipulated by operations of *combination*, to aggregate them, and *extraction*, to extract information regarding a specific question. Initially introduced as axiomatic systems to generalize the local computation schemes for probabilities proposed by Lauritzen and Spiegelhalter [11], they develop as generic mathematical structures to manage information in [2].

In [5], we construct an information algebra of coherent sets of gambles considering a particular model to represent questions of interest called the *multivariate model*. We then generalize it in [6, 7]. Here, we

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broaden the discussion started in [6, 7], introducing an information algebra of coherent lower previsions, analysing the marginal problem at this level of generality, constructing an information algebra of subsets of Ω and, finally, showing that this latter information algebra can be embedded both in the information algebra of coherent sets of gambles and in the one of coherent lower previsions. The aforementioned information algebra of subsets of Ω is, in particular, an instance of *set algebras*, i.e., information algebras where pieces of information are represented in the simplest way as sets of answers to questions of interest, manipulable by usual set operations.

Since set algebras are algebraic counterparts of classical propositional logic, the results of this paper detail as well how the latter is formally part of the theory of imprecise probabilities [9]. We refer also to [12] for another aspect of this issue.

The paper is structured as follows. We recall the main concepts of desirability and information algebras, respectively in Section 2 and Section 3. In particular, in the sub-section 3.5, we provide a general procedure to construct set algebras of subsets of a universal set U, considering the definition of information algebra given in our context. This method was already introduced in the draft work [4], however, we proceed in a more direct way considering an alternative definition of information algebra which, in particular, is proven to be equivalent to the one given in [4]. In Section 4, we recall the construction of an information algebra of coherent sets of gambles. In Section 5, we set up an associated information algebra of coherent lower previsions. In Section 6, we show existing links between the two information algebras of coherent sets of gambles and of coherent lower previsions constructed, with a set algebra of subsets of Ω, obtained using the procedure illustrated in Section 3.5. Finally, in Section 7, we treat a version of the marginal problem both for coherent sets of gambles and for coherent lower previsions in this context. Appendix A and Appendix B give some technical results and properties needed to obtain the results shown in the paper. Appendix C treats instead the particular case of information algebras with multivariate models for questions. Finally, Appendix D is totally devoted to the comparison of the results found in [5] about the marginal problem and the ones found in Section 7.

2. Desirability

Here we recall the necessary notation and basic definitions from desirability theory already introduced in [5, 6, 7]. For additional comments about it, we refer to [9, 10, 13]. For results of order theory, we refer instead to [14].

2.1. Coherent sets of gambles

Let us consider a non-empty set Ω of possible mutually exclusive outcomes of an experiment. We call it *possibility space* and we let its cardinality be general, so Ω can be infinite. In this context we give the concept of *gamble*.

Definition 1 (Gamble). Given a possibility space Ω, a *gamble* on Ω is a bounded real-valued function $f : \Omega \to \mathbb{R}$.

A gamble is usually interpreted as an uncertain reward in a linear utility scale. A subject might desire a gamble or not depending on the information they have about the experiment whose possible outcomes are the elements of Ω.

In what follows we use, in particular, the following notation. We denote with $\mathcal{L}(\Omega)$ the set of all gambles defined on Ω. We also denote with $\mathcal{L}^+(\Omega) \coloneqq \{f \in \mathcal{L}(\Omega) : f \geq 0, f \neq 0\}$, the set of non-vanishing, non-negative gambles. We simplify the notation whenever possible by omitting the possibility space Ω. Thus, we shall write $\mathcal{L}$, $\mathcal{L}^+$ in place respectively of $\mathcal{L}(\Omega)$, $\mathcal{L}^+(\Omega)$.

A *rational* agent should always be disposed to accept gambles in $\mathcal{L}^+$, because they can always increase their wealth without the risk of decreasing it. Moreover, as a consequence of the linearity of the utility scale, an agent that is willing to accept $f, g \in \mathcal{L}$ should always be disposed to accept $\lambda f + \mu g$ for every λ, $\mu > 0$. More generally, we can consider the notion of a *coherent set of desirable gambles*, also called *coherent set of gambles* for simplicity.

Definition 2 (Coherence for sets of gambles). We say that a subset $\mathcal{D}$ of $\mathcal{L}$ is a *coherent* set of desirable gambles, or more simply a *coherent* set of gambles, if and only if $\mathcal{D}$ satisfies the following properties:

D1. $\mathcal{L}^+ \subseteq \mathcal{D}$ [Accepting Partial Gains],

D2. $0 \notin \mathcal{D}$ [Avoiding Null Gain],

D3. $f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$ [Additivity],

D4. $f \in \mathcal{D}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{D}$ [Positive Homogeneity].

In what follows let $C(\Omega) \coloneqq \{\mathcal{D} \subseteq \mathcal{L}(\Omega) : \mathcal{D} \text{ is coherent}\}$, or simply C, denote the set of all the coherent sets of gambles. This definition leads to the concept of *natural extension*.

Definition 3 (Natural extension for sets of gambles). Given a set $\mathcal{K} \subseteq \mathcal{L}$, we call $\mathcal{E}(\mathcal{K}) \coloneqq \mathsf{posi}(\mathcal{K} \cup \mathcal{L}^+)$, where

$$\mathsf{posi}(\mathcal{K}') \coloneqq \left\{ \sum_{j=1}^{r} \lambda_j f_j : f_j \in \mathcal{K}', \lambda_j > 0, r \geq 1 \right\}$$

for every set $\mathcal{K}' \subseteq \mathcal{L}$, its *natural extension*.

The natural extension of a set of gambles $\mathcal{K} \subseteq \mathcal{L}$, $\mathcal{E}(\mathcal{K})$, is coherent if and only if $0 \notin \mathcal{E}(\mathcal{K})$. In this case, it is the smallest coherent set containing $\mathcal{K}$.

$$\mathcal{E}(\mathcal{K}) = \bigcap \{\mathcal{D}' \in C : \mathcal{K} \subseteq \mathcal{D}'\}.$$

Sets of gambles form, in particular, a partially ordered set with respect to inclusion $(\mathcal{P}(\mathcal{L}), \subseteq)$.[1] The natural extension operator is a *closure* or *consequence operator* on this poset $(\mathcal{P}(\mathcal{L}), \subseteq)$, see [14].

Coherent sets are also closed under intersection. In particular, by standard order theory [14], $(C, \subseteq)$ induces a meet-semilattice where meet is the intersection. When a family of coherent sets of gambles $\{\mathcal{D}_j\}_{j \in J}$, where J is an index set, have an upper bound among coherent sets, we can also define its join as follows:

$$\bigvee_{j \in J} \mathcal{D}_j \coloneqq \bigcap \{\mathcal{D}' \in C : \bigcup_{j \in J} \mathcal{D}_j \subseteq \mathcal{D}'\}.$$

So, if $\mathcal{E}(\bigcup_{j \in J} \mathcal{D}_j)$ is coherent, we have:

$$\bigvee_{j \in J} \mathcal{D}_j = \mathcal{E}(\bigcup_{j \in J} \mathcal{D}_j).$$

To obtain a complete lattice, we need to add $\mathcal{L}(\Omega)$ to $C(\Omega)$. We denote the resulting set with $\Phi(\Omega) \coloneqq C(\Omega) \cup \{\mathcal{L}(\Omega)\}$. In what follows we can simply refer to it with Φ when there is no possible ambiguity. In particular, $(\Phi, \subseteq)$ induces a complete lattice, where meet is intersection and join is defined for any family of sets $\{\mathcal{D}_j\}_{j \in J}$ with $\mathcal{D}_j \in \Phi$ for every $j \in J$, as:

$$\bigvee_{j \in J} \mathcal{D}_j \coloneqq \bigcap \{\mathcal{D}' \in \Phi : \bigcup_{j \in J} \mathcal{D}_j \subseteq \mathcal{D}'\}.$$

Starting from this definition of join, we can construct another closure operator on $(\mathcal{P}(\mathcal{L}), \subseteq)$ similar to the natural extension operator:

$$(\forall \mathcal{K} \subseteq \mathcal{L})\ \mathcal{C}(\mathcal{K}) \coloneqq \bigcap \{\mathcal{D}' \in \Phi : \mathcal{K} \subseteq \mathcal{D}'\}. \tag{1}$$

Notice that, given $K \subseteq \mathcal{L}$:

[1] We indicate with $\mathcal{P}(\mathcal{L})$ the power-set of $\mathcal{L}$.

- if $0 \notin \mathcal{E}(\mathcal{K})$, $\mathcal{C}(\mathcal{K}) = \mathcal{E}(\mathcal{K})$;
- if $0 \in \mathcal{E}(\mathcal{K})$, $\mathcal{C}(\mathcal{K}) = \mathcal{L}$ while it is possible to have $\mathcal{E}(\mathcal{K}) \neq \mathcal{L}$.[2]

We refer to [12] for a similar order-theoretic view of desirability.

The most informative cases of coherent sets of gambles, i.e., coherent sets that are not proper subsets of other coherent sets, are called *maximal*.

Proposition 1 (Maximal set of gambles). *A coherent set of gambles $\mathcal{D} \subseteq \mathcal{L}$ is* maximal *if and only if*

$$(\forall f \in \mathcal{L} \setminus \{0\})\ f \notin \mathcal{D} \Rightarrow -f \in \mathcal{D}.$$

We denote maximal sets of gambles with M to differentiate them from the general case of coherent sets.

Coherent sets of gambles encompass also a probabilistic model for Ω that will be discussed in detail in the next sub-section. To this aim, we introduce here a slightly modification of the concept of coherent set of gambles whose importance will become clear later on.

Definition 4 (Strictly desirable set of gambles). A coherent set of gambles $\mathcal{D}$ is said to be *strictly desirable* if and only if it satisfies $(\forall f \in \mathcal{D} \setminus \mathcal{L}^+)(\exists \delta > 0)\ f - \delta \in \mathcal{D}$.

We use the notation $\mathcal{D}^+$ for strictly desirable sets of gambles. Strictly desirable sets form a subfamily of coherent ones. In what follows we indicate with $C^+(\Omega)$ or C^+, the set of all strictly desirable sets of gambles. Moreover, from it, we can define $\Phi^+(\Omega) \coloneqq C^+(\Omega) \cup \{\mathcal{L}(\Omega)\}$ that we also denote with Φ^+.

2.2. Lower previsions, upper previsions and credal sets

Coherent and, in particular, strictly desirable sets of gambles have a tight connection with probability theory. In particular from each of these sets it is possible to construct a *coherent lower* and a *coherent upper prevision*, corresponding respectively to a lower and an upper expectation operator, which generalise de Finetti's work on subjective probabilities to the imprecise case.

A *lower prevision* $\underline{P}$ is a function with values in $\mathbb{R} \cup \{+\infty\}$ defined on some class of gambles $dom(\underline{P})$ called the *domain* of $\underline{P}$.[3] However, it is also possible to interpret each of its values $\underline{P}(f)$, with $f \in dom(\underline{P})$, as the supremum buying price that a subject, characterized by a set $\mathcal{K}$ of desirable gambles, is willing to spend for the gamble f. Analogously, each value of an upper prevision $\overline{P} : dom(\overline{P}) \to \mathbb{R} \cup \{-\infty\}$ can be interpreted as the infimum selling price that an agent is disposed to set for a gamble $f \in dom(\overline{P})$.

Definition 5 (Lower and upper prevision). Given a non-empty set $\mathcal{K} \subseteq \mathcal{L}$, we can associate to it a *lower prevision* (operator) $\underline{P} : dom(\underline{P}) \to \mathbb{R} \cup \{+\infty\}$, defined as

$$\underline{P}(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in \mathcal{K}\} \tag{2}$$

for every $f \in dom(\underline{P})$, and an *upper prevision* (operator) $\overline{P} : dom(\overline{P}) \to \mathbb{R} \cup \{-\infty\}$ defined as

$$\overline{P}(f) \coloneqq -\underline{P}(-f) \tag{3}$$

for every $f \in dom(\overline{P})$, where $dom(\underline{P}), dom(\overline{P}) = -dom(\underline{P}) \subseteq \mathcal{L}$.

So, $dom(\underline{P})$ is constituted by all the gambles f for which $\{\mu \in \mathbb{R} : f - \mu \in \mathcal{K}\}$ is non-empty and $dom(\overline{P}) = -dom(\underline{P})$.

Since upper previsions can be defined in terms of lower ones, in what follows we concentrate only on lower previsions.

In the definition above we have not made explicit the dependence on $\mathcal{K}$. However, when it is important to indicate it, we can also see $\underline{P}$ as the outcome of a function σ applied to a set of gambles $\mathcal{K}$ and write $\underline{P} = \sigma(\mathcal{K})$. We can also denote the set of gambles for which $\underline{P}$ is defined as $dom(\sigma(\mathcal{K}))$.

If $\mathcal{K} \subseteq \mathcal{L}$ is a coherent set of gambles, its associated lower prevision is called *coherent*, its domain is the whole $\mathcal{L}$, see Lemma 7 in Appendix A, and it is characterized by the following properties. For every $f, g \in \mathcal{L}$:

[2] For example, if $\mathcal{K}$ is a coherent set of *almost desirable gambles* [13]: $0 \in \mathcal{E}(\mathcal{K}) \neq \mathcal{L}$.

[3] Usually lower previsions are functions with values in $\mathbb{R}$. We consider here an extended version of this concept.

1. $\underline{P}(f) \geq \inf_{\omega \in \Omega} f(\omega)$,
2. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$, $\forall \lambda > 0$,
3. $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$.

In what follows we denote the set of all the coherent lower previsions as $\underline{\mathbb{P}}(\Omega)$ or $\underline{\mathbb{P}}$. There is a one-to-one correspondence between coherent lower previsions and strictly desirable sets of gambles. Starting from a coherent lower prevision $\underline{P}$, i.e., a lower prevision $\underline{P}$ satisfying the properties above, the set:

$$\mathcal{D}^+ \coloneqq \tau(\underline{P}) \coloneqq \{f \in \mathcal{L} : \underline{P}(f) > 0\} \cup \mathcal{L}^+, \tag{4}$$

is strictly desirable and moreover induces $\underline{P}$ through Eq. (2). Vice versa, starting from a strictly desirable set $\mathcal{D}^+$, $\sigma(\mathcal{D}^+)$ is coherent and induces again $\mathcal{D}^+$ through Eq. (4). We can see therefore the map $\tau : \underline{\mathbb{P}} \to C^+$ introduced in Eq. (4) as the inverse of σ restricted to strictly desirable sets. These considerations cannot be extended to arbitrary coherent sets of gambles. Several different coherent sets of gambles $\mathcal{D}$ may induce, in fact, the same coherent lower prevision $\underline{P}$ by means of Eq. (2). All of them induce in turn the same strictly desirable set of gambles $\mathcal{D}^+ \coloneqq \tau(\sigma(\mathcal{D}))$, which can also be recovered directly from each $\mathcal{D}$ by:

$$\mathcal{D}^+ = \mathcal{L}^+ \cup \{f \in \mathcal{L} : (\exists \delta > 0)\ f - \delta \in \mathcal{D}\}. \tag{5}$$

This set corresponds in particular to the relative interior of $\mathcal{D}$ in the sup-norm topology of the linear space $\mathcal{L}$, plus the non-negative non-zero gambles [9]. Moreover, we have $\sigma(\mathcal{D}^+) = \sigma(\mathcal{D})$, see again [9]. While a coherent set of gambles $\mathcal{D}$ represents the set of 'really desirable' gambles, the induced strictly desirable set of gambles $\mathcal{D}^+ \coloneqq \tau(\sigma(\mathcal{D}))$ corresponds to the gambles *strictly* desirable, i.e., gambles f such that $f \in \mathcal{L}^+$ or there exists $\delta > 0$ such that $f - \delta \in \mathcal{D}$.

An important class of coherent lower previsions are the *linear* ones.

Definition 6. Consider a coherent lower prevision $\underline{P}$. Consider also its associated upper prevision $\overline{P}$, constructed from $\underline{P}$ through Eq.(3). If $\underline{P}(f) = \overline{P}(f)$ for some $f \in \mathcal{L}$, we call the common value the *prevision* of f and we denote it by $P(f)$. If this happens for all $f \in \mathcal{L}$, we call the functional P a *linear prevision.*

So, in particular, a linear prevision is a linear functional on $\mathcal{L}$. In what follows, we denote the set of all the linear previsions as $\mathbb{P}(\Omega)$ or $\mathbb{P}$.

Coherent lower and upper previsions are particularly important because they are respectively lower and upper expectation operators calculated with respect to a set of finitely additive probabilities [9]. Indeed, every coherent lower prevision $\underline{P}$ is the lower envelope of its set of dominating linear previsions:

$$(\forall f \in \mathcal{L})\ \underline{P}(f) = \min\{P(f) : P \in \mathcal{M}(\underline{P})\} \tag{6}$$

where:

$$\mathcal{M}(\underline{P}) \coloneqq \{P \in \mathbb{P} : (\forall f \in \mathcal{L})\ P(f) \geq \underline{P}(f)\}, \tag{7}$$

which turns out to be non-empty, convex and closed under the *weak* topology* [9]. Additionally, every linear prevision corresponds to an expectation operator calculated with respect to a finitely additive probability, which in turn can be obtained by making the restriction of the linear prevision to indicators of events [9]. It is possible therefore to interpret $\mathcal{M}(\underline{P})$ also as a set of probabilities (a so-called *credal set*). If $\underline{P}$ is in particular linear, $\mathcal{M}(\underline{P}) = \{\underline{P}\}$.

On lower previsions it is also possible to introduce a partial order relation.

Definition 7. Given two (not necessarily coherent) lower previsions $\underline{P}, \underline{Q}$ defined respectively on $dom(\underline{P}), dom(\underline{Q}) \subseteq \mathcal{L}$, we say that $\underline{Q}$ *dominates* $\underline{P}$, if and only if $dom(\underline{P}) \subseteq dom(\underline{Q})$ and $\underline{P}(f) \leq \underline{Q}(f)$ for all $f \in dom(\underline{P})$.

