Computation of Parameter Dependent Robust Invariant Sets for LPV Models with Guaranteed Performance ⋆

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Abstract

This paper presents an iterative algorithm to compute a *Robust Control Invariant* (RCI) set, along with an invariance-inducing control law, for *Linear Parameter-Varying* (LPV) systems. As real-time measurements of the scheduling parameters are typically available, we allow the RCI set description and the invariance-inducing controller to be scheduling parameter dependent. Thus, the considered formulation leads to parameter-dependent conditions for the set invariance, which are replaced by sufficient *Linear Matrix Inequalities* (LMIs) via Polya's relaxation. These LMI conditions are then combined with a novel volume maximization approach in a *Semidefinite Programming* (SDP) problem, which aims at computing the desirably large RCI set. Besides ensuring invariance, it is also possible to guarantee performance within the RCI set by imposing a chosen quadratic performance level as an additional constraint in the SDP problem. Using numerical examples, we show that the presented iterative algorithm can generate RCI sets for large parameter variations where commonly used robust approaches fail.

Key words: Linear matrix inequality, Invariant set, Semi-definite program, Linear parameter-varying systems.

1 Introduction

RCI set is a set of system states where a feasible control input always exists, which restricts the future states within the set in the presence of disturbances. These sets have become an essential tool for controller synthesis and stability analysis of linear and nonlinear systems Blanchini & Miani (2015), Raković & Baric (2010), Fiacchini et al. (2010), Bravo et al. (2005).

When computing RCI sets for LPV systems, a common practice is to treat the scheduling parameters as bounded uncertainties Hanema et al. (2020), Miani & Savorgnan (2005), Gupta et al. (2019), Nguyen et al. (2015). Moreover, the invariance inducing control laws are typically assumed to be only state-dependent, without exploiting the observed scheduling parameter information. In this way, the obtained RCI sets can be potentially conservative and, in the worst case, even empty. Thus, to exploit the information on the scheduling parameters, we propose a new algorithm to compute scheduling parameter-dependent RCI sets and invariance inducing control laws for LPV systems. In this paper, such sets are termed as parameter-dependent RCI (PD-RCI) sets and parameter-dependent control laws (PDCLs), respectively. The advantages of using a PDCL and PD-RCI set are:

- *PDCL:* these control laws can stabilize LPV systems that may not be stabilizable by treating the parameters as unknown bounded uncertainties Blanchini et al. (2007). Moreover, when computing the RCI sets, keeping PDCL as an optimization variable provides extra degrees of freedom. We remark that a similar construction was proposed in a robust framework in Blanco et al. (2010), Gupta et al. (2019), Liu et al. (2019).
- *PD-RCI sets*: Scheduling parameters affect the system's time evolution, and thus the set of states for which invariance can be achieved. Therefore, only considering fixed (or parameter-independent) RCI set description for all scheduling parameters could be restrictive and may lead to conservative (volume-wise) sets. This restrictiveness motivates us to allow the RCI set description to be parameter-dependent.

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This paper presents an iterative algorithm to compute a PD-RCI set of desirably large volume and PDCL for the LPV systems. We also present a method to compute PD-RCI sets within which a desired quadratic performance can be guaranteed. The representational complexity of the PD-RCI sets can be predefined. The related LMI conditions for invariance are derived by employing Finsler's lemma and Polya's relaxation. These conditions are constructed to ensure invariance for all future (unknown) values of the scheduling parameters. In order to obtain an RCI set with a desirably large volume, we present a volume maximization heuristic based on the theory of Monte-Carlo integration and its convex relaxations.

The paper is organized as follows: In Section 3, we formalize the problem of computing PD-RCI set and PDCL. Sufficient parameter-dependent conditions for invariance and performance are derived in Section 4, and corresponding LMI conditions in Section 5. Using these conditions, an iterative algorithm to compute desirably large RCI sets is proposed in Section 6. Two case studies are reported in Section 7

Notation: We use $\mathbb{D}^n_+ \in \mathbb{R}^{n\times n}$ to denote the set of all diagonal matrices with positive diagonal entries. I and e_i represent the identity matrix and its i-th column, and vector of ones is denoted by $\mathbf{1}$, with dimension defined by the context. $X \succ 0\,(\succeq 0)$ denotes a positive (semi) definite matrix X. For compactness, in the text $*$'s will represent matrix's entries that are uniquely identifiable from symmetry, and for some square matrix X, $\mathsf{He}(X) = X + X^T$. We use X^k and x^k to represent a matrix and a vector of appropriate dimension indexed with 'k'. Let $L(X^k, Y^l, \bar{\Theta}, \Theta)$ be a matrix-valued function, where X^k and Y^l represent all matrices indexed with 'k' and 'l', and $\bar{\Theta}, \Theta$ are some arbitrary matrices. We use $\boldsymbol{L}^{k,l}(\bar{\Theta}, \Theta) = L(X^k, Y^l, \bar{\Theta}, \Theta)$, $\boldsymbol{L}^{l,k}(\bar{\Theta}, \Theta) = L(X^l, Y^k, \bar{\Theta}, \Theta)$ and $\boldsymbol{L}^{k,k}(\bar{\Theta}, \Theta) = L(X^k, Y^k, \bar{\Theta}, \Theta)$.

2 Preliminaries

We recall two existing results which will be used in the paper.

Lemma 1 (Finsler's Lemma) *Let $\xi \in \Xi \subseteq \mathbb{R}^{N_\xi}$, $\Phi : \Xi \to \mathbb{R}^{n\times n}$ and $\Delta : \Xi \to \mathbb{R}^{m\times n}$. Then the following statements are equivalent (Ishihara et al. 2017):*

i. For each $\xi \in \Xi$, $y^T\Phi(\xi)y \succ 0,\ \forall \Delta(\xi)y = 0,\ y \neq 0$.
ii. For each $\xi \in \Xi$, $\exists \Psi \in \mathbb{R}^{n\times m}$ such that
$$\Phi(\xi) + \Psi(\xi)\Delta(\xi) + \Delta(\xi)^T\Psi(\xi)^T \succ 0.$$

Lemma 2 (Linearization Lemma) *Let $\mathcal{L} \in \mathbb{R}^{m\times n}$ be any arbitrary matrix and $\mathcal{M} \in \mathbb{R}^{m\times m}$ be a positive definite matrix. The following relation always holds for any arbitrary matrix $\mathcal{Y} \in \mathbb{R}^{m\times n}$ (Gupta et al. 2019)*

$$\mathcal{L}^T\mathcal{M}^{-1}\mathcal{L} \succeq \mathcal{L}^T\mathcal{Y} + \mathcal{Y}^T\mathcal{L} - \mathcal{Y}^T\mathcal{M}\mathcal{Y} \tag{1}$$

The result can be easily verified by adding a residual term $(\mathcal{L} - \mathcal{M}\mathcal{Y})^T\mathcal{M}^{-1}(\mathcal{L} - \mathcal{M}\mathcal{Y})$ on the *r.h.s* of (1). This lemma is utilized to resolve the *non-linear* matrix inequalities, i.e., the non-linear term on the l.h.s. of (1) can be replaced with *linear* (in variables $\mathcal{L}, \mathcal{M}$) matrix term on r.h.s., by appropriate choice of $\mathcal{Y}$.

