EXPLICIT COUNTEREXAMPLES TO SCHÄFFER'S CONJECTURE

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Abstract. In 1970 J.J. Schäffer proved that for any invertible $n \times n$ matrix T and for any operator norm $\|\cdot\|$, the inequality

$$|\det T| \left\|T^{-1}\right\| \leq \mathcal{S} \left\|T\right\|^{n-1}$$

holds with $\mathcal{S} = \mathcal{S}(n) \leq \sqrt{en}$. He conjectured that in fact this inequality holds for all T with an $\mathcal{S}$ independent of n. This conjecture was refuted in the early 90's by E. Gluskin, M. Meyer and A. Pajor who have shown that for certain $T = T(n)$ the inequality can only hold when $\mathcal{S}$ is growing with n. Subsequent contributions of J. Bourgain and H. Queffélec provided increasing lower estimates on $\mathcal{S}$. Those results rely on probabilistic and number theoretic arguments for the existence of sequences $T(n)$ with growing $\mathcal{S}$. Explicit counterexamples to Schäffer's conjecture were not available since 1995.

In this article we propose a new and entirely constructive approach to Schäffer's conjecture. As a result, we present an explicit sequence of Toeplitz matrices T_λ with singleton spectrum $\{\lambda\} \subset \mathbb{D} \setminus \{0\}$ such that $\mathcal{S} \geq c(\lambda)\sqrt{n}$.

1. Introduction

It is a classical task in matrix analysis and operator theory to find good estimates for inverses. The following well-established line of research was initiated by studies of B. L. van der Waerden [Sc, p. 331] and J. J. Schäffer [Sc]. Let $\mathcal{M}_n$ be the set of $n \times n$ complex matrices acting on the Banach space $\mathbb{C}^n$ equipped with a certain norm. Let $\|\cdot\|$ denote the operator norm induced on $\mathcal{M}_n$ by the norm of the Banach space. What is the best $\mathcal{S} = \mathcal{S}(n)$ so that

$$|\det T| \left\|T^{-1}\right\| \leq \mathcal{S} \left\|T\right\|^{n-1}$$

holds for *any* invertible $T \in \mathcal{M}_n$ and *any* operator norm $\|\cdot\|$? P. Halmos observed that if we restrict ourselves to Hilbert space norms on $\mathbb{C}^n$, then $\mathcal{S} = 1$, see [Q1]. Schäffer [Sc, Theorem 3.8] proved that for Banach space norms on $\mathbb{C}^n$

$$\mathcal{S} \leq \sqrt{en},$$

and he conjectured that, as in the case of Hilbert spaces, $\mathcal{S}$ is bounded. This claim was refuted by lower estimates on $\mathcal{S}$ obtained by E. Gluskin, M. Meyer and A. Pajor [GMP] and by J. Bourgain[1] [GMP]. The currently best known lower estimate is due to H. Queffélec [Q2]:

$$\mathcal{S} \geq \sqrt{n}(1 - \mathcal{O}(1/n)).$$

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[1]The same article [GMP] contains an appendix with a stronger estimate due to Bourgain.

The common point in the mentioned lower bounds is that they rely on an inequality of Bourgain, see (2.2) below, that relates Schäffer's problem to a geometric property of the spectrum of T: For $\mathcal{S}$ to grow the eigenvalues of T should satisfy a Turán-type power sum inequality. The construction of explicit solutions to such inequalities appears to be a well-studied but open problem in number theory [T, M, ER, A1, A2]. More precisely, Bourgain's inequality (2.2) relates Schäffer's question to Turán's tenth problem [A1, T]. The latter has no constructive solution and relies on deep number-theoretic existence arguments [A1, M, Q2]. In [GMP] as well as in [Q1, question 5] the construction of explicit matrices with growing $\mathcal{S}$ is formulated as an open problem. The main goal of the present paper is to answer the open question. To this aim we introduce an entirely constructive and deterministic approach to Schäffer's conjecture that avoids the hard track through power sum theory. While previous contributions focus on the existence of *spectra* (without reference to explicit matrix representations) in this article we systematically determine the optimal class of operators for the study of $\mathcal{S}$. Computing explicit matrix representations we present a sequence of *Toeplitz matrices* $T_\lambda \in \mathcal{M}_n$ with *singleton spectrum* $\{\lambda\} \in \mathbb{D}\backslash\{0\}$ such that