Starting from this order relation it is possible to introduce, also for lower previsions, a natural extension operator $\underline{E}$. Similarly to coherent sets of gambles, given a lower prevision $\underline{P}$, if $\underline{E}(\underline{P})$ is coherent, it corresponds to the minimal coherent lower prevision dominating $\underline{P}$:

$$(\forall f \in \mathcal{L})\ \underline{E}(\underline{P})(f) = \min\{\underline{P}' \in \underline{\mathbb{P}} : \underline{P} \leq \underline{P}'\}(f).$$

Here however, as for sets of gambles, we consider a slightly different operator $\underline{E}^*$ having for lower previsions the same role of the operator $\mathcal{C}$ for sets of gambles. So, it coincides with $\underline{E}$ when it results in a coherent lower prevision, i.e., when it is applied to a lower prevision for which there exists at least a coherent lower prevision dominating it, and otherwise it corresponds to $\sigma(\mathcal{L})$, where $\sigma(\mathcal{L})(f) \coloneqq +\infty$ for every gamble $f \in \mathcal{L}$.

Definition 8. Given a lower prevision $\underline{P}$ defined on a domain $dom(\underline{P}) \subseteq \mathcal{L}$, we define

$$\underline{E}^*(\underline{P}) \coloneqq \begin{cases} \underline{E}(\underline{P}) & \text{if } \exists \underline{P}' \in \underline{\mathbb{P}} : \underline{P} \leq \underline{P}', \\ \sigma(\mathcal{L}) & \text{otherwise.} \end{cases}$$

In view of Section 5, again similarly to sets of gambles, it is convenient to consider $\underline{\Phi}(\Omega) \coloneqq \underline{\mathbb{P}}(\Omega) \cup \{\sigma(\mathcal{L}(\Omega))\}$, which we sometimes abbreviate to $\underline{\Phi}$ if there is no possible ambiguity.

In Appendix A, for the sake of convenience, we collected some other complementary results preliminary to the rest of the work, already presented in [5].

We can now provide an example to clarify some of the notions above and show the variety of possibilities the tools presented in this section offer to an agent for expressing their opinions about a possibility space.

Example 1. You and two of your friends, Alice and Bob, decide to plan an holiday in a beautiful hotel on a little island. Shortly after your arrival a murder is committed.

The three of you have always been together and none of you can be the murderer. Moreover, no one is arrived or is gone from the island. The list of the suspected people boils down to seven persons $\Omega' = \{\text{a,b,c,d,e,f,g}\}$. In the following list are summarized the preliminary information you and your friend have about the sex, the aspect and the possible crime motive of the suspects:

- subject a: male, blond short hair, economic motive;
- subject b: male, brown long hair, motive of passion;
- subject c: female, blond long hair, economic motive;
- subject d: female, blond long hair, economic motive;
- subject e: female, red long hair, motive of passion;
- subject f: male, blond short hair, economic motive;
- subject g: male, red short hair, motive of passion.

From now on, you and your friends decide to investigate separately. In particular:

- you found some footprints near the victim that cannot be left before or after the crime and that do not belong to the victim. From their size, you conclude that they must definitely belong to a man. You think therefore that the murderer has to be a,b,f or g but you are completely unsure about who the murderer is among them. Your beliefs can then be modeled by the coherent lower prevision $\underline{P}_1$, defined as:
$$\underline{P}_1(f) \coloneqq \min_{\omega' \in \{\text{a,b,f,g}\}} f(\omega'),$$
for every $f \in \mathcal{L}$. Indeed, it corresponds to the credal set composed by all the probability distributions on Ω' that assign probability zero to the events $\{\text{c}\}, \{\text{d}\}, \{\text{e}\}$;

- your first friend, Alice, collects information on the crime motive. Given the modalities of the crime, she thinks a motive of passion is more (or at least equal) probable than an economic one. Her beliefs therefore, can be modeled by the coherent set of gambles:
$$\mathcal{D}_2 \coloneqq \mathcal{E}(\{\mathbb{I}_{\{\text{b,e,g}\}} - \mathbb{I}_{\{\text{a,c,d,f}\}}\}) = \mathsf{posi}(\{\mathbb{I}_{\{\text{b,e,g}\}} - \mathbb{I}_{\{\text{a,c,d,f}\}}\} \cup \mathcal{L}^+),$$

where $\mathbb{I}_{\{b,e,g\}}$ and $\mathbb{I}_{\{a,c,d,f\}}$ are the indicator functions respectively on {b,e,g} and {a,c,d,f}. It is the minimal coherent set of gambles that can be constructed from $\mathcal{K} := \{\mathbb{I}_{\{b,e,g\}} - \mathbb{I}_{\{a,c,d,f\}}\}$ trough the natural extension operator, see Definition 3. Moreover it is associated, through its related coherent lower prevision, to the credal set composed by all the probability distributions that assign a probability to the event {b,e,g}, greater (or equal) than the one of its complementary;

- your second friend, Bob, found a long brown hair on the crime scene. It cannot be left there before or after the crime and it does not belong to the victim. He is then sure that the murderer is the subject b. His beliefs can be modeled by the linear prevision P_3 defined as:

$$P_3(f) = f(\mathrm{b}),$$

for every $f \in \mathcal{L}$. Indeed, it corresponds to the credal set composed by the probability distribution that assigns probability one to the option b.

3. Information algebras

We give here a short introduction of information algebras theory. We refer to [2],[3] and the draft work [4] for a much more complete treatment of the subject.

Information algebras are algebraic structures describing basic modes of information processing. They involve several formalisms in computer science, such as relational databases, multiple systems of formal logic, numerical problems of linear algebra and so on.

The basic idea behind information algebras is that, essentially, in many different formalisms to manage information, information comes in pieces that refer to different questions of interest. These pieces can then be aggregated together or the part relative to specific questions can be extracted. This leads to an algebraic structure capturing these aspects, composed by 'pieces of information' that can be manipulated by essentially two basic operations: *combination* to aggregate them and *extraction* to extract part of the information related to specific questions.

The first idea of a generic structure to manage information can be found in [15], where it is introduced an abstract axiomatic system to generalise the local computation scheme for probabilities proposed in [11]. In [2], after realizing that many formalisms in computer science are essentially instances of this axiomatic system, a slightly different version of this system is developed and it is explicitly formulated as a generic structure for inference.

In [2], and in our initial work [5], questions are represented as sets of *logically independent* variables meaning that for each question, the set of its possible answers is represented as the Cartesian product of the sets of possible values (possibility spaces) of the variables involved (*multivariate model*). In [3] and in the draft work [4], the structure of questions is further generalized considering semilattices of domains. In our works [6, 7], we generalize [5] and we assumed a definition of information algebra equivalent to the most general one presented in the draft work [4]. To avoid working with too abstract structures however, we only focused on a particular class of semilattices of questions that are *semilattices of partitions* of a universal set U (*partitive models*). This is the model for questions considered also in the present work.

Given their definitions, multivariate models can be regarded as particular cases of partitive models where the universal set U corresponds to a Cartesian product of other possibility spaces, see Appendix C. In this case we say that U is a *multivariate space*.

Modeling questions as more general partitions allowed us to manage them more efficiently. The free use of Cartesian products made by multivariate models can cause indeed a proliferation of notation, see Example 2. General partitions avoid this problem assuming a set U not necessarily multivariate and already detailed enough to take into account all the possible answers to questions we want to consider in a particular problem. Additionally, modelling questions by partitions opens new possibilities. For instance, diagnostic trees and, more generally, dependent variables constrained by given relations among them can be modeled by partitions, see [16] for details. In the context of imprecise probabilities moreover, the use of partitive models allow us to be as general as Walley [9], since it permits us to work directly with any possibility space. It allows us also to differentiate between impossible events and events with zero probability, see the following Section 4 and Section 5.

To conclude, we also recall that there exist two equivalent formulations of information algebras: the *labeled* and the *domain-free* one. The main difference is that in labeled information algebras, pieces of information are explicitly linked to the questions they refer while in the domain-free ones they are treated as abstract entities, unrelated to particular questions. The labeled form is in general more convenient for computational considerations while the domain-free one is more suitable for theoretical issues. In this work, we limit ourselves to the domain-free formulation leaving the labeled one for future considerations.

In the rest of the section, we recall the definition of domain-free information algebra used in our works [6, 7], already introduced in an equivalent form in the draft work [4], and some concepts related. As previously noticed, this definition of information algebra involves questions more general than partitions. At the end of the section however, we point out how information algebras modeling questions as partitions constitute an important instance of these more general models.

3.1. Domain-free information algebras

In what follows we denote with $\boldsymbol{\Phi}$, the set whose elements are considered to represent pieces of information, which in turn are denoted with lower case Greek letters.

As model for questions, we consider instead a *quasi-separoid* $(\boldsymbol{Q}, \vee, \perp)$. *Quasi-separoids* $(\boldsymbol{Q}, \vee, \perp)$, also denoted with the term *q-separoids*, are tuples composed by a join-semilattice $(\boldsymbol{Q}, \vee)$, whose elements are indicated with lower case letters $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}...$, and a three-place relation $\perp$ satisfying the following conditions:

C1 $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{y}$, for any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{Q}$;

C2 $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$ implies $\boldsymbol{y} \perp \boldsymbol{x} | \boldsymbol{z}$, for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{Q}$;

C3 $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$ and $\boldsymbol{w} \leq \boldsymbol{y}$ imply $\boldsymbol{x} \perp \boldsymbol{w} | \boldsymbol{z}$, for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{Q}$;

C4 $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$ implies $\boldsymbol{x} \perp \boldsymbol{y} \vee \boldsymbol{z} | \boldsymbol{z}$, for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{Q}$.

These properties abstract away the ones satisfied by a logical *conditional independence relation*. They are required to express the idea that, if $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$, only the part relative to $\boldsymbol{z}$ of an information relative to $\boldsymbol{x}$ is relevant as an information relative to $\boldsymbol{y}$, and vice versa.

Usually, in literature, two additional conditions are assumed for a relation of conditional independence [17]:

C5 $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$ and $\boldsymbol{w} \leq \boldsymbol{y}$ imply $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z} \vee \boldsymbol{w}$, for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{Q}$;

C6 $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$ and $\boldsymbol{x} \perp \boldsymbol{w} | \boldsymbol{y} \vee \boldsymbol{z}$ imply $\boldsymbol{x} \perp \boldsymbol{y} \vee \boldsymbol{w} | \boldsymbol{z}$ for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{Q}$;

in this case we obtain a *separoid* $(\boldsymbol{Q}, \vee, \perp)$.

In [4], systems of questions are assumed only to form q-separoids. This is because three-place relations satisfying properties C1 to C4 are sufficient to generate conditional independence structures allowing for local computation schemes. Join semilattices are instead required because, with the order relation they subsume, they allow us to compare questions by their granularity and, given two questions $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{Q}$, they allow us to work with their combined question $\boldsymbol{x} \vee \boldsymbol{y}$.

Partitions provide a very intuitive example of q-separoid. We focus on them in the following subsection 3.4.

Now, we are ready to give the formal definition of domain-free information algebra used in what follows.

Definition 9 (Domain-free information algebra). A *domain-free information algebra* is a two-sorted structure $(\boldsymbol{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \boldsymbol{0}, \boldsymbol{1}, \boldsymbol{\epsilon})$, where:

- $(\boldsymbol{\Phi}, \cdot, \boldsymbol{0}, \boldsymbol{1})$ is a commutative semigroup with $\cdot : \boldsymbol{\Phi} \times \boldsymbol{\Phi} \rightarrow \boldsymbol{\Phi}$, where $\cdot((\boldsymbol{\phi}, \boldsymbol{\psi}))$ is denoted by $\boldsymbol{\phi} \cdot \boldsymbol{\psi}$ for each $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \boldsymbol{\Phi} \times \boldsymbol{\Phi}$, and with $\boldsymbol{0}$ and $\boldsymbol{1}$ as the null and the unit elements respectively,

- $(\boldsymbol{Q}, \vee, \perp)$ is a q-separoid,
- $\boldsymbol{\epsilon} : \boldsymbol{\Phi} \times \boldsymbol{Q} \to \boldsymbol{\Phi}$, where $\boldsymbol{\epsilon}((\boldsymbol{\phi}, \boldsymbol{x}))$ is denoted by $\boldsymbol{\epsilon_x}(\boldsymbol{\phi})$ for each $(\boldsymbol{\phi}, \boldsymbol{x}) \in \boldsymbol{\Phi} \times \boldsymbol{Q}$,

satisfying moreover the following properties:

1. *Existential Quantifier:* For any $\boldsymbol{x} \in \boldsymbol{Q}$, $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi} \in \boldsymbol{\Phi}$:
   (a) $\boldsymbol{\epsilon_x}(\boldsymbol{0}) = \boldsymbol{0}$,
   (b) $\boldsymbol{\epsilon_x}(\boldsymbol{\phi}) \cdot \boldsymbol{\phi} = \boldsymbol{\phi}$,
   (c) $\boldsymbol{\epsilon_x}(\boldsymbol{\epsilon_x}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\phi}_2) = \boldsymbol{\epsilon_x}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\epsilon_x}(\boldsymbol{\phi}_2)$.
2. *Extraction:* For any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{Q}$, $\boldsymbol{\phi} \in \boldsymbol{\Phi}$, such that $\boldsymbol{x} \vee \boldsymbol{z} \perp \boldsymbol{y} \vee \boldsymbol{z} | \boldsymbol{z}$ and $\boldsymbol{\epsilon_x}(\boldsymbol{\phi}) = \boldsymbol{\phi}$, we have: $\boldsymbol{\epsilon_{y \vee z}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon_{y \vee z}}(\boldsymbol{\epsilon_z}(\boldsymbol{\phi}))$.
3. *Support:* For any $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ there is an $\boldsymbol{x} \in \boldsymbol{Q}$ so that $\boldsymbol{\epsilon_x}(\boldsymbol{\phi}) = \boldsymbol{\phi}$, i.e., a *support* of $\boldsymbol{\phi}$ [6], and whenever $\boldsymbol{\epsilon_x}(\boldsymbol{\phi}) = \boldsymbol{\phi}$, then $\boldsymbol{\epsilon_y}(\boldsymbol{\phi}) = \boldsymbol{\phi}$ for every $\boldsymbol{y} \geq \boldsymbol{x}$, $\boldsymbol{y} \in \boldsymbol{Q}$.

Note that, by Existential Quantifier we also have $\boldsymbol{\epsilon_x}(\boldsymbol{1}) = \boldsymbol{\epsilon_x}(\boldsymbol{1}) \cdot \boldsymbol{1} = \boldsymbol{1}$.

In what follows, when we do not need to specify every element of a domain-free information algebra, it is sufficient to refer to it with the set of its pieces of information $\boldsymbol{\Phi}$.

The first binary operation $\cdot$ defined on $\boldsymbol{\Phi}$ is called *combination*. For every pair of pieces of information $\boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{\Phi}$, $\boldsymbol{\phi} \cdot \boldsymbol{\psi}$ represents the information obtained aggregating $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. $(\boldsymbol{\Phi}, \cdot, \boldsymbol{0}, \boldsymbol{1})$ is then required to be a commutative semigroup in order to mimick the intuitive properties of aggregation of information. The null element represents contradiction, hence combined with any piece of information it generates again contradiction. The unit element instead represents vacuous information that combined with any piece of information does not add new information.

The second binary operation $\boldsymbol{\epsilon}$ is called *Extraction*. Given a piece of information $\boldsymbol{\phi}$ and a question $\boldsymbol{x} \in \boldsymbol{Q}$, $\boldsymbol{\epsilon_x}(\boldsymbol{\phi})$ represents the information regarding the question $\boldsymbol{x}$ extracted from the piece of information $\boldsymbol{\phi}$. If $\boldsymbol{x} \in \boldsymbol{Q}$ is in particular a support of $\boldsymbol{\phi}$, i.e., if $\boldsymbol{\epsilon_x}(\boldsymbol{\phi}) = \boldsymbol{\phi}$, it means essentially that $\boldsymbol{\phi}$ is an information about the question $\boldsymbol{x}$.

The intuition behind the other axioms is the following. Property (c) of Existential Quantifier is fundamental for local computation, as noted by [15], see also [4, 5]. Property (a) of Existential Quantifier says that extraction from the contradiction still gives contradiction. Property (b) says that combining a piece of information with part of it gives nothing new. It has important consequences, especially, it allows to introduce an information order among pieces of information, see next sections. Extraction is again useful for local computation schemes. Finally, Support is necessary to construct an equivalent labeled version of a domain-free information algebra, see [2, 4].

Some consequences of the axioms above, useful in what follows, are shown in Appendix B. In particular, we introduce Theorem 9 in Appendix B, which proves that the axiomatic system introduced in this section coincides with the one introduced in [4]. We use this alternative formulation because it is closer to the one introduced in our initial work [5], where questions are modeled using multivariate models, and allows a simpler construction of *set algebras*, as shown in Section 3.5. In Section 3.4 we treat the sub-case of domain-free information algebras with questions modeled as q-separoids of partitions.

3.2. Information algebras homomorphisms and subalgebras

In what follows, we are interested in *homomorphisms* between domain-free information algebras. As seen in the previous section, a domain-free information algebra is a two-sorted structure consisting of a commutative semigroup with a null and a unit element and a q-separoid, with a further extraction operation. A homomorphism therefore, should preserve all the operations and relations defined on the structure.

For the aims of this article however, we introduce only the definition of homomorphism for domain-free information algebras constructed considering the same q-separoid of questions and assuming that each question is mapped to itself. More general definitions can be considered in other situations [3].

Definition 10 ((Domain-free) information algebras homomorphism). Let $(\mathbf{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$ and $(\mathbf{\Phi}', \boldsymbol{Q}, \vee, \perp, \cdot', \mathbf{0}', \mathbf{1}', \boldsymbol{\epsilon}')$ be two domain-free information algebras. A mapping $f : \mathbf{\Phi} \to \mathbf{\Phi}'$ is called a *(domain-free) information algebras homomorphism* if:

1. $f(\boldsymbol{\phi} \cdot \boldsymbol{\psi}) = f(\boldsymbol{\phi}) \cdot' f(\boldsymbol{\psi})$, for every $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathbf{\Phi}$;
2. $f(\mathbf{0}) = \mathbf{0}'$ and $f(\mathbf{1}) = \mathbf{1}'$;
3. $f(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}'_{\boldsymbol{x}}(f(\boldsymbol{\phi}))$, for all $\boldsymbol{\phi} \in \mathbf{\Phi}$ and $\boldsymbol{x} \in \boldsymbol{Q}$.