Remark 1 *[Successive linearization] Though $\mathcal{Y}$ can be any arbitrary matrix of compatible dimension, in order to reduce conservatism due to linearization, we suggest successive linearization approach. An appropriate choice would be to select $\mathcal{Y} = \mathcal{M}_0^{-1}\mathcal{L}_0$, where $\mathcal{M}_0$, $\mathcal{L}_0$ are the values of $\mathcal{M}$, $\mathcal{L}$ obtained from previous iteration. Thus the nonlinearity can be resolved iteratively in which the residual term shrinks with each iteration. Notice that (1) holds with equality if $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{M} = \mathcal{M}_0$, this will be a key property towards proving recursive feasibility of our iterative schemes proposed in this paper.*

3 Problem Statement

Let us consider a discrete-time polytopic LPV system:

$$x(t+1) = \mathcal{A}(\xi(t))x(t) + \mathcal{B}(\xi(t))u(t) + \mathcal{E}(\xi(t))w(t), \tag{2}$$
$$z(t) = \mathcal{C}(\xi(t))x(t) + \mathcal{D}(\xi(t))u(t), \tag{3}$$

where time index $t \in \mathbb{Z}_+$, $x(t) \in \mathbb{R}^{n_x}$ and $z(t) \in \mathbb{R}^{n_z}$ are the current state and the output vectors, $x(t+1)$ is the successor state, and $u(t) \in \mathbb{R}^{n_u}$ and $w(t) \in \mathbb{R}^{n_w}$ are the control and the disturbance input vectors, respectively. The system matrices $\mathcal{A}(\xi(t))$, $\mathcal{B}(\xi(t))$, $\mathcal{C}(\xi(t))$, $\mathcal{D}(\xi(t))$ and $\mathcal{E}(\xi(t))$ depend on the time-varying scheduling parameter $\xi(t)$, which takes value in unit simplex,

$$\Xi = \left\{ \xi \in \mathbb{R}^{N_\xi} : \sum_{k=1}^{N_\xi} \xi_k = 1,\quad \xi_k \geq 0 \right\}. \tag{4}$$

It is assumed that the current value of $\xi(t)$ is always available. The polytopic system matrices are given by

$$\begin{bmatrix} \mathcal{A}(\xi(t))\mathcal{B}(\xi(t))\mathcal{E}(\xi(t)) \\ \mathcal{C}(\xi(t))\mathcal{D}(\xi(t)) \quad 0 \end{bmatrix} = \sum_{k=1}^{N_\xi} \xi_k(t) \begin{bmatrix} A^k & B^k & E^k \\ C^k & D^k & 0 \end{bmatrix}, \tag{5}$$

with A^k, B^k, C^k, D^k, E^k real matrices of compatible dimensions. The system is subjected to the following polytopic state/input constraints and bounded disturbance:

$$\begin{aligned} \boldsymbol{\mathcal{X}}_u &= \{(x,u) : H_x x(t) + H_u u(t) \leq \mathbf{1}\}, \\ \boldsymbol{\mathcal{W}} &= \{w : -\mathbf{1} \leq G w(t) \leq \mathbf{1}\}. \end{aligned} \tag{6}$$

where $H_x \in \mathbb{R}^{n_h\times n_x}$, $H_u \in \mathbb{R}^{n_h\times n_u}$ and $G \in \mathbb{R}^{n_g\times n_w}$ are given matrices. In this paper, we want to compute a 0-symmetric PD-RCI set with a predefined complexity n_p described as

$$\boldsymbol{\mathcal{S}}(\xi(t)) = \left\{ x \in \mathbb{R}^{n_x} : -\mathbf{1} \leq \mathcal{P}(\xi(t))W^{-1}x(t) \leq \mathbf{1} \right\}, \tag{7}$$

where $\mathcal{P}(\xi(t)) \triangleq \sum_{k=1}^{N_\xi} \xi_k(t) P^k$, $P^k \in \mathbb{R}^{n_p \times n_x}$ and $W \in \mathbb{R}^{n_x \times n_x}$. The presented parameterization of PD-RCI set will be justified when we formalize the problem. Note that, if $P^k = P$ for all $k = 1, \ldots, N_\xi$, then $\mathcal{P}(\xi(t)) = P$, which is similar to the (parameter-independent) RCI set description considered in Gupta & Falcone (2019), Liu et al. (2019). In order to have a non-empty and bounded set $\boldsymbol{\mathcal{S}}(\xi(t))$, the matrix W should be invertible and $\mathsf{Rank}(\mathcal{P}(\xi(t))) = n_x,\ \forall \xi \in \boldsymbol{\Xi}$. This will be later guaranteed by proper LMI conditions. Furthermore, invariance in the set $\boldsymbol{\mathcal{S}}(\xi(t))$ is achieved with a PDCL, which is not known a priori and expressed as

$$u(t) = \mathcal{K}(\xi(t)) x(t), \tag{8}$$

where $\mathcal{K}(\xi(t)) \triangleq \sum_{k=1}^{N_\xi} \xi_k(t) K^k$ and $K^k \in \mathbb{R}^{n_u \times n_x}$. The closed-loop representation of the system (2) and (3) with the controller (8) can be written as

$$x(t{+}1) = \overbrace{(\mathcal{A}(\xi(t)) + \mathcal{B}(\xi(t))\mathcal{K}(\xi(t)))}^{\mathcal{A}_{\mathcal{K}}(\xi(t))} x(t) + \mathcal{E}(\xi(t)) w(t), \tag{9}$$

$$z(t) = \overbrace{(\mathcal{C}(\xi(t)) + \mathcal{D}(\xi(t))\mathcal{K}(\xi(t)))}^{\mathcal{C}_{\mathcal{K}}(\xi(t))} x(t). \tag{10}$$

To the best of authors' knowledge, there is no related work which computes the described PD-RCI set. Thus we first formalize the definition of the set by adapting the standard definition of the RCI set to the LPV setting in the sequel.

Definition 1 *We say a set $\boldsymbol{\mathcal{S}}(\xi(t))$ is a PD-RCI set if for any given $\xi(t) \in \boldsymbol{\Xi}$ and each $x(t) \in \boldsymbol{\mathcal{S}}(\xi(t))$*

$$\mathcal{A}_{\mathcal{K}}(\xi(t)) \boldsymbol{\mathcal{S}}(\xi(t)) \oplus \mathcal{E}(\xi(t)) \boldsymbol{\mathcal{W}} \subseteq \boldsymbol{\mathcal{S}}(\xi(t{+}1)), \forall \xi(t{+}1) \in \boldsymbol{\Xi} \tag{11}$$

Condition (11) should be satisfied for $\forall \xi(t+1) \in \boldsymbol{\Xi}$ since $\xi(t+1)$ is unknown at time t, which also implies $x(t+1) \in \bigcap_{\forall \xi(t+1) \in \boldsymbol{\Xi}} \boldsymbol{\mathcal{S}}(\xi(t+1)))$. The computed set $\boldsymbol{\mathcal{S}}(\xi(t))$ and the PDCL $\mathcal{K}(\xi(t))$ should obey the system constraints (6), which implies

$$\boldsymbol{\mathcal{S}}(\xi(t)) \subseteq \boldsymbol{\mathcal{X}}(\xi(t)), \tag{12}$$

where $\boldsymbol{\mathcal{X}}(\xi(t)) = \{x : (H_x + H_u \mathcal{K}(\xi(t))) x(t) \leq \mathbf{1}\}$. In classical formulation, RCI set description is independent of the scheduling parameters and is fixed for all $\xi \in \boldsymbol{\Xi}$. On the other hand, for PD-RCI set (7), different values of initial parameter $\xi(t) \in \boldsymbol{\Xi}$ provide different slices [1] of the set $\boldsymbol{\mathcal{S}}(\xi(t))$ for which the invariance can be achieved. Thus, PD-RCI set provides more flexibility in finding the set of initial states for which invariance can be achieved.

In some applications (e.g., MPC), it may be desirable to have a guaranteed performance within the PD-RCI sets for the closed loop system (9) and (10). For this purpose, we consider quadratic performance constraint

$$\sum_{t=0}^{\infty} \|z(t)\|_2^2 \leq \gamma,\ 0 \leq \gamma < \infty. \tag{13}$$

Here *w.l.o.g.*, we assume that performance is measured from time $t = 0$. Note that (13) can be only satisfied if $w(t) = 0,\ \forall t \geq 0$ (or $w(t)$ eventually becomes zero after certain time). Hence, we will assume $w(t) = 0$ only when performance constraints are considered.

Our aim is to compute $\mathcal{P}(\xi(t))$, W and $\mathcal{K}(\xi(t))$, which define the PD-RCI set (7) and the invariance inducing controller (8). We remark that, with $W = I$, computation of $\mathcal{P}(\xi(t))$ and $\mathcal{K}(\xi(t))$ results in a highly non-linear problem. Indeed, introduction of the matrix W helps overcome the nonlinearity by decomposing the problem into two subproblems described as follows. The first subproblem aims to compute W and $\mathcal{K}(\xi(t))$ for given *parameter-independent* matrix P. The second subproblem, aims to compute the parameter-dependent matrix $\mathcal{P}(\xi(t))$ and updated controller $\mathcal{K}(\xi(t))$, for a given matrix W obtained from solving the first subproblem. The two subproblems are formalized as follows.