$$|\lambda|^n \left\|T_\lambda^{-1}\right\| \geq c(\lambda)\sqrt{n}\,\|T_\lambda\|^{\,n-1},$$

see Theorem 8. It should also be mentioned that in [GMP, Q1, Q2] the asymptotic growth of order $\sqrt{n}$ is achieved when both $n \to \infty$ and the spectrum approaches the unit circle $\partial\mathbb{D}$, whereas in our construction growth of order $\sqrt{n}$ is achieved for any fixed singleton spectrum $\{\lambda\}$. In other words the dependency of the spectrum on n is removed. To put our work in perspective within the research in this area over the last 25 years, we highlight the following points.

(1) Our construction of T_λ answers an open problem raised in 1991 [GMP] and in 1995 [Q1, question 5].
(2) In 2006 N. K. Nikolski presented an upper/lower estimate on *analytic capacities* for subsets of $\mathbb{D}$ of cardinality n in the so-called *Wiener algebra* W of absolutely converging Fourier series [N1, p. 665], see Section 3 for the definition of W. The distance between the two bounds being of order $\sqrt{n}$, Nikolski asked for a better estimate. Our approach answers his question: we find that the upper bound is actually sharp, even for subsets at a fixed distance from the boundary, see Section 5, Remark 13 for details.
(3) The methods developed in Section 5 allow us to answer another open question of Queffélec [Q1, question 3] about the asymptotic behavior of the Fourier coefficients of the power of a Blaschke factor. The corresponding result appeared in a separate publication [SZ1].

Outline of the paper. The paper is organized as follows. In Section 2 we recall Gluskin–Meyer–Pajor's purely analytic description of $\mathcal{S}$. Then we describe Bourgain's approach to estimate $\mathcal{S}$ from below.

In Section 3 we prove our analogue (3.2) of Bourgain's lower estimate (2.2) on $\mathcal{S}$. We compare it with Bourgain's original result and show how it allows us to circumvent Turán's problem. Our lower estimate (3.2) relates Schäffer's question to the study of the asymptotic behavior of the Fourier coefficients of powers of a Blaschke factor, see Lemma 1. The proof

of Lemma 1 is technical and is postponed to Section 6.
In Section 4 we use an interpolation-theoretic approach to determine a class of "worst" operators that maximizes the quantity

$$\frac{|\det T|\,\|T^{-1}\|}{\|T\|^{\,n-1}}$$

for $T \in \mathcal{M}_n$ with given spectrum. We end Section 4 by stating Theorem 8 which exhibits our counterexamples T_λ.
Section 5 lists a series of remarks to give more background on our work. We conclude the article with Section 6, which is concerned with a self-contained analysis of the asymptotic growths of $\left\|(1-z^2)(\frac{z-\lambda}{1-\bar{\lambda}z})^n\right\|_{l^A_\infty}$ as n gets large.

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2. Gluskin–Meyer–Pajor's approach to Schäffer's problem and Bourgain's trick

Gluskin, Meyer and Pajor [GMP] gave an analytic expression for $\mathcal{S}$ in terms of a "max-min-type" optimization problem, which we shall discuss in detail in the main body of the paper. Speaking briefly, $\mathcal{S}$ can be written as

$$\mathcal{S} = \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{D}^n} \phi\left(\lambda_1,\dots,\lambda_n\right), \tag{2.1}$$

where $\mathbb{D}$ is the open unit disk and ϕ is given by

$$\phi\left(\lambda_1,\dots,\lambda_n\right) := \inf\left\{\sum_{k=1}^{\infty}|a_k| \;\middle|\; f(z)=\prod_{i=1}^{n}\lambda_i+\sum_{k=1}^{\infty}a_k z^k,\ f(\lambda_i)=0,\ i=1\dots n\right\}.$$