If f is one-to-one, it is a *(domain-free) information algebras embedding* and the information algebra $\mathbf{\Phi}$ is said to be *embedded* into the information algebra $\mathbf{\Phi}'$; if f is bijective, it is a *(domain-free) information algebras isomorphism* and the two structures are called *isomorphic*.

Another concept used later on is the notion of *substructure* or *subalgebra*. Again, we consider only the case in which the set of questions is the same for an information algebra and for its *information subalgebra*. In this context, an information subalgebra $\mathbf{\Phi}'$ of an information algebra $\mathbf{\Phi}$ is essentially another information algebra where the pieces of information constitute a subset of the pieces of information of $\mathbf{\Phi}$ and the operations are the same of $\mathbf{\Phi}$.

Definition 11 ((Domain-free) information subalgebra). Let $(\mathbf{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$ be a domain-free information algebra. Let us consider a set $\mathbf{\Phi}'$. If:

1. $\mathbf{\Phi}' \subseteq \mathbf{\Phi}$,
2. $\boldsymbol{\phi}', \boldsymbol{\psi}' \in \mathbf{\Phi}'$ implies $\boldsymbol{\phi}' \cdot \boldsymbol{\psi}' \in \mathbf{\Phi}'$,
3. $\mathbf{0}, \mathbf{1} \in \mathbf{\Phi}'$,
4. $\boldsymbol{\phi}' \in \mathbf{\Phi}', \boldsymbol{x} \in \boldsymbol{Q}$ implies $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}') \in \mathbf{\Phi}'$,

then the structure $(\mathbf{\Phi}', \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$ is a domain-free information algebra and it is said to be a *(domain-free) information subalgebra* of $\mathbf{\Phi}$.

3.3. Information order and Atoms

Information can be more or less precise. This is reflected by a partial order relation among pieces of information, called *information order*.

Definition 12 (Information order). Consider a domain-free information algebra $(\mathbf{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$. Given $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathbf{\Phi}$ we say that $\boldsymbol{\phi} \leq \boldsymbol{\psi}$ if and only if $\boldsymbol{\phi} \cdot \boldsymbol{\psi} = \boldsymbol{\psi}$.

This is a partial order with the vacuous information $\mathbf{1}$ as least and the contradiction $\mathbf{0}$ as greatest elements. Its definition strongly depends on the item (b) of Existential Quantifier axiom defining a domain-free information algebra. It implies in particular that a piece of information $\boldsymbol{\psi}$ is more informative than another piece of information $\boldsymbol{\phi}$, if and only if adding $\boldsymbol{\phi}$ gives nothing new. In Appendix B, a list of properties deriving from this definition are provided.

In some information algebras there exist also maximal elements with respect to this order.

Definition 13 (Atoms). Given a domain-free information algebra $(\mathbf{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$, an element $\boldsymbol{\alpha} \in \mathbf{\Phi}$ is called an *atom* (of $\mathbf{\Phi}$) if and only if

- $\boldsymbol{\alpha} \neq \mathbf{0}$,
- for all $\boldsymbol{\phi} \in \mathbf{\Phi}$, $\boldsymbol{\alpha} \leq \boldsymbol{\phi}$ implies either $\boldsymbol{\phi} = \boldsymbol{\alpha}$ or $\boldsymbol{\phi} = \mathbf{0}$.

Since $\mathbf{0}$ does not represent information, if they are present, atoms are the *maximal informative* elements of a domain-free information algebra, i.e., maximal with respect to the information order. Again, elementary properties of atoms can be found in Appendix B.

Denote with $At(\mathbf{\Phi})$ the set of all atoms of a domain-free information algebra $\mathbf{\Phi}$. Let us define also, for every element $\boldsymbol{\phi} \in \mathbf{\Phi}$,

$$At(\boldsymbol{\phi}) := \{\boldsymbol{\alpha} \in At(\mathbf{\Phi}) : \boldsymbol{\phi} \leq \boldsymbol{\alpha}\}. \tag{8}$$

This motivates the following definitions.

Definition 14 (Atomic information algebras). 1. A domain-free information algebra $(\mathbf{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$ is called *atomic*, if and only if for all $\boldsymbol{\phi} \in \mathbf{\Phi}$, $\boldsymbol{\phi} \neq \mathbf{0}$, $At(\boldsymbol{\phi})$ is non-empty.

2. It is called *atomistic* or *atomic composed*, if and only if it is atomic and if for all $\boldsymbol{\phi} \in \mathbf{\Phi}$, $\boldsymbol{\phi} \neq \mathbf{0}$,

$$\boldsymbol{\phi} = \bigwedge At(\boldsymbol{\phi}).$$

3.4. Example of q-separoids: Partitions

Consider a universal set U, it can be any non-empty set. Questions about U can directly be modeled listing their possible answers, i.e., questions can be modeled as partitions $\mathcal{P}_x$ of U with x in some index set Q_U, whose blocks represent worlds $u \in U$ where $\mathcal{P}_x$ has the same answer. Partitions can also be equivalently expressed with equivalence relations $\equiv_x$ on U: given $u, u' \in U$, $u \equiv_x u'$ if and only if u, u are in the same block of partition $\mathcal{P}_x$. For simplicity, in what follows, we identify partitions and equivalence relations with their indexes $x \in Q_U$. In the special case where the questions of interest cover all the possible partitions of U, we denote Q_U as T_U.

Questions modeled in this way can be ordered with respect to granularity: $x \leq y$ if and only if y is finer than x, i.e., for every $u, u' \in U$, $u \equiv_y u'$ implies $u \equiv_x u'$ or, equivalently, block $[u]_y$ of partition $\mathcal{P}_y$ is contained in some block $[u]_x$ of partition $\mathcal{P}_x$.[4]

The set of all the partitions of U, $\mathcal{P}_x$ with $x \in T_U$, with this order relation induces a lattice [18]. In this lattice, the join of two partitions $\mathcal{P}_x, \mathcal{P}_y$ corresponds to the partition obtained as the non-empty intersections of blocks of $\mathcal{P}_x$ with blocks of $\mathcal{P}_y$. We indicate it with $x \vee y$. Hence, $(T_U, \vee)$, where $\vee$ is the join operation defined above, is a join-semilattice. The definition of meet instead is somewhat involved [18].

We can also define a conditional independence relation on partitions which, in particular for $n = 2$, satisfies properties C1 to C4 listed at the beginning of this section. For a proof of this fact see [4].[5]

Definition 15 (Conditionally Independent Partitions). Consider a finite set of partitions of U, $\mathcal{P}_1, \ldots \mathcal{P}_n$, and a block B of another partition $\mathcal{P}$ of U (contained or not in the list $\mathcal{P}_1, \ldots, \mathcal{P}_n$), then define for $n \geq 1$,

$$R_B(\mathcal{P}_1, \ldots, \mathcal{P}_n) := \{(B_1, \ldots, B_n) : B_i \in \mathcal{P}_i, \cap_{i=1}^n B_i \cap B \neq \emptyset\}.$$

We call $\mathcal{P}_1, \ldots, \mathcal{P}_n$ *conditionally independent given* $\mathcal{P}$, if and only if for all blocks B of $\mathcal{P}$, $R_B(\mathcal{P}_1, \ldots, \mathcal{P}_n) = R_B(\mathcal{P}_1) \times \cdots \times R_B(\mathcal{P}_n)$.

$\mathcal{P}_x, \mathcal{P}_y$ are conditionally independent given $\mathcal{P}_z$ if and only if, for every answer B_z to the question $\mathcal{P}_z$, knowing also an answer to $\mathcal{P}_x$ (or $\mathcal{P}_y$) compatible with B_z, does not give us additional information regarding the answer to $\mathcal{P}_y$ (respectively $\mathcal{P}_x$), except that it must be again compatible with B_z. This is true if and only if for every $B_x \in \mathcal{P}_x, B_y \in \mathcal{P}_y, B_z \in \mathcal{P}_z$ such that $B_x \cap B_z \neq \emptyset$ and $B_y \cap B_z \neq \emptyset$, we have $B_x \cap B_y \cap B_z \neq \emptyset$. In this case we write $x \perp y | z$. We may also say that $x \perp y | z$ if and only if $u \equiv_z u'$ implies the existence of an element $u'' \in U$ such that $u \equiv_{x \vee z} u''$ and $u' \equiv_{y \vee z} u''$. Analogous considerations can be made for more than two partitions conditionally independent given another one.

[4] In the literature usually the inverse order between partitions is considered. However, this order better corresponds to our natural order of questions by granularity.

[5] In [16] the same relation among partitions is introduced but it is denoted with the term *qualitative conditional independence* relation.

Given the above considerations, we can conclude that $(T_U, \vee)$ with the conditional independence relation $\perp$ defined above forms a q-separoid $(T_U, \vee, \perp)$. It is possible to show that also every join-subsemilattice $(Q_U, \vee)$ of $(T_U, \vee)$ induces a q-separoid $(Q_U, \vee, \perp)$, where $\perp$ corresponds again to the conditional independence relation among partitions [4, Theorem 2.6]. In this case, for simplicity, we call $(Q_U, \vee, \perp)$ a *sub-q-separoid* of $(T_U, \vee, \perp)$.

As previously noticed, multivariate models form a particular instance of q-separoids of partitions. In Appendix C, they are briefly introduced.

Example 2. Let us consider again Ω' of the previous example and the following partitions of $U = \Omega'$:

$$\mathcal{P}_{sex} \coloneqq \{\{\text{a,b,f,g}\}, \{\text{c,d,e}\}\},$$

$$\mathcal{P}_{motive} \coloneqq \{\{\text{a,c,d,f}\}, \{\text{b,e,g}\}\},$$
$$\mathcal{P}_{hair\ colour} \coloneqq \{\{\text{a,c,d,f}\}, \{\text{b}\}, \{\text{e,g}\}\},$$
$$\mathcal{P}_{hairstyle} \coloneqq \{\{\text{a,f}\}, \{\text{b}\}, \{\text{c,d}\}, \{\text{e}\}, \{\text{g}\}\}.$$

They respectively divide the space of the suspects, Ω', according to the possible answers to the questions: 'what is the sex of the murderer?', 'what is their motive?', 'what is their hair colour?','what is their hairstyle (i.e., colour and length of the hair)?'.

It is possible to notice that $\mathcal{P}_{hairstyle}$ is a refinement of $\mathcal{P}_{hair\ colour}$. Therefore, $\mathcal{P}_{hair\ colour} \leq \mathcal{P}_{hairstyle}$. We can observe moreover that $\mathcal{P}_{sex} \perp \mathcal{P}_{motive} | \mathcal{P}_{hairstyle}$. Intuitively, this means that assuming the murderer has a certain hairstyle, knowing their sex compatible to the hairstyle does not give us any additional information about the motive, except that it must be again compatible with the hairstyle and vice versa.

If we had wanted to adopt instead a multivariate model for the same questions, we could have proceeded in a very natural way by introducing the following variables:

- X_1: representing the gender of the murderer with set of possible values $\mathcal{X}_1 \coloneqq$\{male, female\};
- X_2: representing the motive of the murderer with set of possible values $\mathcal{X}_2 \coloneqq$\{economic, passion\};
- X_3: representing the hair colour of the murderer with set of possible values $\mathcal{X}_3 \coloneqq$\{blond, brown, red\};
- X_4: representing the length of the hair of the murderer with set of possible values $\mathcal{X}_4 \coloneqq$\{short, long\};
- X_5: representing the hair style of the murderer with set of possible values $\mathcal{X}_5 \coloneqq \mathcal{X}_3 \times \mathcal{X}_4$.

In this context, the previous questions could have been modeled with X_1, X_2, X_3, X_5, respectively. Notice in particular that the set of possible values of X_5, which corresponds to the question obtained as the conjuction of X_3 and X_4,[6] corresponds to the Cartesian product $\mathcal{X}_3 \times \mathcal{X}_4$. Similar reasonings can be made for every conjoined question, causing a proliferation of notation. Notice moreover that some of the elements contained in $\mathcal{X}_3 \times \mathcal{X}_4$ do not even have a correspondence with the real subjects, i.e., they correspond to *impossible events* (e.g., no suspect has brown short hair).

Modeling questions with general partitions avoids these problems: all and only the possible answers to questions of interest are already contained in Ω'.

3.5. Example of information algebras: Set algebras

Archetypes of information algebras are so-called *set algebras.* They are information algebras where pieces of information are modeled in the simplest way as subsets of some universe, combination is set intersection, and extraction is related to the so-called *saturation operators* [2, 3].

Here we show how, in our context, starting from a universal set U, it is possible to create a set algebra where pieces of information correspond to subsets of U and questions of interest are represented by partitions. This construction becomes particularly important in Section 6.

[6] See Appendix C.

Let us consider a universal set U. For any partition $\mathcal{P}_x$ of U, it is possible to construct a *saturation operator* defined for any set $S \subseteq U$ as:

$$\sigma_x(S) := \{u \in U : (\exists u' \in S)\ u \equiv_x u'\}, \tag{9}$$

equivalent to

$$\sigma_x(S) := \cup\{B_x : B_x \text{ block of } \mathcal{P}_x,\ B_x \cap S \neq \emptyset\}.$$

Saturation operators satisfy the following properties. Similar results can be found in [4].

Lemma 1. *For all $S, T \subseteq U$ and any partition $\mathcal{P}_x$ with $x \in T_U$, we have:*

1. $\sigma_x(\emptyset) = \emptyset$,
2. $S \subseteq \sigma_x(S)$,
3. $\sigma_x(\sigma_x(S) \cap T) = \sigma_x(S) \cap \sigma_x(T)$,
4. $\sigma_x(\sigma_x(S)) = \sigma_x(S)$,
5. $S \subseteq T \Rightarrow \sigma_x(S) \subseteq \sigma_x(T)$,
6. $\sigma_x(\sigma_x(S) \cap \sigma_x(T)) = \sigma_x(S) \cap \sigma_x(T)$.

Now, let us consider the algebraic structure identified by the signature:

$$(P_{Q_U}(U), Q_U, \vee, \perp, \cap, \emptyset, U, \sigma),$$

where:

- $(Q_U, \vee, \perp)$ is a sub-q-separoid of $(T_U, \vee, \perp)$;
- $P_{Q_U}(U) := \{S \subseteq U :\ \exists\, x \in Q_U,\ \sigma_x(S) = S\}$ is the set of subsets of U *saturated with respect to* Q_U;
- $\sigma : P_{Q_U}(U) \times Q_U \to P_{Q_U}(U)$ is defined as $\sigma((S, x)) := \sigma_x(S)$ for every $S \in P_{Q_U}(U),\ x \in Q_U$.

We claim it is a domain-free information algebra where $P_{Q_U}(U)$ is the set of pieces of information, $(Q_U, \vee, \perp)$ is the q-separoids of questions of interest, $\cap$ is the combination operation (with $\emptyset$ and U as the null and the unit elements respectively) and σ, defined above, is the extraction operation. Since pieces of information are sets, combination is set intersection and extraction is constructed from saturation operators, it is a set algebra.

We can prove the previous claim step by step as follows:

- $(Q_U, \vee, \perp)$ is a q-separoid by hypothesis.
- Support axiom is satisfied. Indeed, by definition of $P_{Q_U}(U)$, for every $S \in P_{Q_U}(U)$ there exists $x \in Q_U$, such that $\sigma_x(S) = S$. Moreover, if $x \leq y$ and $y \in Q_U$, then $u \equiv_y u'$ implies $u \equiv_x u'$, so that $\sigma_y(S) \subseteq \sigma_x(S)$. Therefore, if $\sigma_x(S) = S$, then $S \subseteq \sigma_y(S) \subseteq \sigma_x(S) = S$.
- $(P_{Q_U}(U), \cap, \emptyset, U)$ is a commutative semigroup with $\emptyset$ as the null element and U as the unit one. Clearly, $\emptyset, U \in P_{Q_U}(U)$ and they are the null and the unit element respectively, with respect to set intersection. Therefore, the only property left to prove is that $P_{Q_U}(U)$ is closed with respect to intersection. But this is true. Consider indeed two sets $S, T \in P_{Q_U}(U)$, such that $\sigma_x(S) = S$ and $\sigma_y(T) = T$ for some $x, y \in Q_U$, respectively. Then, thanks to the Support axiom, we know that $\sigma_{x \vee y}(S) = S$ and $\sigma_{x \vee y}(T) = T$ with $x \vee y \in Q_U$, since $(Q_U, \vee)$ is a join-semilattice. Therefore, thanks to Lemma 1, we have
$$\sigma_{x \vee y}(S \cap T) = \sigma_{x \vee y}(\sigma_{x \vee y}(S) \cap \sigma_{x \vee y}(T)) = \sigma_{x \vee y}(S) \cap \sigma_{x \vee y}(T) = S \cap T.$$
- $P_{Q_U}(U)$ is closed also with respect to extraction thanks to item 4 of Lemma 1.

- The Existential Quantifier axiom follows from items 1-3 of Lemma 1.

It remains only to prove the Extraction axiom but it follows from the theorem below.

Theorem 1. *Given a sub-q-separoid* $(Q_U, \vee, \perp)$ *of* $(T_U, \vee, \perp)$*, consider* $x, y, z \in Q_U$*, such that* $x \vee z \perp y \vee z | z$*. Then, for any* $S \subseteq U$*,*

$$\sigma_{y \vee z}(\sigma_x(S)) = \sigma_{y \vee z}(\sigma_z(\sigma_x(S))).$$

PROOF. From $\sigma_z(\sigma_x(S)) \supseteq \sigma_x(S)$ we obtain $\sigma_{y\vee z}(\sigma_z(\sigma_x(S)) \supseteq \sigma_{y\vee z}(\sigma_x(S))$. Consider therefore an element $u \in \sigma_{y\vee z}(\sigma_z(\sigma_x(S)))$. Then, there are elements w, w' and u' so that $u \equiv_{y\vee z} w \equiv_z w' \equiv_x u'$ and $u' \in S$. This means that u, w belong to some block $B_{y\vee z}$ of partition $\mathcal{P}_{y\vee z}$, w, w' to some block B_z of partition $\mathcal{P}_z$ and w', u' to some block B_x of partition $\mathcal{P}_x$. It follows that $B_x \cap B_z \neq \emptyset$ and $B_{y\vee z} \cap B_z \neq \emptyset$. Then, $x \vee z \perp y \vee z | z$ implies, thanks to properties of a quasi-separoid, that $x \perp y \vee z | z$. Therefore, we have $B_x \cap B_{y\vee z} \cap B_z \neq \emptyset$, and in particular, $B_x \cap B_{y\vee z} \neq \emptyset$. So there is a $v \in B_x \cap B_{y\vee z}$ such that $u \equiv_{y\vee z} v \equiv_x u' \in S$, hence $u \in \sigma_{y\vee z}(\sigma_x(S))$. So, we have $\sigma_{y\vee z}(\sigma_x(S)) = \sigma_{y\vee z}(\sigma_z(\sigma_x(S)))$.