Problem 1 *For a given matrix $P_{init} \in \mathbb{R}^{n_p \times n_x}$ such that $\mathcal{P}(\xi(t)) = P_{init}$ and the discrete-time system (2) subject to constraints (6), find a matrix W and the control law $\mathcal{K}(\xi(t))$ that satisfies conditions* (11), (12) *and* (13) *for any arbitrary variation of $\xi(t) \in \boldsymbol{\Xi}, \forall t \geq 0$.*

Problem 2 *For a given matrix W and the discrete-time system (2) subject to constraints (6), find the matrix $\mathcal{P}(\xi(t))$ and the control law $\mathcal{K}(\xi(t))$ that satisfies conditions* (11), (12) *and* (13) *for any arbitrary variation of $\xi(t) \in \boldsymbol{\Xi}, \forall t \geq 0$.*

Observe that by solving **Problem 1**, we obtain an RCI set which is *independent* of the parameter $\xi(t)$ since $\mathcal{P}(\xi(t)) = P_{init}$. In order to obtain a PD-RCI set $\boldsymbol{\mathcal{S}}(\xi(t))$, we need to solve **Problem 1** and **Problem 2** sequentially. In both problems, (13) is imposed only if performance is desired. Even though we present our formulation in the form of feasibility problems, our final goal is to design algorithms to compute a desirably large PD-RCI set. In the next section, we derive matrix inequality conditions for (11), (12) and (13). These conditions will be later used to obtain LMI conditions which solve **Problem 1** and **Problem 2**.

4 Sufficient Parameter Dependent Conditions for Invariance and Performance

For brevity, we will suppress the time dependent representation of the considered signals and use superscript '+' to indicate successor of $x(t)$ and $\xi(t)$. The arguments of the matrices $\mathcal{A}_{\mathcal{K}}(\xi), \mathcal{E}(\xi), \mathcal{C}_{\mathcal{K}}(\xi), \mathcal{P}(\xi)$ and the set $\boldsymbol{\mathcal{S}}(\xi)$ will be suppressed and recalled whenever necessary.

[1] For a fixed $\bar{\xi} \in \boldsymbol{\Xi}$, a *slice* $\boldsymbol{\mathcal{S}}(\bar{\xi})$ is defined as $\boldsymbol{\mathcal{S}}(\bar{\xi}) = \{x \in \mathbb{R}^{n_x} : -\mathbf{1} \leq \mathcal{P}(\bar{\xi}) W^{-1} x \leq \mathbf{1}\}$

4.1 Parameter dependent conditions for invariance and system constrains

From (7) and (11), a set $\boldsymbol{\mathcal{S}}(\xi)$ is invariant, if for a given $\xi \in \boldsymbol{\Xi}$ and for each $x \in \boldsymbol{\mathcal{S}}(\xi)$, for $i = 1, \ldots, n_p$

$$(1-(e_i^T\mathcal{P}(\xi^+)W^{-1}x^+)^2) \geq 0, \forall(w,\xi^+) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Xi}), \quad (14)$$

Using S-procedure (Pólik & Terlaky (2007)), (6) and (7), we can rewrite condition (14) as,

$$\begin{aligned}&\phi_i(1-(e_i^T\mathcal{P}(\xi^+)W^{-1}x^+)^2) \geq \\ &(\mathbf{1}-\mathcal{P}W^{-1}x)^T\Lambda_i(\mathbf{1}+\mathcal{P}W^{-1}x)+(\mathbf{1}-Gw)^T\Gamma_i(\mathbf{1}+Gw), \\ &\forall(w,\xi^+) \in (\boldsymbol{\mathcal{W}}, \boldsymbol{\Xi}), i = 1, \ldots, n_p, \quad (15)\end{aligned}$$

where $\phi_i \in \mathbb{R}_+$, $\Lambda_i \in \mathbb{D}_+^{n_p}$ and $\Gamma_i \in \mathbb{D}_+^{n_g}$. The vector x^+ in (15) should satisfy (9), hence (15) can be written as

$$\chi_1^T \begin{bmatrix} r_i & 0 & 0 & 0 \\ 0 & W^{-T}\mathcal{P}^T\Lambda_i\mathcal{P}W^{-1} & 0 & 0 \\ 0 & 0 & G^T\Gamma_iG & 0 \\ 0 & 0 & 0 & -p_i \end{bmatrix} \chi_1 \succeq 0,$$
$$\forall \begin{bmatrix} 0 & -\mathcal{A}_{\mathcal{K}} & -\mathcal{E} & I \end{bmatrix} \chi_1 = 0, \quad (16)$$

where $\chi_1 = \begin{bmatrix} 1 & x^T & w^T & (x^+)^T \end{bmatrix}^T$, $r_i = \phi_i - \mathbf{1}^T\Lambda_i\mathbf{1} - \mathbf{1}^T\Gamma_i\mathbf{1}$ and $p_i = W^{-T}\mathcal{P}^T(\xi^+)e_i\phi_ie_i^T\mathcal{P}(\xi^+)W^{-1}$. We will utilize **Lemma 1** to derive sufficient condition for (16). In particular, by choosing $\Psi_i(\xi) = \begin{bmatrix} 0 & 0 & 0 & \mathcal{V}_i(\xi)^{-1} \end{bmatrix}^T$ in **Lemma 1**, where $\mathcal{V}_i(\xi) = \sum_{k=1}^{N_\xi} \xi_k V_i^k$, with $V_i^k \in \mathbb{R}^{n_x \times n_x}$, and by using congruence transform, we get a sufficient condition for (16) as follows

$$\begin{bmatrix} r_i & 0 & 0 & 0 \\ 0 & \mathcal{P}^T\Lambda_i\mathcal{P} & 0 & \mathcal{A}_{\bar{\mathcal{K}}}^T \\ 0 & 0 & G^T\Gamma_iG & \mathcal{E}^T \\ 0 & * & * & \mathsf{He}(\mathcal{V}_i) - \mathcal{V}_i^Tp_i\mathcal{V}_i \end{bmatrix} \succ 0, \forall\xi^+ \in \boldsymbol{\Xi},$$
$$i = 1, \ldots, n_p, \quad (17)$$

where $\mathcal{A}_{\bar{\mathcal{K}}} = \mathcal{A}_{\mathcal{K}}W$ and $\bar{\mathcal{K}}(\xi) = \mathcal{K}(\xi)W \triangleq \sum_{k=1}^{N_\xi} \xi_k\bar{K}^k$. With the intention to resolve the nonlinearity in the (4, 4)-block of (17), we now introduce a positive-definite matrix variable X_i that satisfies

$$X_i^{-1} - p_i \succ 0. \quad (18)$$

Thus, from (17) and (18), we obtain a sufficient parameter dependent matrix inequality condition for (11) as

$$\begin{bmatrix} W^TX_i^{-1}W & * \\ \phi_ie_i^T\mathcal{P}(\xi^+) & \phi_i \end{bmatrix} \succ 0, \quad (19a)$$

$$\begin{bmatrix} r_i & 0 & 0 & 0 \\ 0 & \mathcal{P}^T\Lambda_i\mathcal{P} & 0 & \mathcal{A}_{\bar{\mathcal{K}}}^T \\ 0 & 0 & G^T\Gamma_iG & \mathcal{E}^T \\ 0 & * & * & \mathsf{He}(\mathcal{V}_i) - \mathcal{V}_i^TX_i^{-1}\mathcal{V}_i \end{bmatrix} \succ 0, \quad (19b)$$
$$\forall\xi^+ \in \boldsymbol{\Xi}, i = 1, \ldots, n_p.$$

In the next lemma, we present sufficient parameter dependent conditions for the invariance (11) and system constraints (12) .

Lemma 3 (Gupta et al. (2022)) *For some arbitrary matrices $Y_i \in \mathbb{R}^{n_x \times n_x}$, $\bar{\Lambda}_i \in \mathbb{D}_+^{n_p}$, $i = 1, \ldots, n_p$, $\bar{\Pi}_j \in \mathbb{D}_+^{n_p}$, $j = 1, \ldots, n_h$ and $P_0^k \in \mathbb{R}^{n_p \times n_x}$, $k = 1, \ldots, N_\xi$, if there exist matrices W, P^k, $\bar{K}^k$, V_i^k, X_i, diagonal semi-definite matrices Λ_i, Γ_i, Π_j and scalar $\phi_i > 0$ satisfying conditions* (20a),(20b), (20c) *and* (21) *reported below, then a PD-RCI set can be obtained as in (7) and the PDCL as $\mathcal{K}(\xi) = \bar{\mathcal{K}}(\xi)W^{-1}$:*

$$\begin{bmatrix} W^TY_i + Y_i^TW - Y_i^TX_iY_i & * \\ \phi_ie_i^T\mathcal{P}(\xi^+) & \phi_i \end{bmatrix} \succ 0, \quad (20a)$$