Notice that in this expression the optimization over operator norms has disappeared. As we will discuss later $(\lambda_1,\dots,\lambda_n)$ can be interpreted as the spectra of a sequence of operators $T=T(n)\in\mathcal{M}_n$ with $\phi\left(\lambda_1,\dots,\lambda_n\right)=\|(\det T)\cdot T^{-1}\|$. Any choice of sequence $(\lambda_1,\dots,\lambda_n)$ provides a lower bound to the supremum in (2.1). Thus to show that $\mathcal{S}$ grows unboundedly Gluskin–Meyer–Pajor employed a probabilistic method establishing the existence of a sequence $(\lambda_1,\dots,\lambda_n)$ with

$$\phi(\lambda_1,\dots,\lambda_n)\geq c_1\sqrt{\frac{n}{\log n}}\frac{1}{\log\log n},\qquad c_1>0.$$

The argument was refined by a short and elegant computation of Bourgain, see [GMP, proof of Theorem 5], that yields

$$\phi(\lambda_1,\dots,\lambda_n)\geq \frac{n\prod_{i=1}^{n}|\lambda_i|}{\max_{k\geq1}\left|\sum_{i=1}^{n}\lambda_i^k\right|}-\prod_{i=1}^{n}|\lambda_i|. \tag{2.2}$$

The key to obtain a lower bound on this expression lies in finding $(\lambda_1,\dots,\lambda_n)$ such that $\max_{k\geq1}\left|\sum_{i=1}^{n}\lambda_i^k\right|$ remains bounded by $\sqrt{n}$. In essence this is Turán's tenth problem, which to date has no constructive solution [A1, T]. Moreover $(\lambda_1,\dots,\lambda_n)$ must depend on n or else

$\prod_{i=1}^n |\lambda_i|$ would decay exponentially. Bourgain established existence of suitable $(\lambda_1, ..., \lambda_n)$ by a probabilistic argument and thereby proved that

$$\phi(\lambda_1, ..., \lambda_n) \geq c_2 \sqrt{\frac{n}{\log n}}, \qquad c_2 > 0.$$

The currently strongest estimates are due to Queffélec [Q2] and build on the above inequality by Bourgain. Queffélec uses a number theoretic method of H. Montgomery [M, Example 6, p. 101] to approach the power sum problem: first he shows that $(\lambda_1, ..., \lambda_n)$ can be chosen so that

$$\phi(\lambda_1, ..., \lambda_n) \geq \sqrt{\frac{n}{e}}, \qquad n = p - 1 \text{ and } p \text{ prime},$$

and concludes

$$\mathcal{S} \geq \sqrt{\frac{n}{2e}}, \qquad \mathcal{S} \geq \sqrt{n}(1 - \mathcal{O}(1/n)).$$

Gluskin–Meyer–Pajor explicitly mention [GMP, p. 2] that they do not know a concrete example of $(\lambda_1, \ldots, \lambda_n)$ for which $\phi(\lambda_1, ..., \lambda_n)$ is growing. For p prime and $n = p - 1$ the example of Queffélec cannot be made explicit even assuming the generalized Riemann hypothesis [Q1, Remark 4.7] [Q2, Remark].

The main contribution of this article may be viewed as a new approach to estimate $\phi(\lambda_1, ..., \lambda_n)$ from below that is not related to Bourgain's work. As a consequence

(1) we find that the trivial choice of fixed singleton spectrum $\lambda_1 = \cdots = \lambda_n = \lambda \in \mathbb{D} \setminus \{0\}$ suffices to prove that $\phi(\lambda, ..., \lambda)$ grows like $\sqrt{n}$ and we circumvent Turán's problem. This answers the open problem raised in [GMP] and
(2) we provide the first explicit class of counterexamples to Schäffer's conjecture: a sequence of invertible Toeplitz matrices $T_\lambda \in \mathcal{M}_n$ with singleton spectrum $\{\lambda\}$ and

$$|\lambda|^n \|T_\lambda^{-1}\| \geq c(\lambda) \sqrt{n} \, \|T_\lambda\|^{n-1},$$

see Theorem 8 for details. This answers [Q1, question 5].