4. Domain-free information algebra of coherent sets of gambles

Coherent sets of gambles represent beliefs, hence information, about a possibility space Ω. Therefore, we can ask ourselves if it is possible to construct an information algebra where pieces of information are coherent sets of gambles and questions of interest are partitions of Ω. The answer is yes.

Let us consider $\Phi(\Omega) \coloneqq C(\Omega) \cup \{\mathcal{L}(\Omega)\}$, or more simply Φ, as defined in Section 2 and a q-separoid of partitions $(Q_\Omega, \vee, \perp)$ sub-q-separoid of $(T_\Omega, \vee, \perp)$, as defined in Section 3.4. Since from now on we work with a fixed possibility space Ω, we denote $(Q_\Omega, \vee, \perp)$ more simply as $(Q, \vee, \perp)$. Given $x \in Q$, a gamble f on Ω is called *x-measurable*, iff for all $\omega \equiv_x \omega'$ we have $f(\omega) = f(\omega')$, that is, if f is constant on every block of the partition $\mathcal{P}_x$. It could then also be considered as a function (a gamble) on the set of blocks of $\mathcal{P}_x$. In what follows, we denote with $\mathcal{L}_x(\Omega)$, or more simply with $\mathcal{L}_x$, the set of all x-measurable gambles.

In [6], we showed that $(\Phi, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ is a domain-free information algebra, where the combination operation $\cdot$ and the extraction operation ϵ are defined as follows.

- Combination: $\cdot : \Phi \times \Phi \to \Phi$, where $\cdot((\mathcal{D}_1, \mathcal{D}_2))$ is denoted by $\mathcal{D}_1 \cdot \mathcal{D}_2$ and defined as

  $$\mathcal{D}_1 \cdot \mathcal{D}_2 \coloneqq \mathcal{C}(\mathcal{D}_1 \cup \mathcal{D}_2),$$

  for every $\mathcal{D}_1, \mathcal{D}_2 \in \Phi$, where $\mathcal{C}$ operator is defined in Eq.(1),

- Extraction. $\epsilon : \Phi \times Q \to \Phi$, where $\epsilon((\mathcal{D}, x))$ is denoted by $\epsilon_x(\mathcal{D})$ and defined as

  $$\epsilon_x(\mathcal{D}) \coloneqq \mathcal{C}(\mathcal{D} \cap \mathcal{L}_x),$$

  for every $\mathcal{D} \in \Phi$, $x \in Q$.

With a little abuse of nomenclature we call $(\Phi, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$, a *domain-free information algebra of coherent sets of gambles.* Since we model questions as general partitions of Ω, this construction can be reproduced for every possibility space Ω without restrictions. The same will be not true if we restrict systems of questions to multivariate models, see Appendix C and [5].

To conclude, notice that Φ^+, so essentially strictly desirable sets of gambles, induces an information subalgebra of the information algebra of coherent sets of gambles Φ: $(\Phi^+, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$.

Example 3. Let us consider again Example 1. We can equivalently represent yours and your friends' beliefs with the following coherent sets of gambles.

$$\mathcal{D}_1^+ \coloneqq \tau(\underline{P}_1) \coloneqq \{f \in \mathcal{L} : \min_{\omega' \in \{\mathrm{a,b,f,g}\}} f(\omega') > 0\} \cup \mathcal{L}^+;$$

$$\mathcal{D}_2 \coloneqq \mathsf{posi}(\{\mathbb{I}_{\{\mathrm{b,e,g}\}} - \mathbb{I}_{\{\mathrm{a,c,d,f}\}}\} \cup \mathcal{L}^+);$$

$$\mathcal{D}_3^+ \coloneqq \tau(P_3) \coloneqq \{f \in \mathcal{L} : f(\mathrm{b}) > 0\} \cup \mathcal{L}^+.$$

It is possible to observe that, in particular, $\mathcal{D}_1^+$, $\mathcal{D}_3^+$ are the strictly desirable sets of gambles equivalent to $\underline{P}_1$ and P_3 respectively, see Eq. 4. $\mathcal{D}_2$ instead is a coherent not strictly desirable set of gambles.[7]

Let us now construct, as described in this section, a domain-free information algebra of coherent sets of gambles on Ω': $(\Phi(\Omega'), Q, \vee, \perp, \cdot, \mathcal{L}(\Omega'), \mathcal{L}^+(\Omega'), \epsilon)$, with a set of questions Q including partitions introduced in Example 2.

The combination operation $\cdot$ provides us a way to collect together all the information obtained. As expected, the result of this operation is Bob's conclusions. Indeed:

$$\mathcal{D} \coloneqq \mathcal{D}_1^+ \cdot \mathcal{D}_2 \cdot \mathcal{D}_3^+ \coloneqq \mathcal{C}(\mathcal{D}_1^+ \cup \mathcal{D}_2 \cup \mathcal{D}_3^+) = \mathcal{D}_3^+.$$

Extraction ϵ instead, gives us a way to extract the information obtained about a specific question of interest. Let's say about the motive of the crime. Clearly, since collecting together all the information you obtained a unique murderer - the subject b -, the extracted information tells you that the motive of the murderer has to be a motive of passion, i.e., b's motive:

$$\epsilon_{motive}(\mathcal{D}) = \epsilon_{motive}(\mathcal{D}_3^+) \coloneqq \mathcal{C}(\mathcal{D}_3^+ \cap \mathcal{L}_{motive}) = \{f \in \mathcal{L} : \min_{\omega' \in \{\mathrm{b,e,g}\}} f(\omega') > 0\} \cup \mathcal{L}^+.$$

We can also extract the information you have about the motive before talking with your friends. In this case we obtain:

$$\epsilon_{motive}(\mathcal{D}_1^+) = \mathcal{L}^+.$$

This means that you have no information about the motive. This is reasonable since you are completely unsure about who the murderer is among a and f, having an economic motive, and b and g, having instead a motive of passion for the crime.

4.1. Atoms

As previously noticed in Section 3.3, it is also possible to define a partial order on Φ as $\mathcal{D}_1 \le \mathcal{D}_2$ if and only if $\mathcal{D}_1 \cdot \mathcal{D}_2 = \mathcal{D}_2$. It is easy to show that $\mathcal{D}_1 \le \mathcal{D}_2$ if and only if $\mathcal{D}_1 \subseteq \mathcal{D}_2$. Therefore, $\mathcal{D}_1 \le \mathcal{D}_2$ if and only if $\mathcal{D}_2$ contains more gambles than $\mathcal{D}_1$.

In our previous work [6], we showed moreover that the information algebra Φ admits the presence of maximal elements with respect to this order. Specifically, maximal coherent sets form the atoms of Φ. Using notation introduced in Section 3.3, we indicate with $At(\Phi)$ the whole set of atoms of Φ and with $At(\mathcal{D})$, for every $\mathcal{D} \in \Phi$, the set of atoms $M \in At(\Phi)$ such that $\mathcal{D} \le M$. Φ is in particular *atomic*, i.e., for any $\mathcal{D} \in \Phi$ such that $\mathcal{D} \neq \mathcal{L}$ the set $At(\mathcal{D})$ is non-empty, and *atomistic*, i.e., for any $\mathcal{D} \in \Phi$ such that $\mathcal{D} \neq \mathcal{L}$, $\mathcal{D} = \bigcap At(\mathcal{D})$.

5. Domain-free information algebra of coherent lower previsions

In this section we prove that, similarly to coherent sets of gambles, also coherent lower previsions induce a domain-free information algebra. Analogous results for multivariate models are given in [5].

Coherent lower previsions, as well as coherent sets of gambles, still represent beliefs, hence, information about a possibility space Ω.

Let us consider $\underline{\Phi}(\Omega) \coloneqq \underline{\mathbb{P}}(\Omega) \cup \{\sigma(\mathcal{L}(\Omega))\}$, or simply $\underline{\Phi}$, as defined in Section 2, and a q-separoid of questions $(Q, \vee, \perp)$, sub-q-separoid of $(T_\Omega, \vee, \perp)$.[8]

We now introduce, also in this context, a combination and an extraction operation.

[7] It is possible to notice, in particular, that your friend Bob assigns probability zero to all the sets of suspects not including the suspect b. These events, however, do not have the same meaning of impossible ones as defined in Example 2.

[8] Without loss of generality, we use the same notation used in Section 4 to denote the q-separoid of questions of interest.

Given two coherent sets of gambles $\mathcal{D}_1, \mathcal{D}_2$ which are *consistent*, i.e., such that $\mathcal{D}_1 \cdot \mathcal{D}_2 \neq \mathcal{L}$,[9] with $\underline{P}_1 \coloneqq \sigma(\mathcal{D}_1)$ and $\underline{P}_2 \coloneqq \sigma(\mathcal{D}_2)$, let us consider the lower prevision $\underline{P}' \coloneqq \max\{\underline{P}_1, \underline{P}_2\}$ which assumes the value:

$$\sigma(\mathcal{D}_1 \cup \mathcal{D}_2)(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in \mathcal{D}_1 \cup \mathcal{D}_2\} = \max\{\underline{P}_1(f), \underline{P}_2(f)\} \eqqcolon \underline{P}'(f),$$

for every gamble f in its domain. Following a reasoning similar to the one considered for coherent sets of gambles, we may take $\underline{E}^*(\max\{\underline{P}_1, \underline{P}_2\})$ to define the combination of two lower previsions $\underline{P}_1$ and $\underline{P}_2$ in $\underline{\Phi}$, where $\underline{E}^*$ operator is defined in Definition 8. Regarding the extraction, for every $x \in Q$ and $\underline{P} \in \underline{\Phi}$, we may take $\underline{E}^*(\underline{P}_x)$, where $\underline{P}_x$ is defined as the restriction of $\underline{P}$ to $\mathcal{L}_x$. Thus, in summary, we can define on $\underline{\Phi}$ and Q the following operations.

1. Combination. $\cdot : \underline{\Phi} \times \underline{\Phi} \to \underline{\Phi}$, where $\cdot((\underline{P}_1, \underline{P}_2))$ is denoted by $\underline{P}_1 \cdot \underline{P}_2$ and defined as

$$\underline{P}_1 \cdot \underline{P}_2 \coloneqq \underline{E}^*(\max\{\underline{P}_1, \underline{P}_2\}), \tag{10}$$

   for every $\underline{P}_1, \underline{P}_2 \in \underline{\Phi}$.

2. Extraction. $\underline{e} : \underline{\Phi} \times Q \to \underline{\Phi}$, where $\underline{e}((\underline{P}, x))$ is denoted by $\underline{e}_x(\underline{P})$ and defined as

$$\underline{e}_x(\underline{P}) \coloneqq \underline{E}^*(\underline{P}_x),$$

   for every $\underline{P} \in \underline{\Phi}, x \in Q$.

The following theorem shows that $(\underline{\Phi}, Q, \vee, \perp, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$, where $\cdot, \underline{e}$ are defined above on $\underline{\Phi}$ and Q, forms a domain-free information algebra that, with the same little abuse of nomenclature introduced for coherent sets of gambles, we call a *domain-free information algebra of coherent lower previsions*. To do so, it links the combination and the extraction operation defined on $\underline{\Phi}$ and Q with the ones of the domain-free information algebra $(\Phi^+, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$.

Theorem 2. *Let $\mathcal{D}_1^+$, $\mathcal{D}_2^+, \mathcal{D}^+ \subseteq \mathcal{L}$ be strictly desirable sets of gambles and $x \in Q$. Then*

1. *$\sigma(\mathcal{L})(f) = +\infty$, $\sigma(\mathcal{L}^+)(f) = \inf f$ for all $f \in \mathcal{L}$,*
2. *$\sigma(\mathcal{D}_1^+ \cdot \mathcal{D}_2^+) = \sigma(\mathcal{D}_1^+) \cdot \sigma(\mathcal{D}_2^+)$,*
3. *$\sigma(\epsilon_x(\mathcal{D}^+)) = \underline{e}_x(\sigma(\mathcal{D}^+))$.*

Proof.
1. It follows from the definition.
2. Assume firstly that $\mathcal{D}_1^+ \cdot \mathcal{D}_2^+ = \mathcal{L}$ and let $\underline{P}_1 \coloneqq \sigma(\mathcal{D}_1^+)$, $\underline{P}_2 \coloneqq \sigma(\mathcal{D}_2^+)$. Then there can be no coherent lower prevision $\underline{P}$ dominating both $\underline{P}_1$ and $\underline{P}_2$. Indeed, otherwise we would have, $\mathcal{D}_1^+ = \tau(\underline{P}_1) \leq \tau(\underline{P})$ and $\mathcal{D}_2^+ = \tau(\underline{P}_2) \leq \tau(\underline{P})$, where $\tau(\underline{P})$ is a coherent (strictly desirable) set of gambles. But then $\tau(\underline{P}) \geq \mathcal{D}_1^+ \cdot \mathcal{D}_2^+$ and this is a contradiction. So we have $\underline{P}_1 \cdot \underline{P}_2 = \sigma(\mathcal{L})$. Vice versa is also true, hence $\sigma(\mathcal{D}_1^+) \cdot \sigma(\mathcal{D}_2^+) = \sigma(\mathcal{L})$ if and only if $\sigma(\mathcal{D}_1^+ \cdot \mathcal{D}_2^+) = \sigma(\mathcal{L})$.

   Let then $\mathcal{D}_1^+ \cdot \mathcal{D}_2^+ \neq \mathcal{L}$. Then, $\mathcal{D}_1^+ \cdot \mathcal{D}_2^+$, as well as $\mathcal{D}_1^+ \cup \mathcal{D}_2^+$, satisfies the conditions of Theorem 8 in Appendix A. Therefore, applying this theorem, we have

$$\begin{aligned}\sigma(\mathcal{D}_1^+ \cdot \mathcal{D}_2^+) &\coloneqq \sigma(\mathcal{C}(\mathcal{D}_1^+ \cup \mathcal{D}_2^+)) = \underline{E}(\sigma(\mathcal{D}_1^+ \cup \mathcal{D}_2^+)) \\ &= \underline{E}(\max\{\sigma(\mathcal{D}_1^+), \sigma(\mathcal{D}_2^+)\}) \eqqcolon \sigma(\mathcal{D}_1^+) \cdot \sigma(\mathcal{D}_2^+).\end{aligned}$$

[9] See the following Definition 17.

3. We remark that $\mathcal{D}^+ \cap \mathcal{L}_x$ satisfies the conditions of Theorem 8 in Appendix A. Thus we obtain

$$\sigma(\epsilon_x(\mathcal{D}^+)) := \sigma(\mathcal{C}(\mathcal{D}^+ \cap \mathcal{L}_x)) = \underline{E}(\sigma(\mathcal{D}^+ \cap \mathcal{L}_x)).$$

Now,

$$\sigma(\mathcal{D}^+ \cap \mathcal{L}_x)(f) := \sup\{\mu \in \mathbb{R} : f - \mu \in \mathcal{D}^+ \cap \mathcal{L}_x\},\ \forall f \in dom(\sigma(\mathcal{D}^+ \cap \mathcal{L}_x)).$$

But $f - \mu \in \mathcal{D}^+ \cap \mathcal{L}_x$ if and only if f is x-measurable and $f - \mu \in \mathcal{D}^+$. Therefore, we conclude that $\sigma(\mathcal{D}^+ \cap \mathcal{L}_x) = \sigma(\mathcal{D}^+)_x$. Thus, we have indeed $\sigma(\epsilon_x(\mathcal{D}^+)) = \underline{E}(\sigma(\mathcal{D}^+)_x) =: \underline{e}_x(\sigma(\mathcal{D}^+))$.

This theorem proves, in particular, two things:

- $(\underline{\Phi}, Q, \vee, \perp, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$ is a domain-free information algebra;
- $(\underline{\Phi}, Q, \vee, \perp, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{e})$ is isomorphic to $(\Phi^+, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$, information subalgebra of $(\Phi, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$.

To simplify notation, in the rest of the section we consider only domain-free information algebras $\Phi, \Phi^+, \underline{\Phi}$ defined considering the same q-separoids of questions $(Q, \vee, \perp)$.

Given the results of Theorem 2, it is reasonable to ask if a homomorphism exists also between Φ and Φ^+ and hence between Φ and $\underline{\Phi}$. The answer is partially yes.

Consider indeed the map $\mathcal{D} \mapsto \mathcal{D}^+ := \tau(\sigma(\mathcal{D}))$, defined from Φ to Φ^+. The following theorem and corollary and the example below, show that it is possible to establish only a *weak* homomorphism between Φ and Φ^+ and hence between Φ and $\underline{\Phi}$. We call the homomorphism *weak*, because when $\mathcal{D}_1 \in \Phi$ and $\mathcal{D}_2 \in \Phi$ are not consistent, that is if $\mathcal{D}_1 \cdot \mathcal{D}_2 = \mathcal{L}$, then $\mathcal{D}_1 \cdot \mathcal{D}_2$ can be mapped to something different from $\mathcal{D}_1^+ \cdot \mathcal{D}_2^+$.