$$\phi_i - \mathbf{1}^T\Lambda_i\mathbf{1} - \mathbf{1}^T\Gamma_i\mathbf{1} \succ 0, \quad (20b)$$

$$\begin{aligned}&\sum_{k=1}^{N_\xi} \xi_k^2\boldsymbol{M}_i^{k,k}(\bar{\Lambda}_i, \Lambda_i) + \\ &\sum_{k=1}^{N_\xi-1}\sum_{l=k+1}^{N_\xi} \xi_k\xi_l(\boldsymbol{M}_i^{k,l}(\bar{\Lambda}_i, \Lambda_i) + \boldsymbol{M}_i^{l,k}(\bar{\Lambda}_i, \Lambda_i)) \succ 0, \quad (20c)\end{aligned}$$

$$\begin{aligned}&\sum_{k=1}^{N_\xi} \xi_k^2\boldsymbol{R}_j^{k,k}(\bar{\Pi}_j, \Pi_j) + \\ &\sum_{k=1}^{N_\xi-1}\sum_{l=k+1}^{N_\xi} \xi_k\xi_l(\boldsymbol{R}_j^{k,l}(\bar{\Pi}_j, \Pi_j) + \boldsymbol{R}_j^{l,k}(\bar{\Pi}_j, \Pi_j)) \succeq 0, \quad (21)\end{aligned}$$

where,

$$\boldsymbol{P}^{k,l}(\bar{\Lambda}_i, \Lambda_i) = \mathsf{He}((P^k)^T\bar{\Lambda}_iP_0^l) - (P_0^k)^T\bar{\Lambda}_i\Lambda_i^{-1}\bar{\Lambda}_iP_0^l, \quad (22)$$

$$\boldsymbol{M}_i^{k,l}(\bar{\Lambda}_i, \Lambda_i) = \begin{bmatrix} \boldsymbol{P}^{k,l}(\bar{\Lambda}_i, \Lambda_i) & * & * & * \\ 0 & G^T\Gamma_iG & * & * \\ A^kW + B^k\bar{K}^l & E^k & \mathsf{He}(V_i^k) & * \\ 0 & 0 & V_i^k & X_i \end{bmatrix}, \quad (23)$$

$$\boldsymbol{R}_j^{k,l}(\bar{\Pi}_j, \Pi_j) = \begin{bmatrix} 2 - \mathbf{1}^T \Pi_j \mathbf{1} & e_j^T (H_x W + H_u \bar{K}^l) \\ * & \boldsymbol{P}^{k,l}(\bar{\Pi}_j, \Pi_j) \end{bmatrix}. \quad (24)$$

Remark 2 *A feasible solution to inequalities* (20) *and* (21) *for any arbitrary choice of matrices Y_i, $\bar{\Lambda}_i$, $\bar{\Pi}_j$ and P_0^k gives a PD-RCI set $\boldsymbol{\mathcal{S}}$ and an invariance inducing PDCL $\mathcal{K}$. From **Lemma 2**, we know that the ideal choices of these matrices is $Y_i = X_i^{-1} W$, $\bar{\Lambda}_i = \Lambda_i$, $\bar{\Pi}_j = \Pi_j$ and $P_0^k = P^k$. However, the mentioned choices do not resolve the nonlinearities in* (20) *and* (21). *In Section 5, we will present a systematic way to select these matrices resolving the nonlinearity, which also helps us to reduce the conservatism introduced due to linearization.*

4.2 Parameter dependent performance constraints

We next derive parameter dependent matrix inequality conditions for performance constraint (13). Since we consider performance for $w(t) = 0$, $\forall t \geq 0$, we can ignore the matrix $\mathcal{E}$ in (9). Now, let $\mathcal{Q}(\xi) = \sum_{k=1}^{N_\xi} \xi_k Q^k \succeq 0$ with $Q^k \in \mathbb{R}^{n_x \times n_x}$, then the performance constraint (13) is satisfied by the closed-loop system (9) and (10) within the set $\boldsymbol{\mathcal{S}}$ if (Kothare et al. (1996), Liu et al. (2019)):

$$\left\| \mathcal{Q}^{-1/2} x(t) \right\|_2^2 \leq \gamma, \ \forall x(t) \in \boldsymbol{\mathcal{S}}(\xi), \quad (25a)$$

$$\left\| \mathcal{Q}^{-1/2}(\xi^+) x^+ \right\|_2^2 - \left\| \mathcal{Q}^{-1/2} x(t) \right\|_2^2 \leq - \| z(t) \|_2^2. \quad (25b)$$

It is easy to verify that (25) implies (13) by summing both sides of (25b) from $t = 0$ to $t = \infty$. In the next lemma, we present parameter dependent sufficient conditions for (25a) and (25b).

Lemma 4 (Gupta et al. (2022)) *For a given $\gamma > 0$, and some arbitrary matrices $\bar{\Upsilon} \in \mathbb{D}^{n_p}$ and $P_0^k \in \mathbb{R}^{n_p \times n_x}$, $k = 1, \ldots, N_\xi$, the performance constraints* (13) *is fulfilled by the closed-loop system* (9) *and* (10) *within the set $\boldsymbol{\mathcal{S}}(\xi)$, if there exist matrices W, P^k, $\bar{K}^k$, Q^k, Z^k, F^k and diagonal semi-definite matrix Υ satisfying the following conditions:*

$$\sum_{k=1}^{N_\xi} \xi_k^2 \boldsymbol{N}^{k,k} + \sum_{k=1}^{N_\xi - 1} \sum_{l=k+1}^{N_\xi} \xi_k \xi_l (\boldsymbol{N}^{k,l} + \boldsymbol{N}^{l,k}) \succeq 0. \quad (26a)$$

$$\sum_{k=1}^{N_\xi} \xi_k^2 \boldsymbol{L}^{k,k}(\bar{\Upsilon}, \Upsilon) + \sum_{k=1}^{N_\xi - 1} \sum_{l=k+1}^{N_\xi} \xi_k \xi_l (\boldsymbol{L}^{k,l}(\bar{\Upsilon}, \Upsilon) + \boldsymbol{L}^{l,k}(\bar{\Upsilon}, \Upsilon)) \succeq 0. \quad (26b)$$

where,

$$\boldsymbol{N}^{k,l} = \begin{bmatrix} \mathsf{He}(W) - Q^k & * & * & * & * \\ A^k W + B^k \bar{K}^l & \mathsf{He}(Z^k) & * & * & * \\ 0 & Z^k & Q^k & * & * \\ C^k W + D^k \bar{K}^l & 0 & 0 & \mathsf{He}(F^k) & * \\ 0 & 0 & 0 & F^k & I \end{bmatrix},$$

$$\boldsymbol{L}^{k,l}(\bar{\Upsilon}, \Upsilon) = \begin{bmatrix} \gamma - \mathbf{1}^T \Upsilon \mathbf{1} & * & * \\ 0 & \boldsymbol{P}^{k,l}(\bar{\Upsilon}, \Upsilon) & * \\ 0 & W & Q^k \end{bmatrix}.$$

Notice that the performance constraints (26b) depend on matrices $\bar{\Upsilon}$ and P_0^k, and their ideal choices are Υ and P^k, respectively. We will present systematic choices of these matrices in the next section. To summarize, in this section, we have obtained parameter dependent matrix inequality conditions for invariance (11), system constraints (12), and performance constraints (13) which are given by (20), (21) (**Lemma 3**), and (26) (**Lemma 4**), respectively. The parameter dependent conditions are linear if $\boldsymbol{P}^{k,l}$ is linear. Assuming P_0^k is known, the linearity of the matrix $\boldsymbol{P}^{k,l}$ in turn depends on the matrices $\bar{\Lambda}_i$ (and $\bar{\Pi}_j$, $\bar{\Upsilon}$) and P^k. Resolving the nonlinearity in $\boldsymbol{P}^{k,l}$ was one of the main motivating factors behind the presented formulation of **Problem 1** and **Problem 2**.

5 Tractable LMI Feasibility Conditions

The matrix inequality conditions for invariance, system constraints and performance derived in **Lemma 3** and **Lemma 4** are nonlinear and dependent on ξ. Hence, solving them in the current form can be intractable. We resolve the nonlinearity in $\boldsymbol{P}^{k,l}$ (see (22)) by fixing the matrices $P^k = P_0^k = P_{init}$, $k = 1, \ldots, N_\xi$, where P_{init} is some *known* matrix. As explained in **Remark 2**, we can thus allow matrices $\bar{\Lambda}_i = \Lambda_i$, $\bar{\Pi}_j = \Pi_j$, $\bar{\Upsilon} = \Upsilon$ (their ideal choices). In the following theorem, we present one of the main result of this paper which gives tractable LMI feasibility conditions for **Problem 1**.