3. A constructive method to estimate $\mathcal{S}$ from below

Before presenting in detail a model-theoretic approach to Schäffer's conjecture we begin by introducing a simple method to bound ϕ from below. Let $Hol(\mathbb{D})$ be the set of holomorphic functions on $\mathbb{D}$ and let $L^2(\partial\mathbb{D})$ be the usual L^2 space of the boundary $\partial\mathbb{D}$ equipped with the standard scalar product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(e^{i\varphi}) \overline{g(e^{i\varphi})} \frac{\mathrm{d}\varphi}{2\pi},$$

see [N2] for details. The Wiener algebra is the subset of $Hol(\mathbb{D})$ of absolutely convergent Fourier series,

$$W := \{f = \sum_{k \geq 0} \hat{f}(k) z^k | \, \|f\|_W := \sum_{k \geq 0} |\hat{f}(k)| < \infty\}.$$

With this notation we can write the Gluskin–Meyer–Pajor expression for ϕ more concisely
(3.1)

$$\phi\left(\lambda_1,\ldots,\lambda_n\right)=\inf\left\{\|h\|_W-|h(0)|\;\Big|\;h\in W,\;h(0)=\prod_{i=1}^{n}\lambda_i,\;h(\lambda_i)=0,\;i=1\ldots n\right\}.$$

For $f=\sum_k\hat{f}(k)z^k,\;g=\sum_k\hat{g}(k)z^k\in Hol(\mathbb{D})$ it is well known that the $L^2(\partial\mathbb{D})$ scalar product can be written as

$$\langle f,\,g\rangle=\sum_{j\geq 0}\hat{f}(j)\overline{\hat{g}(j)}.$$

To bound ϕ we will apply Hölder's inequality in the form

$$|\,\langle f,\,g\rangle\,|\leq\|f\|_{l^A_\infty}\|g\|_W\,,$$

where $\|f\|_{l^A_\infty}:=\sup_{k\geq 0}|\hat{f}(k)|$. Let $B(z)=\prod_{i=1}^{n}\frac{z-\lambda_i}{1-\bar{\lambda}_i z}$ denote the finite Blaschke product whose zeros are $\lambda_1,\ldots,\lambda_n$. B maps the unit disk onto itself and satisfies $\overline{B(z)}=\frac{1}{B(z)}$ for $z\in\partial\mathbb{D}$, [G]. It is easily verified that if f is in W and $f(\lambda)=0$ with $\lambda\in\mathbb{D}$ then $\left\|\frac{f}{z-\lambda}\right\|_W\leq\frac{\|f\|_W}{1-|\lambda|}$. Hence, $\frac{f}{z-\lambda}$ is also in W. This is sometimes called the division-property of W, see for instance [AZ, p. 22] for more general algebras satisfying this property. Therefore for any $h\in W$ with $h(\lambda_i)=0$ we have $\frac{h}{B}\in W$. We are ready to bound ϕ from below. Let $h\in W$ with $h(\lambda_i)=0$ and $h(0)=\prod_{i=1}^{n}\lambda_i$ and let $g=\frac{h}{B}\in W$. We have

$$\begin{aligned}\left\langle z^2h\middle|(1-z^2)B\right\rangle&=\left\langle(z^2-1)h\middle|B\right\rangle\\&=\left\langle(z^2-1)g\middle|1\right\rangle\\&=-g(0)=(-1)^{n-1}.\end{aligned}$$

Applying Hölder's inequality and observing that $\|z^2h\|_W=\|h\|_W$ we conclude that

$$\begin{aligned}1&\leq\left\|z^2h\right\|_W\left\|(1-z^2)B\right\|_{l^A_\infty}\\&=\|h\|_W\left\|(1-z^2)B\right\|_{l^A_\infty}.\end{aligned}$$

It follows that any candidate function h in the definition of ϕ satisfies

$$\|h\|_W\geq\frac{1}{\|(1-z^2)B\|_{l^A_\infty}}$$

and consequently

$$\phi\left(\lambda_1,\ldots,\lambda_n\right)\geq\frac{1}{\|(1-z^2)B\|_{l^A_\infty}}-\prod_{i=1}^{n}|\lambda_i|. \tag{3.2}$$

This is our analogue of Bourgain's lower estimate to $\phi\left(\lambda_1,\ldots,\lambda_n\right)$. It relates Schäffer's problem to a well-defined question in asymptotic analysis. The task is to determine the asymptotic n-dependence of the Fourier coefficient of $(1-z^2)B$ with slowest decay. We have developed the tools for this in a previous article [SZ2]. Of course the question about the "right" eigenvalues remains but as we will find the trivial choice $\lambda_1=...=\lambda_n=\lambda\in(0,1)$ already reaches $\|(1-z^2)B\|_{l^A_\infty}\leq K(\lambda)\frac{1}{\sqrt{n}}$.