Theorem 3. *Let us consider the map $\mathcal{D} \mapsto \mathcal{D}^+ := \tau(\sigma(\mathcal{D}))$, defined from Φ to Φ^+. Let then $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}$ be coherent sets of gambles and $x \in Q$. We have the following results.*

1. *If $\mathcal{D}_1 \cdot \mathcal{D}_2 \neq \mathcal{L}$, then $\mathcal{D}_1 \cdot \mathcal{D}_2 \mapsto (\mathcal{D}_1 \cdot \mathcal{D}_2)^+ = \mathcal{D}_1^+ \cdot \mathcal{D}_2^+$,*
2. *$\epsilon_x(\mathcal{D}) \mapsto (\epsilon_x(\mathcal{D}))^+ = \epsilon_x(\mathcal{D}^+)$.*

Proof. 1. Note first that $\mathcal{D}_1^+ \subseteq \mathcal{D}_1$ and $\mathcal{D}_2^+ \subseteq \mathcal{D}_2$, see Section 2. So that

$$\mathcal{D}_1^+ \cdot \mathcal{D}_2^+ = \tau(\sigma(\mathcal{D}_1^+ \cdot \mathcal{D}_2^+)) \subseteq \tau(\sigma(\mathcal{D}_1 \cdot \mathcal{D}_2)) =: (\mathcal{D}_1 \cdot \mathcal{D}_2)^+.$$

Further

$$(\mathcal{D}_1 \cdot \mathcal{D}_2)^+ := \tau(\sigma(\mathcal{D}_1 \cdot \mathcal{D}_2)) := \{f \in \mathcal{L} : \sigma(\mathcal{D}_1 \cdot \mathcal{D}_2)(f) > 0\} \cup \mathcal{L}^+.$$

So, if $f \in (\mathcal{D}_1 \cdot \mathcal{D}_2)^+$, then either $f \in \mathcal{L}^+$ or

$$\sigma(\mathcal{D}_1 \cdot \mathcal{D}_2)(f) := \sup\{\mu \in \mathbb{R} : f - \mu \in \mathcal{C}(\mathcal{D}_1 \cup \mathcal{D}_2)\} > 0. \tag{11}$$

In the first case obviously $f \in \mathcal{D}_1^+ \cdot \mathcal{D}_2^+$. Let us consider now $f \notin \mathcal{L}^+$. In this case there is a $\delta > 0$ so that $f - \delta \in \mathcal{C}(\mathcal{D}_1 \cup \mathcal{D}_2)$. This means that $f - \delta = h + \lambda_1 f_1 + \lambda_2 f_2$, where $h \in \mathcal{L}^+ \cup \{0\}$, $f_1 \in \mathcal{D}_1$, $f_2 \in \mathcal{D}_2$ and $\lambda_1, \lambda_2 \geq 0$ not both equal 0. But then

$$f = h + (\lambda_1 f_1 + \delta/2) + (\lambda_2 f_2 + \delta/2).$$

Let us define $f_1' := \lambda_1 f_1 + \delta/2$ and $f_2' := \lambda_2 f_2 + \delta/2$. This, together with $\lambda_1 f_1 = f_1' - \delta/2 \in \mathcal{D}_1$ if $\lambda_1 > 0$ or otherwise $f_1' \in \mathcal{L}^+$, and $\lambda_2 f_2 = f_2' - \delta/2 \in \mathcal{D}_2$ if $\lambda_2 > 0$ or otherwise $f_2' \in \mathcal{L}^+$, show according to Eq. (5) that $f_1' \in \mathcal{D}_1^+$ and $f_2' \in \mathcal{D}_2^+$. So, finally, we have $f \in \mathcal{D}_1^+ \cdot \mathcal{D}_2^+ = \mathcal{C}(\mathcal{D}_1^+ \cup \mathcal{D}_2^+)$. This proves that $(\mathcal{D}_1 \cdot \mathcal{D}_2)^+ = \mathcal{D}_1^+ \cdot \mathcal{D}_2^+$.

2. Note that $\mathcal{D}^+ \subseteq \mathcal{D}$, see again Section 2, so that

$$\epsilon_x(\mathcal{D}^+) = \tau(\sigma(\epsilon_x(\mathcal{D}^+))) \subseteq \tau(\sigma(\epsilon_x(\mathcal{D}))) =: (\epsilon_x(\mathcal{D}))^+.$$

Further

$$(\epsilon_x(\mathcal{D}))^+ \coloneqq \tau(\sigma(\epsilon_x(\mathcal{D}))) \coloneqq \{f \in \mathcal{L} : \sigma(\epsilon_x(\mathcal{D}))(f) > 0\} \cup \mathcal{L}^+,$$

where

$$\sigma(\epsilon_x(\mathcal{D}))(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in \mathcal{C}(\mathcal{D} \cap \mathcal{L}_x)\},$$

for every $f \in \mathcal{L}$. So, if $f \in (\epsilon_x(\mathcal{D}))^+$, then either $f \in \mathcal{L}^+$ in which case $f \in \epsilon_x(\mathcal{D}^+)$ or there is a $\delta > 0$ so that $f - \delta \in \mathcal{C}(\mathcal{D} \cap \mathcal{L}_x) = \mathsf{posi}\{\mathcal{L}^+ \cup (\mathcal{D} \cap \mathcal{L}_x)\}$. In the second case, if $f \notin \mathcal{L}^+$, $f - \delta = h + g$ where $h \in \mathcal{L}^+ \cup \{0\}$ and $g \in \mathcal{D} \cap \mathcal{L}_x$. Then, we have $f = h + g'$ where $g' \coloneqq g + \delta$ is still x- measurable. But, given the fact that $g = g' - \delta \in \mathcal{D} \cap \mathcal{L}_x$, from Eq. (5), we have $g' \in \mathcal{D}^+ \cap \mathcal{L}_x$ and therefore $f \in \epsilon_x(\mathcal{D}^+)$. Thus, we conclude that $(\epsilon_x(\mathcal{D}))^+ = \epsilon_x(\mathcal{D}^+)$.

Corollary 1. *Let $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}$ be coherent sets of gambles so that $\mathcal{D}_1 \cdot \mathcal{D}_2 \neq \mathcal{L}$ and $x \in Q$. Then*

1. $\sigma(\mathcal{D}_1 \cdot \mathcal{D}_2) = \sigma(\mathcal{D}_1) \cdot \sigma(\mathcal{D}_2)$,

2. $\sigma(\epsilon_x(\mathcal{D})) = \underline{e}_x(\sigma(\mathcal{D}))$.

Proof. These claims are immediate consequences of Theorem 2 and Theorem 3.

The weak homomorphism, as previously noticed, does not extend to a pair of not consistent coherent sets of gambles. Consider indeed two coherent sets of gambles $\mathcal{D}_1, \mathcal{D}_2$ such that $\mathcal{D}_1 \cdot \mathcal{D}_2 = \mathcal{L}$. The contradiction in $\mathcal{D}_1 \cdot \mathcal{D}_2$ can be generated by gambles contained in the boundary structure of the two sets not affecting their relative interiors. In such light, see Section 2, $\mathcal{D}_1^+ \cdot \mathcal{D}_2^+$ and $\sigma(\mathcal{D}_1) \cdot \sigma(\mathcal{D}_2) = \sigma(\mathcal{D}_1^+) \cdot \sigma(\mathcal{D}_2^+)$ can still represent *non*-contradictory information, even though $\mathcal{D}_1 \cdot \mathcal{D}_2$ does not. The following example further illustrates this point.

Example 4. Consider again the framework of Example 1. In particular, consider your and your friends' beliefs expressed through coherent sets of gambles as in the Example 3. Coherent lower previsions associated to $\mathcal{D}_1^+, \mathcal{D}_2, \mathcal{D}_3^+$ are respectively the following.

$$(\forall f \in \mathcal{L})\ \sigma(\mathcal{D}_1^+)(f) = \underline{P}_1(f) \coloneqq \min_{\omega' \in \{\text{a,b,f,g}\}} f(\omega'),$$

$$(\forall f \in \mathcal{L})\ \sigma(\mathcal{D}_2)(f) = \min\{P(f) : P \in \mathbb{P},\ P(\mathbb{I}_{\{\text{b,e,g}\}}) \geq P(\mathbb{I}_{\{\text{a,c,d,f}\}})\},$$

$$(\forall f \in \mathcal{L})\ \sigma(\mathcal{D}_3^+)(f) = P_3(f) \coloneqq f(\text{b}).$$

$\mathcal{D}_1^+, \mathcal{D}_2, \mathcal{D}_3^+$ are consistent. Therefore, thanks to Corollary 1, we have:

$$\sigma(\mathcal{D}_1^+) \cdot \sigma(\mathcal{D}_2) \cdot \sigma(\mathcal{D}_3^+) = \sigma(\mathcal{D}_1^+ \cdot \mathcal{D}_2 \cdot \mathcal{D}_3^+) = \sigma(\mathcal{D}_3^+),$$

$$\underline{e}_{motive}(\sigma(\mathcal{D}_1^+) \cdot \sigma(\mathcal{D}_2) \cdot \sigma(\mathcal{D}_3^+)) = \underline{e}_{motive}(\sigma(\mathcal{D}_3^+)) = \sigma(\epsilon_{motive}(\mathcal{D}_3^+)),$$

where

$$(\forall f \in \mathcal{L})\ \sigma(\epsilon_{motive}(\mathcal{D}_3^+))(f) = \min_{\omega' \in \{\text{b,e,g}\}} f(\omega').$$

However, consider the following slightly modifications of $\mathcal{D}_1^+$

$$\mathcal{D}_1' \coloneqq \mathcal{D}_1^+ \cup \{f \in \mathcal{L} : \min_{\omega' \in \{\text{a,b,f,g}\}} f(\omega') = 0, f(\text{c}) > 0\},$$

$$\mathcal{D}_1'' \coloneqq \mathcal{D}_1^+ \cup \{f \in \mathcal{L} : \min_{\omega' \in \{\text{a,b,f,g}\}} f(\omega') = 0, f(\text{d}) > 0\}.$$

They are coherent and not consistent ($\mathcal{D}_1' \cdot \mathcal{D}_1'' = \mathcal{L}$), because $0 = (g + h) \in \mathcal{E}(\mathcal{D}_1' \cup \mathcal{D}_1'')$, where $g(\omega') = h(\omega') = 0$ when $\omega' \in \{\text{a,b,e,f,g}\}$, $g(\text{c}) = 1 = -h(\text{c})$ and $h(\text{d}) = 1 = -g(\text{d})$.

However, by Eq. 5, we know that $(\mathcal{D}_1')^+ = (\mathcal{D}_1'')^+ = \mathcal{D}_1^+$. Hence, we have $\mathcal{L} = (\mathcal{D}_1' \cdot \mathcal{D}_1'')^+ \neq (\mathcal{D}_1')^+ \cdot (\mathcal{D}_1'')^+ = \mathcal{D}_1^+$. Moreover, $\sigma(\mathcal{D}_1') = \sigma(\mathcal{D}_1'') = \sigma(\mathcal{D}_1^+)$, see Section 2. Hence, $\sigma(\mathcal{L}) = \sigma(\mathcal{D}_1' \cdot \mathcal{D}_1'') \neq \sigma(\mathcal{D}_1') \cdot \sigma(\mathcal{D}_1'') = \sigma(\mathcal{D}_1^+)$.

5.1. Atoms

Analogously to what happens with coherent sets of gambles, the information order on lower previsions in $\underline{\Phi}$: $\underline{P}_1 \leq \underline{P}_2$ if and only if $\underline{P}_1 \cdot \underline{P}_2 = \underline{P}_2$, coincides with the usual partial order on lower previsions introduced in Section 2 restricted to $\underline{\Phi}$.

As the domain-free information algebra of coherent sets of gambles, also the domain-free information algebra of coherent lower previsions admits maximal elements with respect to this order. In particular, Lemma 8 in Appendix A permits to conclude that linear previsions are its atoms. Moreover, the latter information algebra is atomistic, as the following theorem shows.

In what follows, as usual, we denote with $At(\underline{\Phi}) := \mathbb{P}$, the set of all linear previsions, atoms of $\underline{\Phi}$, and with $At(\underline{P})$ the set of all linear previsions (atoms) dominating $\underline{P} \in \underline{\Phi}$:

$$At(\underline{P}) := \{P \in At(\underline{\Phi}) : \underline{P} \leq P\}.$$

Theorem 4. *Consider the set of lower previsions $\underline{\Phi}$. If $\underline{P} \in \underline{\mathbb{P}}$, then $At(\underline{P}) \neq \emptyset$ and*

$$\underline{P} = \inf At(\underline{P}).$$

This theorem is due to Walley, for a proof see [9, Theorem 3.3.3].

6. Set algebras

Following the reasoning of Section 3.5, we can show that

$$(P_Q(\Omega), Q, \vee, \perp, \cap, \emptyset, \Omega, \sigma),$$

where:

- $(Q, \vee, \perp)$ is a sub-q-separoid of $(T_\Omega, \vee, \perp)$;
- $P_Q(\Omega) := \{S \subseteq \Omega : \ \exists\, x \in Q,\ \sigma_x(S) = S\}$ is the set of subsets of Ω saturated with respect to Q, where σ_x is the saturation operator associated to $\mathcal{P}_x$ of Ω, $x \in Q$, defined in Eq.(9);
- $\sigma : P_Q(\Omega) \times Q \to P_Q(\Omega)$ is defined as $\sigma((S,x)) := \sigma_x(S)$ for every $S \in P_Q(\Omega)$ and $x \in Q$;

is a set algebra. In particular, we claim that this set algebra can be embedded into the information algebra of strictly desirable sets of gambles Φ^+, constructed assuming the same q-separoid of questions $(Q, \vee, \perp)$ considered for the set algebra $P_Q(\Omega)$. Indeed, for any set $S \in P_Q(\Omega)$, let us define

$$\mathcal{D}_S^+ := \{f \in \mathcal{L}(\Omega) : \inf_{\omega \in S} f(\omega) > 0\} \cup \mathcal{L}^+(\Omega). \tag{12}$$

If $S \neq \emptyset$, this is clearly a strictly desirable set of gambles,[10] otherwise it corresponds to $\mathcal{L}(\Omega)$. The next theorem shows that the map $f : S \mapsto \mathcal{D}_S^+$ is an information algebras homomorphism between $(P_Q(\Omega), Q, \vee, \perp, \cap, \emptyset, \Omega, \sigma)$ and $(\Phi^+, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$.

Theorem 5. *Let $S, T \in P_Q(\Omega)$ and $x \in Q$. Then*

1. $\mathcal{D}_S^+ \cdot \mathcal{D}_T^+ = \mathcal{D}_{S \cap T}^+$,
2. $\mathcal{D}_\emptyset^+ = \mathcal{L}(\Omega)$, $\mathcal{D}_\Omega^+ = \mathcal{L}^+(\Omega)$,
3. $\epsilon_x(\mathcal{D}_S^+) = \mathcal{D}_{\sigma_x(S)}^+$.

[10] Indeed, $\underline{P}(f) := \inf_S(f)$ for every $f \in \mathcal{L}$ with $S \neq \emptyset$ is a coherent lower prevision [9].

PROOF. 1. Note that $\mathcal{D}_S^+ = \mathcal{L}^+$ or $\mathcal{D}_T^+ = \mathcal{L}^+$ if and only if $S = \Omega$ or $T = \Omega$. Clearly in this case we have immediately the result. The same is true if $\mathcal{D}_S^+ = \mathcal{L}$ or $\mathcal{D}_T^+ = \mathcal{L}$, which is equivalent to having $S = \emptyset$ or $T = \emptyset$. Now suppose $\mathcal{D}_S^+, \mathcal{D}_T^+ \neq \mathcal{L}^+$ and $\mathcal{D}_S^+, \mathcal{D}_T^+ \neq \mathcal{L}$. If $S \cap T = \emptyset$, then $\mathcal{D}_{S\cap T}^+ = \mathcal{L}$. Consider $f \in \mathcal{D}_S^+ \setminus \mathcal{L}^+$ and $g \in \mathcal{D}_T^+ \setminus \mathcal{L}^+$. Since S and T are disjoint, we have $\tilde{f} \in \mathcal{D}_S^+$ and $\tilde{g} \in \mathcal{D}_T^+$, where $\tilde{f}$, $\tilde{g}$ are defined in the following way for every $\omega \in \Omega$:

$$\tilde{f}(\omega) := \begin{cases} f(\omega) & \text{for } \omega \in S, \\ -g(\omega) & \text{for } \omega \in T, \\ 0 & \text{for } \omega \in (S \cup T)^c, \end{cases} \qquad \tilde{g}(\omega) := \begin{cases} -f(\omega) & \text{for } \omega \in S, \\ g(\omega) & \text{for } \omega \in T, \\ 0 & \text{for } \omega \in (S \cup T)^c. \end{cases}$$

However, $\tilde{f} + \tilde{g} = 0 \in \mathcal{E}(\mathcal{D}_S^+ \cup \mathcal{D}_T^+)$, hence $\mathcal{D}_S^+ \cdot \mathcal{D}_T^+ := \mathcal{C}(\mathcal{D}_S^+ \cup \mathcal{D}_T^+) = \mathcal{L}(\Omega) = \mathcal{D}_{S\cap T}^+$. Assume then that $S \cap T \neq \emptyset$. If $S \cap T = \Omega$, we already have the result. So suppose $S \cap T \neq \emptyset, \Omega$. Note that $\mathcal{D}_S^+ \cup \mathcal{D}_T^+ \subseteq \mathcal{E}(\mathcal{D}_S^+ \cup \mathcal{D}_T^+) \subseteq \mathcal{D}_{S\cap T}^+$ so that $\mathcal{E}(\mathcal{D}_S^+ \cup \mathcal{D}_T^+)$ is coherent and $\mathcal{D}_S^+ \cdot \mathcal{D}_T^+ = \mathcal{E}(\mathcal{D}_S^+ \cup \mathcal{D}_T^+) \subseteq \mathcal{D}_{S\cap T}^+$. Consider then a gamble $f \in \mathcal{D}_{S\cap T}^+$. If $f \in \mathcal{L}^+$, clearly $f \in \mathcal{D}_S^+ \cdot \mathcal{D}_T^+$. Vice versa, suppose $f \in \mathcal{D}_{S\cap T}^+ \setminus \mathcal{L}^+$. Select a $\delta > 0$ and define the two gambles

$$f_1(\omega) := \begin{cases} 1/2 f(\omega) & \text{for } \omega \in (S \cap T), \\ \delta & \text{for } \omega \in S \setminus T, \\ f(\omega) - \delta & \text{for } \omega \in T \setminus S, \\ 1/2 f(\omega) & \text{for } \omega \in (S \cup T)^c, \end{cases} \qquad f_2(\omega) := \begin{cases} 1/2 f(\omega) & \text{for } \omega \in (S \cap T), \\ f(\omega) - \delta & \text{for } \omega \in S \setminus T, \\ \delta & \text{for } \omega \in T \setminus S, \\ 1/2 f(\omega) & \text{for } \omega \in (S \cup T)^c, \end{cases}$$

for every $\omega \in \Omega$. Then $f = f_1 + f_2$ and $f_1 \in \mathcal{D}_S^+$, $f_2 \in \mathcal{D}_T^+$. Therefore $f \in \mathcal{E}(\mathcal{D}_S^+ \cup \mathcal{D}_T^+) = \mathcal{D}_S^+ \cdot \mathcal{D}_T^+$. Hence $\mathcal{D}_S^+ \cdot \mathcal{D}_T^+ = \mathcal{D}_{S\cap T}^+$.