Theorem 1 *Let $\mathcal{P}(\xi) = \mathcal{P}_0(\xi) = P_{init}$ be a given matrix, then **Problem 1** has a feasible solution if,*

i. there exist matrices W, $\bar{K}^k$, V_i^k, X_i, diagonal semi-definite matrices Λ_i, Γ_i, Π_j and scalar $\phi_i > 0$, where $k = 1, \cdots, N_\xi$, $i = 1, \ldots, n_p$ and $j = 1, \ldots, n_h$ satisfying:

$$\begin{bmatrix} \mathsf{He}(W^T Y_i) - Y_i^T X_i Y_i & * \\ \phi_i e_i^T P_{init} & \phi_i \end{bmatrix} \succ 0, \quad (27a)$$

$$\phi_i - \mathbf{1}^T \Lambda_i \mathbf{1} - \mathbf{1}^T \Gamma_i \mathbf{1} \succ 0, \quad (27b)$$

$$\left.\begin{aligned} \boldsymbol{M}_i^{k,k}(\Lambda_i,\Lambda_i)\succ 0, k=1,\dots,N_\xi \\ \boldsymbol{M}_i^{k,l}(\Lambda_i,\Lambda_i)+\boldsymbol{M}_i^{l,k}(\Lambda_i,\Lambda_i)\succ 0,\ k=1,\dots,N_\xi-1, \\ l=k+1,\dots,N_\xi \end{aligned}\right\}, \tag{27c}$$

$$\left.\begin{aligned} \boldsymbol{R}_i^{k,k}(\Pi_j,\Pi_j)\succeq 0, k=1,\dots,N_\xi \\ \boldsymbol{R}_i^{k,l}(\Pi_j,\Pi_j)+\boldsymbol{R}_i^{l,k}(\Pi_j,\Pi_j)\succeq 0,\ k=1,\dots,N_\xi-1, \\ l=k+1,\dots,N_\xi \end{aligned}\right\}, \tag{28}$$

to fulfill conditions (11) *and* (12).

ii. there exist W, $\bar{K}^k$, Q^k, Z^k, F^k *and* Υ, *where* $k = 1,\cdots,N_\xi$ *for a given performance bound* γ *satisfying*

$$\left.\begin{aligned} \boldsymbol{N}_i^{k,k}\succeq 0,\ k=1,\dots,N_\xi \\ \boldsymbol{N}_i^{k,l}+\boldsymbol{N}_i^{l,k}\succeq 0,\ k=1,\dots,N_\xi-1, \\ l=k+1,\dots,N_\xi \end{aligned}\right\}, \tag{29a}$$

$$\left.\begin{aligned} \boldsymbol{L}_i^{k,k}(\Upsilon,\Upsilon)\succeq 0, k=1,\dots,N_\xi \\ \boldsymbol{L}_i^{k,l}(\Upsilon,\Upsilon)+\boldsymbol{L}_i^{l,k}(\Upsilon,\Upsilon)\succeq 0,\quad k=1,\dots,N_\xi-1, \\ l=k+1,\dots,N_\xi \end{aligned}\right\}, \tag{29b}$$

to fulfill condition (13).

An RCI set can then be obtained as in (7) and the PDCL $\mathcal{K}(\xi)=\bar{\mathcal{K}}(\xi)W^{-1}$.

PROOF.

i. Considering $\mathcal{P}(\xi)=\mathcal{P}_0(\xi)=P_{init}$, (27a) and (27b) are directly obtained from (20a) and (20b), respectively. Next, we consider (20c), which is a homogeneous matrix valued polynomial of degree 2 and choose $\bar{\Lambda}_i=\Lambda_i$. The *l.h.s* of (20c) is a matrix valued polynomial in ξ_k, $k=1,\dots,N_\xi$. Since $\xi_k\geq 0$, a sufficient condition for (20c) can be obtained by imposing each coefficient matrix of the polynomial to be positive-definite, which is given by (27c). Similarly, letting $\bar{\Pi}_j=\Pi_j$, a sufficient condition for (21) is (28).

ii. We can prove (29a) and (29b) are sufficient for (26a) and (26b) by using similar arguments as mentioned in part-*i*. Notice that in (29b) we substitute $\bar{\Upsilon}=\Upsilon$.

Note that, even if P^k's are assumed to be constant in **Theorem 1**, the variable matrix W allows to reshape the RCI set. A similar construction to find initial RCI set was also proposed in Liu et al. (2019), Gupta & Falcone (2019). We formulate feasibility conditions for **Problem 2** in the next theorem. In the theorem, matrices P^k's are treated as variables and thus, inline with **Remark 1**, we fix $\bar{\Lambda}_i=\Lambda_i^0$, $\bar{\Pi}_j=\Pi_j^0$, $\bar{\Upsilon}=\Upsilon^0$.

Theorem 2 *Let* $\mathcal{P}_0(\xi)$ *and* W *be given matrices, then* ***Problem 2*** *has a feasible solution if,*

i. there exist matrices P^k, $\bar{K}^k$, V_i^k, X_i, *diagonal semi-definite matrices* Λ_i, Γ_i, Π_j *and scalar* $\phi_i>0$, *where* $k=1,\cdots,N_\xi$, $i=1,\dots,n_p$ *and* $j=1,\dots,n_h$ *satisfying:*

$$\begin{bmatrix} \mathsf{He}(W^TY_i)-Y_i^TX_iY_i & * \\ e_i^TP^k & \phi_i^{-1}\end{bmatrix}\succ 0, \tag{30a}$$

$$\begin{bmatrix} \phi_i^{-1}-\mathbf{1}^T\bar{\Gamma}_i\mathbf{1} & * \\ \phi_i^{-1}\mathbf{1} & \Lambda_i^{-1}\end{bmatrix}\succeq 0, \tag{30b}$$

$$\left.\begin{aligned} \bar{\boldsymbol{M}}_i^{k,k}(\Lambda_i^0,\Lambda_i)\succ 0, k=1,\dots,N_\xi \\ \bar{\boldsymbol{M}}_i^{k,l}(\Lambda_i^0,\Lambda_i)+\bar{\boldsymbol{M}}_i^{l,k}(\Lambda_i^0,\Lambda_i)\succ 0,\ k=1,\dots,N_\xi-1, \\ l=k+1,\dots,N_\xi \end{aligned}\right\} \tag{30c}$$

$$\left.\begin{aligned} \boldsymbol{R}_j^{k,k}(\Pi_j^0,\Pi_j)\succeq 0, k=1,\dots,N_\xi \\ \boldsymbol{R}_j^{k,l}(\Pi_j^0,\Pi_j)+\boldsymbol{R}_j^{l,k}(\Pi_j^0,\Pi_j)\succeq 0,\ k=1,\dots,N_\xi-1, \\ l=k+1,\dots,N_\xi \end{aligned}\right\} \tag{31}$$

to fulfill conditions (11) *and* (12), *where*

$$\bar{\boldsymbol{M}}_i^{k,l}(\Lambda_i^0,\Lambda_i)=\begin{bmatrix} \boldsymbol{P}^{k,l}(\Lambda_i^0,\Lambda_i) & * & * & * \\ 0 & G^T\bar{\Gamma}_iG & * & * \\ A^kW+B^k\bar{K}^l & \phi_i^{-1}E^k & \mathsf{He}(V_i^k) & * \\ 0 & 0 & V_i^k & X_i \end{bmatrix} \tag{32}$$

ii. there exist P^k, $\bar{K}^k$, Q^k, Z^k, F^k *and* Υ, *where* $k = 1,\cdots,N_\xi$ *for a given performance bound* γ *satisfying*

$$\left.\begin{aligned} \boldsymbol{N}_i^{k,k}\succeq 0,\ k=1,\dots,N_\xi \\ \boldsymbol{N}_i^{k,l}+\boldsymbol{N}_i^{l,k}\succeq 0,\ k=1,\dots,N_\xi-1, \\ l=k+1,\dots,N_\xi \end{aligned}\right\} \tag{33a}$$

$$\left.\begin{aligned} \boldsymbol{L}^{k,k}(\Upsilon^0,\Upsilon)\succeq 0, k=1,\dots,N_\xi \\ \boldsymbol{L}^{k,l}(\Upsilon^0,\Upsilon)+\boldsymbol{L}^{l,k}(\Upsilon^0,\Upsilon)\succeq 0,\quad k=1,\dots,N_\xi-1, \\ l=k+1,\dots,N_\xi \end{aligned}\right\} \tag{33b}$$

to fulfill condition (13).