Lemma 1. *Given* $\lambda \in (0,1)$ *there is* $K = K(\lambda) > 0$ *such that*

$$\left\|(1-z^2)\left(\frac{z-\lambda}{1-\lambda z}\right)^n\right\|_{l^A_\infty} \leq K\frac{1}{\sqrt{n}},$$

for all $n \geq 1$.

The asymptotic analysis for the proof of the lemma is conducted in Section 6. We conclude this section with a constructive lower estimate on $\mathcal{S}$. Theorem 2 is an immediate consequence of Equation (2.1), our lower estimate (3.2) and Lemma 1.

Theorem 2. *Given any fixed* $\lambda \in \mathbb{D}\setminus\{0\}$ *we have*

$$\mathcal{S} \geq \phi(\lambda,\dots,\lambda) \geq c(\lambda)\sqrt{n}$$

where $c(\lambda) > 0$ *depends only on* λ.

Not only does this circumvent Turán's problem, but the estimate holds for any fixed λ. This avoids the n-dependence of the spectrum present in previous lower bounds.

4. An interpolation-theoretic approach to Schäffer's question

This section's goal is to determine a class of "worst" operators that achieve the bound $\phi(\lambda_1,\dots,\lambda_n)$. To this aim we start with a detailed discussion of Equation (2.1) along the lines of Gluskin–Meyer–Pajor.

4.1. Gluskin-Meyer-Pajor's max-min problem.

By homogeneity Schäffer's problem consists in finding the best $\mathcal{S}$ such that

$$\|(\det T)\cdot T^{-1}\| \leq \mathcal{S}$$

holds for any invertible T with $\|T\| \leq 1$. For a given T let $\mathcal{N} = \mathcal{N}(T)$ denote the collection of norms on $\mathbb{C}^n$ such that for the induced norm we have $\|T\| \leq 1$. It is clear that $\mathcal{N}$ is not empty if and only if the set $\{T^l \mid l \geq 0\}$ is bounded. Notice that this does not depend on the operator norm considered. Operators that have this property are commonly called power bounded, i.e. there exists a constant C such that each power of T can be bounded by this constant, $\sup_{l\geq 0}\|T^l\| \leq C$. As a consequence $\mathcal{S}$ can be written as a double supremum [GMP]

$$\mathcal{S} = \sup\left\{\sup\left\{\left\|(\det T)\cdot T^{-1}\right\| \mid \|\cdot\| \in \mathcal{N}(T)\right\} \mid T \text{ is power bounded}\right\}. \tag{4.1}$$

For given T the inner supremum is over all norms such that $\|T\| \leq 1$. The outer supremum is over all T, where power-boundedness is added or else the inner supremum would be over an empty set. Gluskin–Meyer–Pajor continue by proving [GMP, Proposition 2] that if T is power-bounded then the inner supremum can be written in terms of the spectrum $\lambda_1,\dots,\lambda_n$ of T. They show

$$\sup\left\{\left\|(\det T)\cdot T^{-1}\right\| \mid \|\cdot\| \in \mathcal{N}(T)\right\} = \phi(\lambda_1,\dots,\lambda_n).$$

An operator on finite dimensional space is power bounded iff its spectrum is contained in the closed unit disk and no eigenvalues on the boundary carry degeneracy. This reduces the task of bounding $\mathcal{S}$ from below to the "max-min-type" optimization problem stated in Equation (2.1). The problem can be split into two components: *i*) Given $(\lambda_1,\dots,\lambda_n) \in \mathbb{D}^n$

find the least Wiener-norm function h with $h(\lambda_i) = 0$ and $h(0) = 1$, see Equation (3.1). This is a Nevanlinna-Pick interpolation problem in the Wiener algebra W. *ii*) Find a suitable sequence $(\lambda_1, ..., \lambda_n) \in \mathbb{D}^n$. The articles [GMP] and [Q2] focus on the latter leaving the computation of T an open task. Below we explicitly solve *i*) using an operator-theoretic approach in terms of the norm of the so-called model operator. Computing matrix representations of this model will provide us with explicit matrices T that achieve the bound $\phi(\lambda_1, ..., \lambda_n)$.