2. Both have been noted above.

3. If S is empty, then $\epsilon_x(\mathcal{D}_\emptyset^+) = \mathcal{L}(\Omega)$ so that item 3 holds in this case. Hence, assume $S \neq \emptyset$. Then we have that $\mathcal{D}_S^+$ is coherent, and therefore:

$$\epsilon_x(\mathcal{D}_S^+) := \mathcal{C}(\mathcal{D}_S^+ \cap \mathcal{L}_x) = \mathcal{E}(\mathcal{D}_S^+ \cap \mathcal{L}_x) := \mathsf{posi}(\mathcal{L}^+(\Omega) \cup (\mathcal{D}_S^+ \cap \mathcal{L}_x)).$$

Consider a gamble $f \in \mathcal{D}_S^+ \cap \mathcal{L}_x$. If $f \in \mathcal{L}^+(\Omega) \cap \mathcal{L}_x$ then clearly $f \in \mathcal{D}_{\sigma_x(S)}^+$. Otherwise, $\inf_S f > 0$ and f is x-measurable. If $\omega \equiv_x \omega'$ for some $\omega' \in S$ and $\omega \in \Omega$, then $f(\omega) = f(\omega')$. Therefore $\inf_{\sigma_x(S)} f = \inf_S f > 0$, hence $f \in \mathcal{D}_{\sigma_x(S)}^+$. Then $\mathcal{C}(\mathcal{D}_S^+ \cap \mathcal{L}_x) \subseteq \mathcal{C}(\mathcal{D}_{\sigma_x(S)}^+) = \mathcal{D}_{\sigma_x(S)}^+$. Conversely, consider a gamble $f \in \mathcal{D}_{\sigma_x(S)}^+$. $\mathcal{D}_{\sigma_x(S)}^+$ is a strictly desirable set of gambles. Hence, if $f \in \mathcal{D}_{\sigma_x(S)}^+$, $f \in \mathcal{L}^+(\Omega)$ or there is $\delta > 0$ such that $f - \delta \in \mathcal{D}_{\sigma_x(S)}^+ \setminus \mathcal{L}^+(\Omega)$. If $f \in \mathcal{L}^+(\Omega)$, then $f \in \epsilon_x(\mathcal{D}_S^+)$. Otherwise, let us define for every $\omega \in \Omega$, $g(\omega) := \inf_{\omega' \equiv_x \omega} f(\omega') - \delta$. If $\omega \in S$, then $g(\omega) > 0$ since $\inf_{\sigma_x(S)}(f - \delta) > 0$. So, we have $\inf_S g \geq 0$ and g is x-measurable. However, $\inf_S(g + \delta) = \inf_S g + \delta > 0$ hence $(g + \delta) \in \mathcal{D}_S^+ \cap \mathcal{L}_x$ and $f \geq g + \delta$. Therefore, $f \in \mathcal{E}(\mathcal{D}_S^+ \cap \mathcal{L}_x) = \mathcal{C}(\mathcal{D}_S^+ \cap \mathcal{L}_x) =: \epsilon_x(\mathcal{D}_S^+)$.

The map $f : S \mapsto \mathcal{D}_S^+$ is in particular one-to-one, indeed if $\mathcal{D}_S^+ = \mathcal{D}_T^+$, then $S = T$. So $(P_Q(\Omega), Q, \vee, \perp, \cap, \emptyset, \Omega, \sigma)$ is embedded into $(\Phi^+, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ and hence also into $(\Phi, Q, \vee, \perp, \cdot, \mathcal{L}, \mathcal{L}^+, \epsilon)$ and $(\underline{\Phi}, Q, \vee, \perp, \cdot, \sigma(\mathcal{L}), \sigma(\mathcal{L}^+), \underline{\epsilon})$.

7. Marginal problem

The *marginal problem* is the problem of establishing the compatibility of a number of marginal assessments with a global model. The problem has a long history in literature, see for example [1, 19, 20, 21, 22]. In particular in [1], it has been faced using tools of desirability theory. In our previous work [5], we gave an alternative avenue of the results found in [1] (in the unconditional case), using only properties of the even more general structures of information algebras. In [1] and in [5], the problem has been analyzed considering only marginal assessments about logically independent variables. Here, we further generalize the discussion. As in [5], we face the problem using properties of information algebras. For a comparison with the results found in [5], see Appendix D.

We model variables through the questions they represent, which in turn are modeled as partitions $\mathcal{P}_x$ of a possibility space Ω, with $x \in Q$ and $(Q, \vee, \perp)$ a sub-q-separoid of $(T_\Omega, \vee, \perp)$. Assessments on variables of interest are instead represented as coherent sets of gambles or coherent lower previsions respectively. In what follows therefore, we use the tools provided by the two information algebras $(\Phi(\Omega), Q, \vee, \perp, \cdot, \mathcal{L}(\Omega), \mathcal{L}^+(\Omega), \epsilon)$ and $(\underline{\Phi}(\Omega), Q, \vee, \perp, \cdot, \sigma(\mathcal{L}(\Omega)), \sigma(\mathcal{L}^+(\Omega)), \underline{\epsilon})$. In [5], we use a similar framework assuming only multivariate models for questions.

Given a family of coherent sets of gambles $\mathcal{D}_1, .., \mathcal{D}_n$ corresponding to assessments about variables represented by questions $x_1, ..., x_n \in Q$ respectively (i.e, $\mathcal{D}_i$ with support x_i, for every i), the goal is finding a join model, i.e., another coherent set of gambles $\mathcal{D}$, from which we can reproduce by extraction respect to x_i the information represented by $\mathcal{D}_i$, for every $i = 1, ..., n$ respectively. A similar reasoning is valid also for coherent lower previsions.

We start by recalling the necessary definitions of [1] and [5] adapted to this more general framework, then we state our main results.

Let us firstly consider the case of assessments modeled by coherent sets of gambles.

Definition 16 (Compatibility for coherent sets of gambles). A finite family of coherent sets of gambles $\mathcal{D}_1, \ldots, \mathcal{D}_n$, where $\mathcal{D}_i$ has support x_i for every $i = 1, \ldots, n$ respectively, is called *compatible*, or $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are called *compatible*, if and only if there is a coherent set of gambles $\mathcal{D}$ such that $\epsilon_{x_i}(\mathcal{D}) = \mathcal{D}_i$ for $i = 1, \ldots, n$.

A minimal requirement for a set of marginal assessments $\mathcal{D}_1, ..., \mathcal{D}_n$ to be compatible is that they are *consistent*.

Definition 17 (Consistency for coherent sets of gambles). A finite family of coherent sets of gambles $\mathcal{D}_1, \ldots, \mathcal{D}_n$ is *consistent*, or $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are *consistent*, if and only if $\mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_n \neq \mathcal{L}$.

Consistency is a necessary condition for a finite family of coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$ to be compatible.

Lemma 2. *Consider a finite family of coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$ having supports $x_1, .., x_n$ respectively. If they are compatible, they are consistent.*

Proof. Let us suppose that $\mathcal{D}_1, ..., \mathcal{D}_n$ with supports $x_1, ..., x_n$ respectively, are compatible. Then, there exists a coherent set of gambles $\mathcal{D}$ such that $\epsilon_{x_i}(\mathcal{D}) = \mathcal{D}_i$, for every i. By item (b) of Existential Quantifier axiom, we have also: $\mathcal{D} \geq \epsilon_{x_i}(\mathcal{D}) = \mathcal{D}_i$ for every $i = 1, ..., n$. From this fact, we can prove recursively that $\mathcal{D} \geq \mathcal{D}_1 \cdot ... \cdot \mathcal{D}_n$. Indeed, if $n = 1$ we already have the result. Otherwise, from $\mathcal{D} \geq \mathcal{D}_1$, we derive $\mathcal{D} = \mathcal{D} \cdot \mathcal{D}_2 \geq \mathcal{D}_1 \cdot \mathcal{D}_2$, by item 3 of Lemma 10 in Appendix B. This can be then repeated for every $\mathcal{D}_i$ with $i \leq n$, hence $\mathcal{D} \geq \mathcal{D}_1 \cdot ... \cdot \mathcal{D}_n$. Thus, $\mathcal{D}_1 \cdot ... \cdot \mathcal{D}_n \neq \mathcal{L}$.

Another necessary condition for a set of marginal assessments $\mathcal{D}_1, ..., \mathcal{D}_n$ to be compatible is that they are *pairwise compatible* [1, 5]. We translate this definition in our context.

Definition 18 (Pairwise compatibility for coherent sets of gambles). Two coherent sets of gambles $\mathcal{D}_i$ and $\mathcal{D}_j$, where $\mathcal{D}_i$ has support x_i and $\mathcal{D}_j$ support x_j respectively, are called *pairwise compatible* if and only if

$$\epsilon_{x_i}(\mathcal{D}_i \cdot \mathcal{D}_j) = \mathcal{D}_i,$$

$$\epsilon_{x_j}(\mathcal{D}_i \cdot \mathcal{D}_j) = \mathcal{D}_j.$$

Analogously, a finite family of coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$, where $\mathcal{D}_i$ has support x_i for every $i = 1, ..., n$ respectively, is pairwise compatible, or again $\mathcal{D}_1, ..., \mathcal{D}_n$ are pairwise compatible, if and only if pairs $\mathcal{D}_i, \mathcal{D}_j$ are pairwise compatible for every $i, j \in \{1, ...n\}$.

It means essentially that pieces of information are two-by-two compatible, i.e., given any pair $\mathcal{D}_i, \mathcal{D}_j \in \mathcal{D}_1, ...\mathcal{D}_n$ with supports x_i, x_j respectively, $\mathcal{D}_j$ does not give us any new information regarding question x_i that is not already contained in $\mathcal{D}_i$ and vice versa. As previously claimed, this is a necessary condition for compatibility.

Lemma 3. *Consider a finite family of coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$ having supports $x_1, .., x_n$ respectively. If they are compatible, they are pairwise compatible.*

Proof. Let us suppose that $\mathcal{D}_1, ..., \mathcal{D}_n$ with supports $x_1, ..., x_n$ respectively, are compatible. Then, there exists a coherent set $\mathcal{D}$ such that $\epsilon_{x_i}(\mathcal{D}) = \mathcal{D}_i$, for every i. As previously noticed in the proof of Lemma 2,

$$\mathcal{D} \geq \mathcal{D}_1 \cdot ... \cdot \mathcal{D}_n.$$

Hence,

$$\mathcal{D}_i = \epsilon_{x_i}(\mathcal{D}) \geq \epsilon_{x_i}(\mathcal{D}_1 \cdot ... \cdot \mathcal{D}_n) \geq \epsilon_{x_i}(\mathcal{D}_i \cdot \mathcal{D}_j) \geq \epsilon_{x_i}(\mathcal{D}_i) = \mathcal{D}_i,$$

for every $i, j = 1, ..., n$, thanks to Lemma 10 in Appendix B.

In the multivariate case, treated in [5], we gave another definition of pairwise compatibility. However, we can show that if we assume a multivariate model for questions, the definition of pairwise compatibility given in this section and the one given in [5] coincide, see Appendix D. In this latter case, it is also well-known that consistency and pairwise compatibility do not, in general, imply compatibility. A sufficient condition found to obtain the compatibility of a consistent and pairwise compatible family of coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$ is that their supports satisfy the *running intersection property* [5]. Here we consider a generalization of this property for q-separoids.

Definition 19 (Hypertree). Let $(\boldsymbol{Q}, \vee, \perp)$ be a q-separoid. A n-elements subset $\boldsymbol{S}$ of $\boldsymbol{Q}$ is called a *hypertree* if there is a numbering of its elements $\boldsymbol{S} = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$ such that for all $i = 1, ..., n-1$ there is an element $b(i) > i$ in the numbering so that

$$\boldsymbol{x}_i \perp \bigvee_{j=i+1}^{n} \boldsymbol{x}_j | \boldsymbol{x}_{b(i)}.$$

Now we can formulate the most important result of this section, i.e., proving that consistency and pairwise compatibility combined with a family of supports forming a hypertree guarantee global compatibility of a set of marginal assessments. This result is a generalisation of [1, Theorem 2, Proposition 1] and [5, Theorem 14] in our context.

Theorem 6. *Consider a finite family of consistent coherent sets of gambles $\mathcal{D}_1, \ldots, \mathcal{D}_n$ with $n > 1$ where $\mathcal{D}_i$ has support x_i for every $i = 1, ..., n$ respectively. If the set $S := \{x_1, \ldots, x_n\}$ forms a hypertree and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are pairwise compatible, then they are compatible and $\epsilon_{x_i}(\mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_n) = \mathcal{D}_i$ for $i = 1, \ldots, n$.*

Proof. Let us suppose without loss of generality that the numbering $\{x_1, ..., x_n\}$ guarantees:

$$(\forall i = 1, ..., n-1, \; \exists\, b(i) > i) \; x_i \perp \bigvee_{j=i+1}^{n} x_j | x_{b(i)}. \tag{13}$$

Now, let $y_i := x_{i+1} \vee \ldots \vee x_n$ for $i = 1, \ldots, n-1$ and $\mathcal{D} := \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_n$. Then, we have:

$$\epsilon_{y_1}(\mathcal{D}) = \epsilon_{y_1}(\mathcal{D}_1) \cdot \mathcal{D}_2 \cdot \ldots \cdot \mathcal{D}_n,$$

thanks to Existential Quantifier axiom, which can be used because $\mathcal{D}_2 \cdot ... \cdot \mathcal{D}_n$ has support y_1, see Lemma 9 in Appendix B.

Now, by Eq.(13) and Theorem 9 in Appendix B, we have:

$$\epsilon_{y_1}(\mathcal{D}_1) \cdot \mathcal{D}_2 \cdot \ldots \cdot \mathcal{D}_n = \epsilon_{y_1}(\epsilon_{x_{b(1)}}(\mathcal{D}_1)) \cdot \mathcal{D}_2 \cdot \ldots \cdot \mathcal{D}_n = \epsilon_{x_{b(1)}}(\mathcal{D}_1) \cdot \mathcal{D}_2 \cdot \ldots \cdot \mathcal{D}_n$$

where, to show the second equality we used Lemma 9 in Appendix B.

Now, by Existential Quantifier and pairwise compatibility, we have:

$$\begin{aligned}\epsilon_{x_{b(1)}}(\mathcal{D}_1) \cdot \mathcal{D}_2 \cdot \ldots \cdot \mathcal{D}_n &= \epsilon_{x_{b(1)}}(\mathcal{D}_1 \cdot \mathcal{D}_{b(1)}) \cdot \ldots \cdot \mathcal{D}_{b(1)-1} \cdot \mathcal{D}_{\min\{b(1)+1,n\}} \cdot \ldots \cdot \mathcal{D}_n \\ &= \mathcal{D}_2 \cdot \ldots \cdot \mathcal{D}_n.\end{aligned}$$

By induction on i, one shows exactly in the same way that

$$\epsilon_{y_i}(\mathcal{D}) = \mathcal{D}_{i+1} \cdot \ldots \cdot \mathcal{D}_n, \ \forall i = 1, ..., n-1.$$

So, we obtain $\epsilon_{y_{n-1}}(\mathcal{D}) \coloneqq \epsilon_{x_n}(\mathcal{D}) = \mathcal{D}_n$.

Now, we claim that $\epsilon_{x_i}(\mathcal{D}) = \epsilon_{x_i}(\epsilon_{x_{b(i)}}(\mathcal{D})) \cdot \mathcal{D}_i$ for every $i = 1, ..., n-1$. Since $x_{b(i)} \leq y_i$ and since $y_i \perp x_i | x_{b(i)}$, we have:

$$\begin{aligned}&\epsilon_{x_i}(\epsilon_{x_{b(i)}}(\mathcal{D})) \cdot \mathcal{D}_i = \mathcal{D}_i \cdot \epsilon_{x_i}(\epsilon_{x_{b(i)}}(\epsilon_{y_i}((\mathcal{D})))) = \mathcal{D}_i \cdot \epsilon_{x_i}(\epsilon_{y_i}(\mathcal{D})) = \\ &\mathcal{D}_i \cdot \epsilon_{x_i}(\mathcal{D}_{i+1} \cdot \ldots \cdot \mathcal{D}_n) = \epsilon_{x_i}(\mathcal{D}_i \cdot \ldots \cdot \mathcal{D}_n),\end{aligned}$$

by Lemma 9 in Appendix B, Theorem 9 in Appendix B and Existential Quantifier axiom. Now, if $i = 1$ we have the result, otherwise, for $i \geq 2$, we have

$$\epsilon_{x_i}(\mathcal{D}_i \cdot \ldots \cdot \mathcal{D}_n) = \epsilon_{x_i}(\epsilon_{y_{i-1}}(\mathcal{D})) = \epsilon_{x_i}(\mathcal{D}),$$

again by Lemma 9 in Appendix B. Then, by backward induction, based on the induction assumption $\epsilon_{x_j}(\mathcal{D}) = \mathcal{D}_j$ for $j > i$, and rooted in $\epsilon_{x_n}(\mathcal{D}) = \mathcal{D}_n$, for $i = n-1, \ldots 1$, we have by pairwise compatibility and Existential Quantifier:

$$\epsilon_{x_i}(\mathcal{D}) = \epsilon_{x_i}(\epsilon_{x_{b(i)}}(\mathcal{D})) \cdot \mathcal{D}_i = \epsilon_{x_i}(\mathcal{D}_{b(i)}) \cdot \mathcal{D}_i = \epsilon_{x_i}(\mathcal{D}_{b(i)} \cdot \mathcal{D}_i) = \mathcal{D}_i.$$

Notice that in all the previous results of this section we used only properties of domain-free information algebras. They are therefore results valid at this level of generality. So, in particular, they remain true also for the information algebra of coherent lower previsions.