A PD-RCI set can then be obtained as in (7) and the PDCL is $\mathcal{K}(\xi)=\bar{\mathcal{K}}(\xi)W^{-1}$.

PROOF.

i. We obtain (30a) from (20a) by application of congruence transform and since the resultant matrix inequality is affinely dependent on the parameter. Using Schur complement on (20b) and substituting $\bar{\Gamma}_i =$

$\phi_i^{-2}\Gamma_i$, we get (30b). By replacing $\Gamma_i = \phi_i^2\bar{\Gamma}_i$ and $\bar{\Lambda}_i = \Lambda_i^0$ in (20c),

$$\sum_{k=1}^{N_\xi}\xi_k^2\bar{\boldsymbol{M}}_i^{k,k}(\Lambda_i^0,\Lambda_i)+ \sum_{k=1}^{N_\xi-1}\sum_{l=k+1}^{N_\xi}\xi_k\xi_l(\bar{\boldsymbol{M}}_i^{k,l}(\Lambda_i^0,\Lambda_i)+\bar{\boldsymbol{M}}_i^{l,k}(\Lambda_i^0,\Lambda_i))\succ 0, \quad (34)$$

where $\bar{\boldsymbol{M}}_i^{k,l}$ is given in (32). Since (34) is homogeneous matrix valued polynomial of degree 2, we can now employ zeroth order Polya's relaxation to obtain (30c). Similarly, (31) is obtained from (21), by substituting $\bar{\Pi}_j = \Pi_j^0$ and using zeroth order Polya's relaxation.

ii. We can prove (33a) and (33b) using similar approach as in part-i. Notice that in (33b), we replace $\bar{\Upsilon} = \Upsilon^0$.

By finding a feasible solution for **Problem 2**, we obtain a PD-RCI set. However, the inequalities (30), (31) and (33) depend on the matrices P_0^k, Λ_i^0, Π_j^0 and Υ^0, which are the initial guess of matrices P^k, Λ_i, Π_j and Υ, respectively. Finding an initial guess for these matrices is not straightforward; we thus obtain them by solving **Problem 1**. It is easy to verify that using solutions from **Problem 1** to initialize **Problem 2**, always preserves feasibility of solutions, see **Remark 1**. Finally, for clarity of exposition, we summarize the main results presented so far and their interrelation with the help of a flowchart, as depicted in Fig. 1.

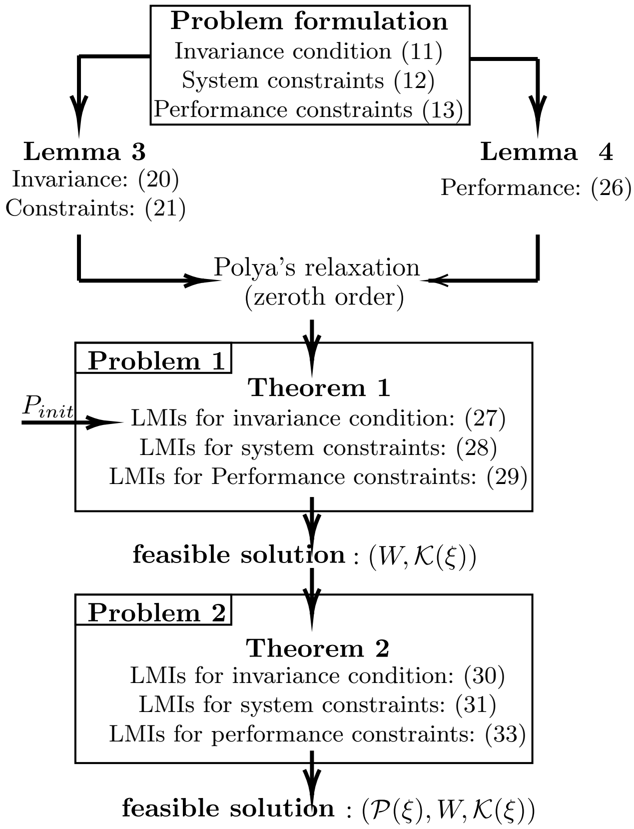

Fig. 1. Flowchart summarizing the main results.

6 Iterative PD-RCI Set Computation

Our primary goal is to compute PD-RCI set (7) of desirably large volume and the PDCL controller (8). Thus, we need to formulate a method which computes a maximum volume set feasible to conditions proposed in **Theorem 1** and **Theorem 2**. In the original form, the conditions in these theorems were nonlinear, and to make them tractable for solving, we linearized them by using **Lemma 2**. As mentioned in the **Remark 1**, the linearization introduces conservatism, which can be reduced by adopting an iterative scheme, where we first consider **Problem 1** in which we assumed $P(\xi) = P^0(\xi) = P_{init}$. As shown in Gupta et al. (2021), the volume of the considered RCI set is proportional to $|\det(W)|$. We next propose an optimization problem that computes desirably large RCI set for **Problem 1**.

6.1 Initial RCI set computation

We develop an iterative scheme in which we solve a determinant maximization problem under LMI conditions presented in **Theorem 1**. Similar to Gupta et al. (2019), we will try to iteratively maximize the volume to avoid enforcing symmetry on W. The basic idea is to maximize the determinant of a different matrix J, which is required to satisfy

$$W^TW \succcurlyeq J \succ 0, \quad (35)$$

Condition (35) ensures that $\det(J) \leq |\det(W)|^2$. Since (35) is not an LMI, it needs to be replaced with a sufficient condition. This is done within the iterative scheme in which the solution of W at the previous step is represented as W^0. A sufficient condition for (35) is formulated in terms of W^0 as (see Gupta et al. (2019))

$$W^TW^0 + (W^0)^TW - (W^0)^TW^0 \succcurlyeq J \succ 0. \quad (36)$$

Note that this condition is necessarily satisfied with $W = W^0$. Thus, maximization of $\det(J)$ under (36) would lead to a solution W that satisfies $|\det(W)| \geq |\det(W^0)|$. Moreover, as described in **Remark 1**, at each iteration we update $Y_i = (X_i^0)^{-1}W^0$ in (27a), where X_i^0 is previous solution of X_i. This allows us to develop the following iterative algorithm to compute RCI sets of increased volume at each step for a priori chosen matrix P_{init}

$$\left.\begin{array}{ll} \max & \mathsf{log\,det}(J) \\ \phi_i, W, \bar{K}^k, V_i^k, X_i, \Lambda_i, \Gamma_i & \\ \Pi_j, Q^k, Z^k, F^k, \Upsilon, J & \\ \text{subject to:} & (27), (28), (29) \text{ and } (36) \end{array}\right\} \quad (37)$$

Initial Optimization to Compute W^0: Condition (36) is removed and $\mathsf{log\,det}(J)$ is changed to $\mathsf{log\,det}(W + W^T)$; (27a) is imposed with $Y_i \to I$.

6.2 Computation of PD-RCI sets

In order to compute a desirably large PD-RCI set, conforming **Problem 2**, we formulate a new optimization problem. In this problem, we fix the matrix W obtained by solving (37), and now treat matrices P^k's as optimization variables. By construction, for each $\xi \in \Xi$, $\boldsymbol{\mathcal{S}}(\xi)$ is an 0-symmetric polytope in the state-space. Thus, an intuitive way to maximize the volume of such a set is to compute matrices P^k's such that the sum of the volumes of each slice of $\boldsymbol{\mathcal{S}}(\xi)$ is maximized. However, maximizing infinite slices of the PD-RCI set would lead to solving semi-infinite problem, which is intractable. Nevertheless, to deal with such intractability, we only maximize the slices $\boldsymbol{\mathcal{S}}(\xi^m) = \left\{x \in \mathbb{R}^{n_x} : -\mathbf{1} \le \mathcal{P}(\xi^m)W^{-1}x \le \mathbf{1}\right\}$, corresponding to the finite set of grid points $\xi^m \in \Xi$, $m = 1, \ldots, N_m$. For example, a possible choice of the grid points can be the vertices of Ξ. We propose a novel volume maximization approach for polytopic sets which leads to the following SDP problem.

Proposition 3 *Given N_m number of grid points, the slices $\boldsymbol{\mathcal{S}}(\xi^m)$, $m = 1, \ldots, N_m$ of desirably large volume characterizing a PD-RCI set, can be obtained by solving the following SDP problem in an iterative manner,*

$$\left.\begin{array}{ll} \min & \sum_{n=1}^{N_\sigma} \sum_{m=1}^{N_m} \sigma_n^m \\ \phi_i, P^k, \bar{K}^k, V_i^k, X_i, \Lambda_i, \Gamma_i & \\ \Pi_j, Q_1^k, Z^k, F^k, \Upsilon, \sigma_n^m & \\ \textit{subject to:} & \sigma_n^m \ge 0, \\ & \begin{bmatrix} \widetilde{P}W^{-1} \\ -\widetilde{P}W^{-1} \end{bmatrix} \widetilde{x}_n - \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \le \begin{bmatrix} \widetilde{\sigma} \\ \widetilde{\sigma} \end{bmatrix}, \\ & (30), (31) \textit{ and } (33). \end{array}\right\} \quad (38)$$

where $\widetilde{P} = [\mathcal{P}(\xi^1)^T, \cdots, (\mathcal{P}(\xi^{N_m})^T]^T \in \mathbb{R}^{(n_p N_m) \times n_x}$, $\widetilde{\sigma} = [\sigma_n^1 \mathbf{1}, \cdots, \sigma_n^{N_m} \mathbf{1}]^T \in \mathbb{R}^{n_p N_m}$ and $\{\widetilde{x}_n\}_{n=1}^{N_\sigma}$ are the vertices of some known n_x dimensional outer bounding box $\mathfrak{B}$ which contains the state constraint set $\boldsymbol{\mathcal{X}}$.

We refer the reader to **Appendix** in Gupta et al. (2022) for the details of the volume maximization algorithm which is based on Monte-Carlo techniques presented in Benavoli & Piga (2016), Piga & Benavoli (2019).

Algorithm 1 : Computing PD-RCI set.

Input: System (2), $\boldsymbol{\mathcal{X}_u}$, $\boldsymbol{\mathcal{W}}$, $\mathcal{P}_0$, W, Y_i Λ_i^0, Γ_j^0, Υ^0
Output: $\mathcal{P}(\xi)$, $\mathcal{K}(\xi)$
while Iteration ≥ 0 **do**
  $[\mathcal{P}, \bar{\mathcal{K}}, X_i, \Lambda_i, \Gamma_j, \Upsilon] \leftarrow$ solve (38)
  Update: $Y_i \leftarrow X_i^{-1}W$, $\mathcal{P}_0 \leftarrow \mathcal{P}$, $\Lambda_i^0 \leftarrow \Lambda_i$, $\Gamma_j^0 \leftarrow \Gamma$, $\Upsilon^0 \leftarrow \Upsilon$
  Iteration $\leftarrow$ Iteration $- 1$
end while

Assuming that the initial values of $\mathcal{P}_0, W, Y_i, \Lambda_i^0, \Gamma_j^0$ and Υ^0 are available after solving (37), we summarize the whole approach to compute PD-RCI set in **Algorithm 1**. As a consequence of the adopted successive linearization approach (see **Remark 1**), **Algorithm 1** always has a feasible solution at the first iteration if initialized using solutions from (37). The update scheme in the algorithm alleviates the conservatism introduced while linearizing the equation (30a), (30c), (31) and (33b) using **Lemma 2**. The systematic update procedure also guarantees that the solutions from the previous iteration are feasible in the current iteration. Thus, at each iteration we find a new PD-RCI set of larger volume until the specified number of iterations are performed, or convergence is achieved. We purposely present the termination of the algorithm based on the number of iteration instead of convergence to emphasize that latter is not necessary.

6.3 Computation of the RCI set for quasi-LPV systems

If the scheduling parameters ξ are function of system states x and input u, then the system is called as *quasi-LPV* (qLPV). For qLPV systems, we need to keep the RCI set description independent of parameter i.e., by restricting $P^k = P$, $\forall k = 1, \ldots, N_\xi$. Alternatively, we can construct a set $\breve{\boldsymbol{\mathcal{S}}} = \bigcap_{\forall \xi \in \Xi} \boldsymbol{\mathcal{S}}(\xi)$. Notice that the set $\breve{\boldsymbol{\mathcal{S}}}$ also satisfies conditions (11) and (12) if $\boldsymbol{\mathcal{S}}(\xi)$ satisfies them, and since it is independent of ξ, we call it simply RCI set. The set $\breve{\boldsymbol{\mathcal{S}}}$ can be possibly larger (volume-wise) than the set obtained by restricting $P^k = P$, due to extra degree of freedom provided by additional variables involved in the overall optimization problem when computing the former. Even though we define $\breve{\boldsymbol{\mathcal{S}}}$ as the intersection of infinite slices of $\boldsymbol{\mathcal{S}}(\xi)$, we have proved that it can be exactly obtained by intersecting the slices generated at vertices $\xi^m, m = 1, \ldots, N_\xi$ of the set Ξ (see, Gupta et al. (2022)), *i.e.*,

$$\breve{\boldsymbol{\mathcal{S}}} \triangleq \bigcap_{\forall \xi \in \Xi} \boldsymbol{\mathcal{S}}(\xi) = \bigcap_{\forall \xi^m \in \Xi} \boldsymbol{\mathcal{S}}(\xi^m) \quad (39)$$

7 Numerical Examples

We now demonstrate the potential of the proposed algorithm through examples. The algorithm is implemented in Matlab on a Intel Core i7-555U CPU with 8 GB RAM using YALMIP (Löfberg 2004) and the solver SeDuMi.

7.1 Double Integrator

Let us consider a parameter-varying double integrator:

$$x^+ = \begin{bmatrix} 1+\theta & 1+\theta \\ 0 & 1+\theta \end{bmatrix} x + \begin{bmatrix} 0 \\ 1+\theta \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w, \tag{40}$$

where $|\theta| \leq 0.25$ is a time-varying parameter. The state and control input constraints, and the disturbance bounds are expressed as $|x| \leq [5,\ 5]^T$, $|u| \leq 1$, $|w| \leq 0.25$. We rewrite (40) in the form (5) with $N_\xi = 2$ and

$$\left[\begin{array}{c|c|c} A^1 & B^1 & E^1 \\ \hline A^2 & B^2 & E^2 \end{array}\right] = \left[\begin{array}{cc|c|c} 1.25 & 1.25 & 0 & 1 \\ 0 & 1.25 & 1.25 & 0 \\ \hline 0.75 & 0.75 & 0 & 1 \\ 0 & 0.75 & 0.75 & 0 \end{array}\right], \tag{41}$$

where $\xi_1 = (0.25+\theta)/0.5$ and $\xi_2 = (0.25-\theta)/0.5$. We then select P_{init} as described in (Gupta et al. 2021, **Remark 1**) and solve (37) iteratively until convergence, which took 10 iterations and thus obtain all the matrices needed to initialize **Algorithm 1**. Finally, the PD-RCI set $\boldsymbol{\mathcal{S}}(\xi)$, shown in Fig. 2, is obtained after performing 60 iterations of **Algorithm 1**. The average computation time is 9.31 seconds per iteration. The obtained matrices characterizing PD-RCI set and PDCL are

$$[\, P^1 \,|\, P^2 \,] = \left[\begin{array}{cc|cc} -0.4111 & -0.1354 & -0.3257 & -0.0854 \\ 0.0303 & -0.5151 & 0.0404 & -0.3823 \\ 0.4867 & -0.2474 & 0.4867 & -0.2474 \\ 0.4884 & -0.0504 & 0.4883 & -0.0506 \end{array}\right],$$

$$\left[\begin{array}{c|c} W & \begin{array}{c} K^1 \\ \hline K^2 \end{array} \end{array}\right] = \left[\begin{array}{cc|cc} 2.4373 & -0.6691 & -0.2246 & -0.7898 \\ \hline -0.7327 & 0.8379 & -0.1506 & -0.5601 \end{array}\right].$$

The RCI set $\breve{\boldsymbol{\mathcal{S}}}$ in (39) can be seen in the Fig. 2 (b) as bounded *colourless* region. The region outside the set $\breve{\boldsymbol{\mathcal{S}}}$, highlighted in cyan, consists of points which can be brought within the RCI set $\breve{\boldsymbol{\mathcal{S}}}$ in one step for some selectable initial value of the parameter θ, thus, enlarging the overall set of safe initial states. To compare the volume gain between **Problem 1** and **Problem 2**, we plot the volume of the set $\breve{\boldsymbol{\mathcal{S}}}$ at each iteration, as shown in Fig. 3a. In the figure, it can be seen that there is an additional (approximately) 29% gain in the volume when the proposed Monte-Carlo based volume maximization approach is utilized. For comparison, we plot

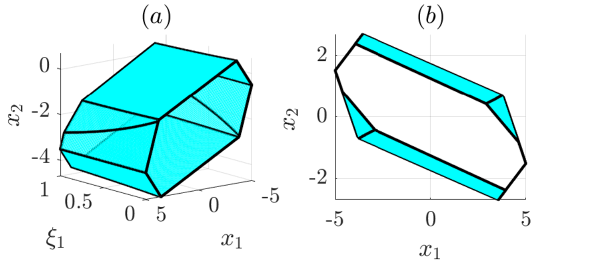

Fig. 2. (a) Plot of the PD-RCI set $\boldsymbol{\mathcal{S}}(\xi)$ in (7) w.r.t ξ_1 and (b) projection $\boldsymbol{\mathcal{S}}(\xi)$ on (x_1, x_2) axis.