Definition 20 (Consistency for coherent lower previsions). A finite family of coherent lower previsions $\underline{P}_1, \ldots, \underline{P}_n$, is *consistent*, or $\underline{P}_1, \ldots, \underline{P}_n$ are *consistent*, if and only if $\underline{P}_1 \cdot \ldots \cdot \underline{P}_n \neq \sigma(\mathcal{L})$.

Definition 21 (Compatibility for coherent lower previsions). A finite family of coherent lower previsions $\underline{P}_1, \ldots, \underline{P}_n$, where $\underline{P}_i$ has support x_i for every $i = 1, \ldots, n$ respectively, is called *compatible*, or $\underline{P}_1, \ldots, \underline{P}_n$ are called *compatible*, if and only if there is a coherent lower prevision $\underline{P}$ such that $\underline{e}_{x_i}(\underline{P}) = \underline{P}_i$ for $i = 1, \ldots, n$.

Definition 22 (Pairwise compatibility for coherent lower previsions). Two coherent lower previsions $\underline{P}_i$ and $\underline{P}_j$, where $\underline{P}_i$ has support x_i and $\underline{P}_j$ support x_j respectively, are called *pairwise compatible*, if and only if

$$\underline{e}_{x_i}(\underline{P}_i \cdot \underline{P}_j) = \underline{P}_i,$$

$$\underline{e}_{x_j}(\underline{P}_i \cdot \underline{P}_j) = \underline{P}_j.$$

Analogously, a finite family of coherent lower previsions $\underline{P}_1, ..., \underline{P}_n$, where $\underline{P}_i$ has support x_i for every $i = 1, ..., n$ respectively, is pairwise compatible, or again $\underline{P}_1, ..., \underline{P}_n$ are pairwise compatible, if and only if pairs $\underline{P}_i, \underline{P}_j$ are pairwise compatible for every $i, j \in \{1, ...n\}$.

We have therefore the following results that can be proven analogously to the ones for coherent sets of gambles.

Lemma 4. *Consider a finite family of coherent lower previsions* $\underline{P}_1, ..., \underline{P}_n$ *having supports* $x_1, .., x_n$ *respectively. If they are compatible, they are consistent.*

Lemma 5. *Consider a finite family of coherent lower previsions* $\underline{P}_1, ..., \underline{P}_n$ *having supports* $x_1, .., x_n$ *respectively. If they are compatible, they are pairwise compatible.*

Theorem 7. *Consider a finite family of consistent coherent lower previsions* $\underline{P}_1, \ldots, \underline{P}_n$ *with* $n > 1$*, where* $\underline{P}_i$ *has support* x_i *for every* $i = 1, ..., n$ *respectively. If the set* $S := \{x_1, \ldots, x_n\}$ *forms a hypertree and* $\underline{P}_1, \ldots, \underline{P}_n$ *are pairwise compatible, then they are compatible and* $\underline{e}_{x_i}(\underline{P}_1 \cdot \ldots \cdot \underline{P}_n) = \underline{P}_i$ *for* $i = 1, \ldots, n$.

Clearly there are connections between compatibility of a family of coherent sets of gambles and a family of coherent lower previsions, as the following result shows.

Lemma 6. *Given a family of compatible coherent sets of gambles* $\mathcal{D}_1, ..., \mathcal{D}_n$ *having supports* $x_1, .., x_n$ *respectively, the family of coherent lower previsions* $\underline{P}_1, ..., \underline{P}_n$ *where* $\underline{P}_i := \sigma(\mathcal{D}_i)$ *for every* $i = 1, ..., n$ *is also compatible. Vice versa, given a family of compatible coherent lower previsions* $\underline{P}_1, ..., \underline{P}_n$ *having supports* $x_1, .., x_n$ *respectively, the family of coherent (strictly desirable) sets of gambles* $\mathcal{D}_1^+, ..., \mathcal{D}_n^+$ *where* $\mathcal{D}_i^+ := \tau(\underline{P}_i)$ *for every* $i = 1, ..., n$ *is compatible.*

Proof. Consider firstly a family of compatible coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$ with supports $x_1, ..., x_n$. Then, there exists a coherent set $\mathcal{D}$ such that $\epsilon_{x_i}(\mathcal{D}) = \mathcal{D}_i$, for every $i = 1, ..., n$. If we define $\underline{P}_i := \sigma(\mathcal{D}_i)$ for every $i = 1, ..., n$ and $\underline{P} := \sigma(\mathcal{D})$, we have that $\underline{e}_{x_i}(\underline{P}_i) := \underline{e}_{x_i}(\sigma(\mathcal{D}_i)) = \sigma(\epsilon_{x_i}(\mathcal{D}_i)) = \sigma(\mathcal{D}_i) =: \underline{P}_i$ and $\underline{e}_{x_i}(\underline{P}) := \underline{e}_{x_i}(\sigma(\mathcal{D})) = \sigma(\epsilon_{x_i}(\mathcal{D})) = \sigma(\mathcal{D}_i) =: \underline{P}_i$ for every $i = 1, .., n$, thanks to Corollary 1.

Vice versa, let us consider a family of compatible coherent lower previsions $\underline{P}_1, ..., \underline{P}_n$ with supports $x_1, ..., x_n$. Then, there exists a coherent lower prevision $\underline{P}$ such that $\underline{e}_{x_i}(\underline{P}) = \underline{P}_i$, for every $i = 1, ..., n$. If we define $\mathcal{D}_i^+ := \tau(\underline{P}_i)$ for every $i = 1, ..., n$ and $\mathcal{D}^+ := \tau(\underline{P})$, we have that $\epsilon_{x_i}(\mathcal{D}_i^+) = \tau(\underline{e}_{x_i}(\underline{P}_i)) = \tau(\underline{P}_i) =: \mathcal{D}_i^+$ and $\epsilon_{x_i}(\mathcal{D}^+) = \tau(\underline{e}_{x_i}(\underline{P})) = \tau(\underline{P}_i) =: \mathcal{D}_i^+$ for every $i = 1, ..., n$, thanks to Theorem 2.

From now on, we refer only to the domain-free information algebra of coherent sets of gambles, since analogous considerations are valid for the information algebra of coherent lower previsions.

Compatibility of a family of pieces of information is not always desirable. As noticed before indeed, it is a kind of irrelevance or (conditional) independence condition. If a family of coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$ satisfies the hypotheses of Theorem 6, it means that $\epsilon_{x_i}(\mathcal{D}_1 \cdot ... \cdot \mathcal{D}_n) = \mathcal{D}_i$, i.e., the pieces of information $\mathcal{D}_j$ for $j \neq i$ give no new information regarding x_i not already contained in $\mathcal{D}_i$. If instead $\mathcal{D}_1, ..., \mathcal{D}_n$ are not compatible but consistent in the sense that $\mathcal{D}_1 \cdot ... \cdot \mathcal{D}_n \neq \mathcal{L}$, then $\epsilon_{x_i}(\mathcal{D}_1 \cdot ... \cdot \mathcal{D}_n) = \mathcal{D}_i \cdot \epsilon_{x_i}(\mathcal{D}_1 \cdot \mathcal{D}_{\{\max i-1,1\}} \cdot \mathcal{D}_{\{\min i+1,n\}} \cdot ...\mathcal{D}_n) \geq \mathcal{D}_i$. This means that $\mathcal{D}_j$ may provide additional information on x_i, for every $i, j \in \{1, ..., n\}$.

Example 5. Consider again the framework of Example 1,Example 2 and Example 3.

First of all, it is possible to notice that $sex, motive, hairstyle$, introduced through the correspondent partitions $\mathcal{P}_{sex}, \mathcal{P}_{motive}, \mathcal{P}_{hairstyle}$ in Example 2, are supports respectively of $\mathcal{D}_1^+, \mathcal{D}_2, \mathcal{D}_3^+$.

Moreover, we have:

$$\mathcal{P}_{sex} \perp (\mathcal{P}_{motive} \vee \mathcal{P}_{hairstyle}) | \mathcal{P}_{hairstyle}.$$

Therefore, {*sex, motive, hairstyle*} forms a hypertree with the numbering 1: sex, 2: motive, 3: hairstyle.

$\mathcal{D}_1^+, \mathcal{D}_2, \mathcal{D}_3^+$ are, in particular, consistent. Therefore, Theorem 6 tells us that they are compatible if they are pairwise compatible. Let us check their pairwise compatibility.

$$\epsilon_{sex}(\mathcal{D}_1^+ \cdot \mathcal{D}_2) = \mathcal{D}_1^+ \cdot \epsilon_{sex}(\mathcal{D}_2) = \mathcal{D}_1^+ \cdot \mathcal{L}^+ = \mathcal{D}_1^+$$

$$\epsilon_{motive}(\mathcal{D}_1^+ \cdot \mathcal{D}_2) = \epsilon_{motive}(\mathcal{D}_1^+) \cdot \mathcal{D}_2 = \mathcal{L}^+ \cdot \mathcal{D}_2 = \mathcal{D}_2$$

$$\epsilon_{sex}(\mathcal{D}_1^+ \cdot \mathcal{D}_3^+) = \mathcal{D}_1^+ \cdot \epsilon_{sex}(\mathcal{D}_3^+) = \mathcal{D}_1^+ \cdot \mathcal{D}_1^+ = \mathcal{D}_1^+$$

$$\epsilon_{hairstyle}(\mathcal{D}_1^+ \cdot \mathcal{D}_3^+) = \epsilon_{hairstyle}(\mathcal{D}_1^+) \cdot \mathcal{D}_3^+ = \mathcal{D}_1^+ \cdot \mathcal{D}_3^+ = \mathcal{D}_3^+$$

$$\begin{aligned}\epsilon_{motive}(\mathcal{D}_2 \cdot \mathcal{D}_3^+) = \mathcal{D}_2 \cdot \epsilon_{motive}(\mathcal{D}_3^+) = \\ = \mathcal{D}_2 \cdot \{f \in \mathcal{L} : \min_{\{\omega' \in \{\text{b,e,g}\}\}} f(\omega') > 0\} \cup \mathcal{L}^+ = \\ = \{f \in \mathcal{L} : \min_{\{\omega' \in \{\text{b,e,g}\}\}} f(\omega') > 0\} \cup \mathcal{L}^+ \supset \mathcal{D}_2.\end{aligned}$$

The last relation can be derived as follows. By definition, $\mathcal{D}_2 \subseteq \{f \in \mathcal{L} : \min_{\{\omega' \in \{\text{b,e,g}\}\}} f(\omega') > 0\} \cup \mathcal{L}^+$. However, consider the gamble f such that $f(\omega') = 1$ if $\omega' \in \{\text{b,e,g}\}$ and $f(\omega') = -2$ otherwise. Then clearly f is acceptable for an agent that thinks that surely the murderer is b,e or g, hence $f \in \{f \in \mathcal{L} : \min_{\{\omega' \in \{\text{b,e,g}\}\}} f(\omega') > 0\} \cup \mathcal{L}^+$. However, it is not acceptable for an agent who only thinks that it is more probable than the murderer is b,e or g than otherwise, because if he thinks for example that $P($ the murderer is b$) = 0.6$ and $P($ the murderer is a$) = 0.4$, then $P(f) = 0.6 * 1 - 0.4 * 2 = -0.2$. Hence, $f \notin \mathcal{D}_2$ and $\mathcal{D}_2 \subset \{f \in \mathcal{L} : \min_{\{\omega' \in \{\text{b,e,g}\}\}} f(\omega') > 0\} \cup \mathcal{L}^+$. So $\mathcal{D}_3^+$ gives more information than $\mathcal{D}_2$ about the motive of the crime.

However, you can notice that $\mathcal{D}_1^+, \mathcal{D}_2$, as well as $\mathcal{D}_1^+, \mathcal{D}_3^+$, are pairwise compatible. Hence compatible. In particular, thanks to Theorem 6, $\mathcal{D}' := \mathcal{D}_1^+ \cdot \mathcal{D}_2$ and $\mathcal{D}'' := \mathcal{D}_1^+ \cdot \mathcal{D}_3^+$ are, what we can call, *compatible joints*, i.e., sets from which we can derive by extraction the marginal information.

Thanks to Lemma 6 then, similar considerations can be made also for coherent lower previsions associated to $\mathcal{D}_1^+, \mathcal{D}_2, \mathcal{D}_3^+$.

We conclude this section highlighting that hypertrees are also particularly interesting to reduce the computational complexity of uncertainty calculi. These structures indeed allow us to construct algorithms of local computation.

As noticed in Section 3, the theory of information algebras was originally motivated by the desire of generalising local computation schema originally proposed for Bayesian networks. In their development however, these algebraic structures become models able to represent essential features of information in general, beyond particular uncertainty calculi. Nevertheless, the goal of generalizing local computation schema is maintained by information algebras managing information about multivariate models of partitions [2] or more general quasi-separoids of questions [4].

To see more details about local computation schema constructed from hypertrees at the level of generality presented in this paper, see [4].

8. Conclusions

This paper presents an extension of our works on information algebras induced by desirability [5, 6, 7]. In particular here, as in the previous works [6, 7], we consider questions modeled as partitions that do not necessarily form a multivariate model. In this general context, we introduce a domain-free information algebra of coherent lower previsions and we analyze its relations, as well as the ones of the domain-free information algebra of coherent sets of gambles, with an instance of set algebras. We left as an open problem to show if, as well-known for atomistic information algebras assuming a multivariate model for questions [2], the domain-free information algebra of coherent sets of gambles, as well as the one of coherent lower previsions, can be in turn embedded into an information algebra of sets of its atoms. We would like to study this problem in future works.

Finally, as an application of these results, we treat the marginal problem, already analyzed for multivariate models in [5]. Specifically, we have shown that we can obtain conditions for compatibility similar to the ones obtained in the multivariate case, with *hypertrees* in place of family of supports satisfying the *running intersection property.*

With respect to other future developments, it could be useful to provide an equivalent expression of the results shown in this paper for the *labeled view* of information algebras, better suited for computational purposes [2, 6].

Another important issue not addressed here is the one of conditioning. Treating this issue would also serve to analyse conditional independence, fundamental for any theory of information.

Conditional independence structures can also be used for efficient local computation schemes in imprecise probability. It would then be interesting to compare our potential results with those of Miranda and and Zaffalon in [1], in a conditional framework.

Appendix A. Complementary results - Desirability

In this appendix we recall some preliminary results on coherent sets of gambles and on lower previsions already presented in [5]. For proofs, we refer to [5].

The following lemma specifies some properties of lower previsions and their domains in different situations.

Lemma 7. *Given a non-empty set of gambles $\mathcal{K} \subseteq \mathcal{L}$, we have:*

1. $\mathcal{K} \subseteq dom(\sigma(\mathcal{K}))$,
2. *if* $0 \notin \mathcal{E}(\mathcal{K})$, *then* $\sigma(\mathcal{K})(f) \in \mathbb{R}$ *for every* $f \in \mathcal{K}$,
3. *if* $\mathcal{K} \in C$, *then* $dom(\sigma(\mathcal{K})) = \mathcal{L}$ *and* $\sigma(\mathcal{K})(f) \in \mathbb{R}$ *for every* $f \in \mathcal{L}$.

The following result highlights instead some further relations among linear previsions and general coherent lower previsions. It will be useful in Section 5.1.

Lemma 8. *Let $\underline{P}$ be an element of $\underline{\Phi}$ and P a linear prevision. Then $P \leq \underline{P}$ implies either $\underline{P} = P$ or $\underline{P} = \sigma(\mathcal{L})$.*

Finally, the following lemma establishes that the map σ commutes with natural extension under certain conditions. This result is fundamental for the whole Section 5.

Theorem 8. *Let $\mathcal{K} \subseteq \mathcal{L}$ be a non-empty set of gambles which satisfies the following two conditions:*

1. $0 \notin \mathcal{E}(\mathcal{K})$,
2. *for all* $f \in \mathcal{K} \setminus \mathcal{L}^+$ *there exists* $\delta > 0$ *such that* $f - \delta \in \mathcal{K}$,

then we have

$$\sigma(\mathcal{C}(\mathcal{K})) = \sigma(\mathcal{E}(\mathcal{K})) = \underline{E}^*(\sigma(\mathcal{K})) = \underline{E}(\sigma(\mathcal{K})).$$

Appendix B. Complementary results - Domain-free information algebras

The following lemma is a direct consequence of the axioms defining a domain-free information algebra.

Lemma 9. *Consider a domain-free information algebra $(\boldsymbol{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$. Given $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{Q}$ and $\boldsymbol{\phi}, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \boldsymbol{\Phi}$, we have:*

1. $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) = \mathbf{0}$ *if and only if* $\boldsymbol{\phi} = \mathbf{0}$,
2. $\boldsymbol{x}$ *is a support of* $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})$,
3. *if* $\boldsymbol{x} \leq \boldsymbol{y}$, *then* $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) \cdot \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})$,
4. *if* $\boldsymbol{x} \leq \boldsymbol{y}$, *then* $\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})$,
5. *if* $\boldsymbol{x} \leq \boldsymbol{y}$, *then* $\epsilon_{\boldsymbol{x}}(\epsilon_{\boldsymbol{y}}(\boldsymbol{\phi})) = \epsilon_{\boldsymbol{x}}(\boldsymbol{\phi})$,
6. *if* $\boldsymbol{x}$ *is a support of both* $\boldsymbol{\phi}_1$ *and* $\boldsymbol{\phi}_2$, *then it is a support of* $\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2$,

7. *if $\boldsymbol{x}$ is a support of $\boldsymbol{\phi}_1$ and $\boldsymbol{y}$ a support of $\boldsymbol{\phi}_2$, then $\boldsymbol{x} \vee \boldsymbol{y}$ is a support of $\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2$ and $\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2 = \boldsymbol{\epsilon}_{\boldsymbol{x}\vee\boldsymbol{y}}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}\vee\boldsymbol{y}}(\boldsymbol{\phi}_2)$.*

Proof. 1. $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{0}) = \boldsymbol{0}$ by item (a) of Existential Quantifier axiom. If instead $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) = 0$, then $\boldsymbol{\phi} = \boldsymbol{\phi} \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) = \boldsymbol{\phi} \cdot \boldsymbol{0} = \boldsymbol{0}$ by item (b) of Existential Quantifier axiom.

2. $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) \cdot \boldsymbol{1}) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{1}) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})$ by item (c) of Existential Quantifier axiom.

3. By item 2 of this lemma, we know that $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})$. Hence, by Support axiom, we have also $\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})$. Therefore, $\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi} \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi} \cdot \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}))) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}) \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi})$ by items (b) and (c) of Existential Quantifier axiom.

4. It directly follows from Support axiom and item 2 of this lemma, as previously noticed.

5. By item 3 of this lemma, we have
$$\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) \cdot \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}).$$
So, applying $\boldsymbol{\epsilon}_{\boldsymbol{x}}$ operator to both sides of the equation, we have
$$\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) \cdot \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})).$$
However, by item (b) and (c) of Existential Quantifier axiom, we have
$$\begin{aligned}&\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) \cdot \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi} \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})) = \\ &\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi} \cdot \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}) \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi} \cdot \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}).\end{aligned}$$

6. $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}_2)) = \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}_2) = \boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2$, from property (c) of Existential Quantifier axiom.

7. From Support axiom, we know $\boldsymbol{x} \vee \boldsymbol{y}$ is a support of both $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$. Hence, by item 6 of this lemma we have the result.

Item 7, in particular, justifies join-semilattices as basic structures for questions.

We show now the following theorem, which proves that the definition of domain-free information algebra provided in this paper coincides with the one given in [4]. In [4] indeed, a definition of domain-free information algebra is given where, in place of the Extraction axiom, we have the following:

- let us consider $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{Q}$. If $\boldsymbol{x}$ is a support of $\boldsymbol{\phi}$ and $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$,
$$\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi})),$$

and in place of item (c) of Existential Quantifier axiom, we have:

- let us consider $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \boldsymbol{\Phi}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{Q}$. If $\boldsymbol{x}$ is a support of $\boldsymbol{\phi}_1$, $\boldsymbol{y}$ of $\boldsymbol{\phi}_2$ and $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$,
$$\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2) = \boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_2).$$

The subsequent theorem proves therefore that the previous axioms can be respectively interchanged.[11]

Theorem 9. *Consider a domain-free information algebra $(\boldsymbol{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \boldsymbol{0}, \boldsymbol{1}, \boldsymbol{\epsilon})$.*

1. *For any $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{Q}$ such that $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) = \boldsymbol{\phi}$ and $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$, we have:*
$$\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi})).$$

[11] In [4] it is required also that $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) = \boldsymbol{0}$ if and only if $\boldsymbol{\phi} = \boldsymbol{0}$ in place of item (a) of Existential Quantifier axiom. However, this is satisfied also by our definition of domain-free information algebra, as Lemma 9 proves.

2. *For any* $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \boldsymbol{\Phi}$ *and* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{Q}$ *such that* $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}_1) = \boldsymbol{\phi}_1$, $\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}_2) = \boldsymbol{\phi}_2$ *and* $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$, *we have:*

$$\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2) = \boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_2).$$

Vice versa, if the structure $(\boldsymbol{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$ *satisfies all the axioms defining a domain-free information algebra except the Extraction axiom and item (c) of Existential Quantifier axiom that are replaced by item 1 and item 2 of this theorem respectively, then it is still a domain-free information algebra.*

Proof. Let us start by assuming that $(\boldsymbol{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$ is a domain-free information algebra and let us prove items 1 and 2. First of all, notice that from $\boldsymbol{x} \perp \boldsymbol{y} | \boldsymbol{z}$ it follows by the properties of a quasi-separoid that $\boldsymbol{x} \vee \boldsymbol{z} \perp \boldsymbol{y} \vee \boldsymbol{z} | \boldsymbol{z}$.

1. By the Extraction axiom, we have:

$$\boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi})).$$

By Lemma 9 in Appendix B, applying $\boldsymbol{\epsilon}_{\boldsymbol{y}}$ to both sides, we have $\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi})$ and $\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}))$. Hence $\boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}))$.

2. By the Existential Quantifier, the Extraction and the Support axioms, we have

$$\boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2) = \boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\phi}_2)) = \boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\phi}_2 = \boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1)) \cdot \boldsymbol{\phi}_2.$$

By Lemma 9 in Appendix B, the last combination equals $\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\phi}_2$. But then, again by the Existential Quantifier axiom and Lemma 9 in Appendix B, we have

$$\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2) = \boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2)) =$$
$$\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\phi}_2) = \boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_1) \cdot \boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi}_2).$$

This concludes the first part of the proof. Vice versa, suppose item 1 and item 2 of this theorem are satisfied in place of the Extraction axiom and item (c) of Existential Quantifier axiom respectively.

- Let us consider $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{Q}$ and $\boldsymbol{\phi} \in \boldsymbol{\Phi}$, such that $\boldsymbol{\epsilon}_{\boldsymbol{x}}(\boldsymbol{\phi}) = \boldsymbol{\phi}$ and $\boldsymbol{x} \vee \boldsymbol{z} \perp \boldsymbol{y} \vee \boldsymbol{z} | \boldsymbol{z}$. Then, by the properties of a quasi-separoid, we have also $\boldsymbol{x} \perp \boldsymbol{y} \vee \boldsymbol{z} | \boldsymbol{z}$. Hence, by item 1, we have

$$\boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_{\boldsymbol{y} \vee \boldsymbol{z}}(\boldsymbol{\epsilon}_{\boldsymbol{z}}(\boldsymbol{\phi})),$$

thus the Extraction axiom is satisfied.

- Item (c) of Existential Quantifier axiom follows from [4, Lemma 5.2].

This concludes the proof.

A similar result for a specific instance of domain-free information algebras can be found in [6].

As seen in Section 3.3, in a domain-free information algebra can be introduced a definition of partial order among pieces of information. It is easy to see that, in particular, the following properties are satisfied, see also [4].

Lemma 10. *Consider a domain-free information algebra* $(\boldsymbol{\Phi}, \boldsymbol{Q}, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$. *The following properties are valid.*

1. $\mathbf{1} \leq \boldsymbol{\phi} \leq \mathbf{0}$, *for every* $\boldsymbol{\phi} \in \boldsymbol{\Phi}$;

2. $\boldsymbol{\phi}, \boldsymbol{\psi} \leq \boldsymbol{\phi} \cdot \boldsymbol{\psi}$, *for every* $\boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{\Phi}$;

3. $\boldsymbol{\phi} \leq \boldsymbol{\psi}$ *implies* $\boldsymbol{\phi} \cdot \boldsymbol{\mu} \leq \boldsymbol{\psi} \cdot \boldsymbol{\mu}$ *for every* $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, $\boldsymbol{\mu} \in \boldsymbol{\Phi}$;

4. $\boldsymbol{\epsilon_x}(\boldsymbol{\phi}) \leq \boldsymbol{\phi}$ *for every* $\boldsymbol{x} \in \boldsymbol{Q},\ \boldsymbol{\phi} \in \boldsymbol{\Phi}$*;*

5. $\boldsymbol{\phi} \leq \boldsymbol{\psi}$ *implies* $\boldsymbol{\epsilon_x}(\boldsymbol{\phi}) \leq \boldsymbol{\epsilon_x}(\boldsymbol{\psi})$ *for every* $\boldsymbol{x} \in \boldsymbol{Q},\ \boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{\Phi}$*;*

6. $\boldsymbol{x} \leq \boldsymbol{y}$ *implies* $\boldsymbol{\epsilon_x}(\boldsymbol{\phi}) \leq \boldsymbol{\epsilon_y}(\boldsymbol{\phi})$*, for every* $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{Q},\ \boldsymbol{\phi} \in \boldsymbol{\Phi}$*.*

If a domain-free information algebra admits the presence of atoms, the following properties follow directly, see also [4]. In [4] in particular, the proof of these properties is given in the context of labeled information algebras. Here therefore we report the proof for domain-free ones.

Lemma 11. *Consider a domain-free information algebra* $(\boldsymbol{\Phi}, Q, \vee, \perp, \cdot, \mathbf{0}, \mathbf{1}, \boldsymbol{\epsilon})$*.*

1. If $\boldsymbol{\alpha} \in \boldsymbol{\Phi}$ *is an atom and* $\boldsymbol{\phi} \in \boldsymbol{\Phi}$*, then* $\boldsymbol{\phi} \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha}$ *or* $\boldsymbol{\phi} \cdot \boldsymbol{\alpha} = \mathbf{0}$*;*

2. If $\boldsymbol{\alpha} \in \boldsymbol{\Phi}$ *is an atom and* $\boldsymbol{\phi} \in \boldsymbol{\Phi}$*, then* $\boldsymbol{\phi} \leq \boldsymbol{\alpha}$ *or* $\boldsymbol{\alpha} \cdot \boldsymbol{\phi} = \mathbf{0}$*;*

3. If $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{\Phi}$ *are atoms, either* $\boldsymbol{\alpha} = \boldsymbol{\beta}$ *or* $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \mathbf{0}$*.*

PROOF. We show only the first item because the others follow directly from it. So, suppose that $\boldsymbol{\alpha}$ is an atom of $\boldsymbol{\Phi}$ and consider $\boldsymbol{\phi} \in \boldsymbol{\Phi}$. Then $\boldsymbol{\alpha} \cdot \boldsymbol{\phi} = \boldsymbol{\psi}$ for some $\boldsymbol{\psi} \in \boldsymbol{\Phi}$. Then, $\boldsymbol{\alpha} \leq \boldsymbol{\psi}$. Hence, $\boldsymbol{\psi} = \boldsymbol{\alpha}$ or $\boldsymbol{\psi} = \mathbf{0}$.

Appendix C. Multivariate model

Multivariate models provide an important instance of partitive models. In this case we suppose that questions of interest regard possible values of a set of logically independent variables. We consider therefore a universal set of a specific form:

$$U = \bigtimes_{i \in I} U_i,$$

where I is an index set and U_i is the set of possible values of a variable X_i, for every $i \in I$.[12] We may think of elements u of U as maps $u : I \to U$ such that $u(i) \in U_i$. In this context, questions regard values of subsets of variables, therefore we can model them as equivalence relations $\equiv_S$ where $S \subseteq I$ and $u \equiv_S u'$ if and only if u coincides with u' on S. A question about a set of variables $S \subseteq I$ has clearly the Cartesian product $\bigtimes_{i \in S} U_i$ as its set of possible answers.

If we use the same identification used in Section 3.4 between partitions/ equivalence relations and indexes, we have that $(\mathcal{P}(I), \vee, \perp)$ is a q-separoid, where $\mathcal{P}(I)$ is the power-set of I and $\vee$ and $\perp$ are respectively the join and the conditional independence relation among partitions defined in Section 3.4, see [4]. In particular, $(\mathcal{P}(I), \vee)$ induces a lattice $(\mathcal{P}(I), \vee, \wedge)$, where $S \vee T = S \cup T$ and $S \wedge T \coloneqq S \cap T$, for every $S, T \in \mathcal{P}(I)$, see again [4]. In this case we have also that $S \perp T | R$ if and only if $(S \cup R) \cap (T \cup R) = R$ [4]. These properties imply that in a domain-free information algebra having a multivariate model as model for questions, it is valid also the following axiom [4, Lemma 5.3]:

Commutative Extraction: For any $S, T \in \mathcal{P}(I),\ \boldsymbol{\phi} \in \boldsymbol{\Phi}$,

$$\boldsymbol{\epsilon}_T(\boldsymbol{\epsilon}_S(\boldsymbol{\phi})) = \boldsymbol{\epsilon}_{S \cap T}(\boldsymbol{\phi}) = \boldsymbol{\epsilon}_S(\boldsymbol{\epsilon}_T(\boldsymbol{\phi})). \tag{C.1}$$

The resulting axiomatic framework obtained incorporating this latter axiom essentially corresponds to the one introduced in [2] and used in our initial work [5].

[12] If needed, we assume the axiom of choice.

Appendix D. Marginal problem - Multivariate model

In [5] we treat a version of the marginal problem where we consider assessments about sets of logically independent variables modeled as coherent sets of gambles or coherent lower previsions.

In this appendix, we will compare the definitions and the results established in [5] with the ones found in Section 7 of this paper. We limit ourselves to consider assessments represented as coherent sets of gambles. Analogous considerations can be made also for coherent lower previsions.

Let us consider the framework of Section 7. Assume however that $\Omega = \bigtimes_{i \in I} \Omega_i$, where Ω_i is the set of values of a variable X_i, for every i in some index set I. Assume moreover $\{X_i\}_{i \in I}$ as the variables of interest, hence $(Q, \vee, \perp) = (\mathcal{P}(I), \vee, \perp)$. We recall from the previous Appendix C that in this case the join-semilattice $(\mathcal{P}(I), \vee)$ forms, in particular, a lattice $(\mathcal{P}(I), \vee, \wedge)$ where $S \vee T = S \cup T$ and $S \wedge T := S \cap T$, $S \perp T | R \iff (S \cup R) \cap (T \cup R) = R$ for every $S, T, R \in \mathcal{P}(I)$ and the derived information algebra $\Phi(\Omega)$ satisfies the *commutative extraction* axiom.

We are now ready to compare the definitions and the results found in Section 7 with the ones found in [5]. In particular, we can observe that the definitions of consistency and compatibility for coherent sets of gambles given in [5] are analogous to the ones given in Section 7 of this paper. The definition of pairwise compatibility instead is a bit different.

Definition 23. Pairwise compatibility for coherent sets of gambles - multivariate model. Two coherent sets of gambles $\mathcal{D}_i$ and $\mathcal{D}_j$, where $\mathcal{D}_i$ has support $S_i \subseteq I$ and $\mathcal{D}_j$ support $S_j \subseteq I$, are called *pairwise compatible* if and only if

$$\epsilon_{S_i \cap S_j}(\mathcal{D}_i) = \epsilon_{S_i \cap S_j}(\mathcal{D}_j). \tag{D.1}$$

Analogously, a finite family of coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$, where $\mathcal{D}_i$ has support $S_i \subseteq I$ for every $i = 1, ..., n$ respectively, is pairwise compatible, or again $\mathcal{D}_1, ..., \mathcal{D}_n$ are pairwise compatible, if and only if pairs $\mathcal{D}_i, \mathcal{D}_j$ are pairwise compatible for every $i, j \in \{1, ...n\}$.

The two definitions, however, can be easily reconciled.

Theorem 10. *Let us consider $\Omega = \bigtimes_{i \in I} \Omega_i$, where Ω_i is the set of values of a variable X_i for every $i \in I$, and the domain-free information algebra $(\Phi(\Omega), \mathcal{P}(I), \vee, \perp, \cdot, \mathcal{L}(\Omega), \mathcal{L}^+(\Omega), \epsilon)$. In this context, consider a family of sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n \in C(\Omega) \subseteq \Phi(\Omega)$ having supports $S_1, ..., S_n \in \mathcal{P}(I)$ respectively. Then*

$$(\forall i, j)\ \epsilon_{S_i \cap S_j}(\mathcal{D}_j) = \epsilon_{S_i \cap S_j}(\mathcal{D}_i) \tag{D.2}$$

if and only if

$$(\forall i, j)\ \epsilon_{S_i}(\mathcal{D}_i \cdot \mathcal{D}_j) = \mathcal{D}_i. \tag{D.3}$$

Proof. Let us consider a family of coherent sets of gambles $\mathcal{D}_1, ..., \mathcal{D}_n$ having supports $S_1, .., S_n$ respectively and satisfying Eq. D.2. Then, we have

$$\begin{aligned} &\epsilon_{S_i}(\mathcal{D}_i \cdot \mathcal{D}_j) = \mathcal{D}_i \cdot \epsilon_{S_i}(\mathcal{D}_j) = \mathcal{D}_i \cdot \epsilon_{S_i}(\epsilon_{S_j}(\mathcal{D}_j)) = \\ &\mathcal{D}_i \cdot \epsilon_{S_i \cap S_j}(\mathcal{D}_j) = \mathcal{D}_i \cdot \epsilon_{S_i \cap S_j}(\mathcal{D}_i) = \mathcal{D}_i, \end{aligned}$$

for every $i, j \in \{1, ..., n\}$, by Existential Quantifier axiom and axiom (C.1).

Vice versa, if they satisfy Eq. D.3, we have

$$\epsilon_{S_i \cap S_j}(\mathcal{D}_j) = \epsilon_{S_i \cap S_j}(\epsilon_{S_j}(\mathcal{D}_i \cdot \mathcal{D}_j)) = \epsilon_{S_i \cap S_j}(\mathcal{D}_i \cdot \mathcal{D}_j),$$

thanks to Lemma 9 in Appendix B. Analogously, we have:

$$\epsilon_{S_i \cap S_j}(\mathcal{D}_i) = \epsilon_{S_i \cap S_j}(\epsilon_{S_i}(\mathcal{D}_i \cdot \mathcal{D}_j)) = \epsilon_{S_i \cap S_j}(\mathcal{D}_i \cdot \mathcal{D}_j).$$

Hence, we have the result.

In [5] is then shown a result similar to Theorem 6, where, in place of requiring that the family of supports of a set of marginal assessments form a hypertree, it is required it satisfies the *running intersection property* (RIP):

RIP A n-elements subset S of $\mathcal{P}(I)$ satisfies the *running intersection property* (RIP) if there is a numbering of its elements $S = \{S_1, ..., S_n\}$ such that for all $i = 1, ..., n-1$ there is an element $b(i) > i$ in the numbering so that

$$S_i \cap S_{b(i)} = S_i \cap (\cup_{j=i+1}^{n} S_j).$$

Notice however that the two properties convey the same idea: for every i only the part relative to question $b(i)$ of an information relative to $(i+1) \vee ... \vee n$ is relevant. In fact, it can easily been shown that in this context if a set of supports $\{S_1, ..., S_n\}$ such that $S_i \in \mathcal{P}(I)$ for every i forms a hypertree, it satisfy the running intersection property and vice versa.[13]

Given the discussion stated here, it is then clear how the results found in [5] are a special case of the ones found in Section 7 for multivariate models of questions.

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[13] It directly follows from: $S \perp T | R \iff (S \cup R) \cap (T \cup R) = R$, for every $S, T, R \subseteq I$.