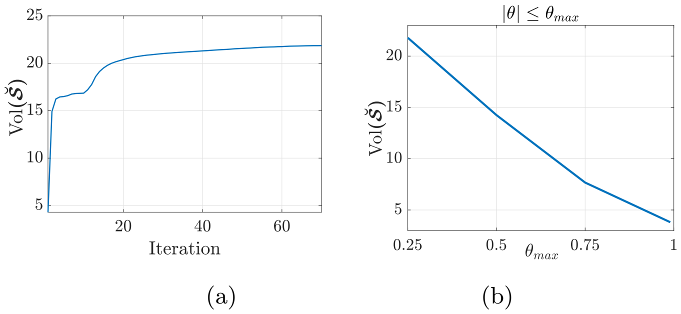

Fig. 3. Volume of the set $\breve{\boldsymbol{\mathcal{S}}}$ plotted against (a) iteration, (b) different scheduling parameter bounds.

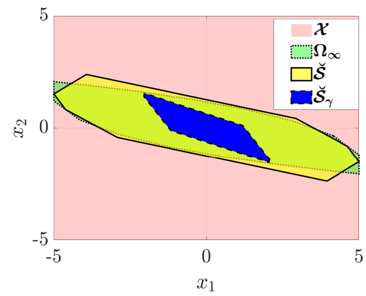

Fig. 4. Admissible set $\boldsymbol{\mathcal{X}}$ (red), maximal RCI set $\boldsymbol{\Omega}_\infty$ (green), and RCI sets $\breve{\boldsymbol{\mathcal{S}}}$ (yellow) and $\breve{\boldsymbol{\mathcal{S}}}_\gamma$ (blue).

the computed set $\breve{\boldsymbol{\mathcal{S}}}$ and the maximal RCI set $\boldsymbol{\Omega}_\infty$ obtained using classical geometric approach Herceg et al. (2013) in Fig. 4. The geometric approach treats parameter as unknown bounded signals, and the control input is free from any state-feedback structure. Not surprisingly, the set $\breve{\boldsymbol{\mathcal{S}}}$ (volume 21.7907) computed using the proposed approach was found to be larger than the maximal RCI set $\boldsymbol{\Omega}_\infty$ (volume 19.3703). Moreover, the overall representational complexity of the set $\breve{\boldsymbol{\mathcal{S}}}$ is just 8, which is exactly half the complexity of the set $\boldsymbol{\Omega}_\infty$, this further demonstrates the benefits of using PD-RCI sets and PDCL in the LPV setting. We also show the set $\breve{\boldsymbol{\mathcal{S}}}_\gamma$ in Fig. 4, which satisfy performance constraints $\sum_{t=0}^{\infty} x(t)^T Q_x x(t) + u(t)^T Q_u u(t) \leq \gamma$, for all x within the set. Where $Q_x = I$, $Q_u = 0.1$ and $\gamma = 10$. Lastly, to demonstrate the main advantage of the presented algorithm, we perform an analysis in which the RCI sets

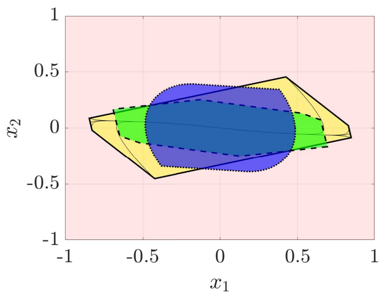

Fig. 5. Admissible set $\boldsymbol{\mathcal{X}}$ (red), RCI set $\breve{\boldsymbol{\mathcal{S}}}$ (yellow; solid), RCI set using Gupta & Falcone (2019) (green; dashed) and RPI set (blue; dotted) for the Van der Pol oscillator system.

are computed by changing the bound θ_{max} on the parameter θ. The volume of the computed set $\breve{\boldsymbol{\mathcal{S}}}$ is plotted against parameter bound in Fig. 3b. As expected from the theory, the volume decreases with an increase in the value of θ_{max}. Nonetheless, it is interesting to observe that the proposed method is able to compute the RCI sets even for a large bound on the scheduling parameter. We remark that the geometric approach Herceg et al. (2013), failed to generate any RCI set for $\theta_{max} \geq 0.4$

7.2 Nonlinear System

One important application of the proposed approach is to compute RCI sets for nonlinear systems. For this purpose, we consider the controlled Van der Pol oscillator system in Hanema et al. (2017):

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \mu(1 - x_1^2)x_2 + u, \qquad (42)$$

where $\mu = 2$. The system should satisfy the input constraints $|u| \leq 1$ and state constraints $|x_1| \leq 1$, $|x_2| \leq 1$. For computation and simulation purpose we discretize the system using Eulers method with sampling time 0.1 units. Further, we rewrite the system in the quasi-LPV form (5) with scheduling parameters $\xi_1 = (2 - \mu(1 - x_1^2))/2$ and $\xi_2 = \mu(1 - x_1^2)/2$. Using the proposed approach we compute the matrix variables defining the RCI set and the invariance inducing controller for the nonlinear system which are given as

$$[P_1|P_2] = \left[\begin{array}{cc|cc} -0.5066 & -0.1205 & -0.5066 & -0.1358 \\ -0.4349 & -0.0135 & -0.4367 & -0.0134 \\ 0.4238 & -0.2686 & 0.4237 & -0.3173 \\ 0.5280 & 0.0385 & 0.5280 & 0.0385 \end{array}\right],$$

$$\left[W \left| \begin{array}{c} K^1 \\ \hline K^2 \end{array}\right.\right] = \left[\begin{array}{cc|cc} 0.4409 & 0.0136 & 0.8341 & -2.3111 \\ \hline -0.0127 & 0.1090 & 0.8727 & -3.0114 \end{array}\right].$$

Since the scheduling parameters ξ_1, ξ_2 are state dependent, in accordance with Section 6.3, we compute RCI set $\breve{\boldsymbol{\mathcal{S}}}$ (39), shown in Fig. 5. The closed-loop trajectories from all the vertices of the set $\breve{\boldsymbol{\mathcal{S}}}$ are also shown in Fig. 5. For comparison, we compute an RCI set (of a representational complexity same as $\breve{\boldsymbol{\mathcal{S}}}$) using the method presented in Gupta & Falcone (2019), which assumes the invariance inducing controller to be linear state-feedback. We show the computed set in Fig. 5 with green color. It can be seen that this set is smaller than the one generated by the proposed algorithm presented in this paper. The geometric approach Herceg et al. (2013) for computing maximal RCI set did not converge even after $24hrs$, so instead, we show a robust positive invariant (RPI) set corresponding to an LQR controller for nominal system and tuning matrices $Q = I$ and $R = 1$. The representational complexity of the RPI set is 50. Clearly, the proposed algorithm is more advantageous here since it can generate visibly larger RCI sets of low complexity.

8 Conclusion

The paper presented a novel iterative algorithm to compute a PD-RCI set and PD-invariance inducing control law for LPV systems. At each iteration of the algorithm, an SDP is solved to obtain a larger PD-RCI set successively until convergence. In the SDP, we introduced the invariance conditions, system constraints and performance constraints as LMIs, which were constructed using Finslers's lemma and zeroth order Polya's relaxation. Besides, we also presented a new approach for volume maximization of polytopes based on Monte-Carlo principles. It was shown that a larger invariant set could be obtained by exploiting the knowledge of parameters in the invariant set description as well as in the controller design. We assumed candidate RCI set to be 0-symmetric. This is a reasonable assumption if the system is linear and the constraints are 0-symmetric. In other cases, this assumption would be potentially conservative. Thus, a natural extension of this work could be to devise a similar algorithm for computing non-symmetrical RCI set.

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