4.2. **Interpolation and the right class of operators.** We begin by recalling two useful definitions:

Definition 3 (Function algebra)**.** *A unital Banach algebra A continuously embedded into the algebra of all holomorphic functions on the open unit disk is called a function algebra, if*

- *A contains all polynomials and* $\lim_{n\to\infty} \|z^n\|_A^{1/n} = 1$.
- $(a \in A, \lambda \in \mathbb{D}, a(\lambda) = 0) \Rightarrow \frac{a}{z-\lambda} \in A$.

Definition 4 (Functional calculus)**.** *Let $X : B \to B$ be an operator on a Banach space B. A bounded algebra homomorphism from a function algebra A into the set of linear operators on B*

$$J_X : A \to L(B),$$

is called a functional calculus for X, if it satisfies $J_X(z) = X$ and $J_X(1) = 1$.

Intuitively J_X captures the notion of "plugging an operator into a function", that is for $a \in A$ we have $a(X) = J_X(a)$ and by the boundedness property there is a constant C_X such that

$$\|a(X)\| \leq C_X \|a\|_A .$$

Let $m = \prod_{i=1}^{|m|}(z - \lambda_i)$ be a polynomial of degree $|m|$ with zeros $\lambda_1, ..., \lambda_{|m|}$ in $\mathbb{D}$. The Blaschke product associated with m is

$$B = \prod_{i=1}^{|m|} b_{\lambda_i}, \qquad b_\lambda = \frac{z - \lambda}{1 - \bar{\lambda} z}$$

and has numerator m. The $|m|$-dimensional *model space* K_B (for W, see [N2]) is defined as the quotient vector space

$$K_B = W/BW,$$

where $BW := \{Bf \mid f \in W\}$. W/BW inherits the Banach algebra properties from W and the norm on K_B is defined as

$$\|a\|_{K_B} := \|a\|_{W/BW} := \inf\{\|f\|_W \mid f = a + mg,\ g \in W\}.$$

We denote by S the multiplication operator by z on W

$$\begin{aligned} S : W &\to W \\ f &\mapsto S(f) = zf. \end{aligned}$$

The *model operator* M_S is "the compression" of S to the model space

$$M_S : \Big(K_B,\, \|\cdot\|_{W/BW}\Big) \to \Big(K_B,\, \|\cdot\|_{W/BW}\Big)$$
$$f \mapsto zf.$$

We will also use the operator norm $\|M_S\| := \|M_S\|_{W/BW \to W/BW}$. As W/BW is an algebra it follows that multiplication by z is an operator on W/BW. It is known [N2] that the minimal polynomial of M_S is equal to the numerator of B and that $\|M_S\| \le 1$.

Interpolation problems in function algebras have been studied in detail in the literature. For us the most interesting result is an extension of the Nagy-Foiaş commutant lifting aproach to interpolation theory [FF, NF, NF1] to general function algebras by N. K. Nikolski [N1, Theorem 3.4]. For completeness the result is stated for general function algebras A (see [N1]) but we will use it only for W.

Lemma 5. [N1, Theorem 3.4] *Let m be a monic polynomial, B the Blaschke product associated with m, A a function algebra and $C \ge 1$. We have for $a \in A$ that*

$$\|a\|_{A/BA} \le \sup \|a(T)\| \le C\, \|a\|_{A/BA}\,,$$

where the supremum is taken over all algebraic operators T with minimal polynomial m obeying an A functional calculus with constant C.

Proof. We begin with the upper bound. If $a \in A$ and T admits an A functional calculus with constant C then we can bound

$$\|a(T)\| \le C\, \|a\|_A\,.$$

By definition m is the monic polynomial of least degree such that $m(T) = 0$. Therefore we have $(a + mg)(T) = a(T)$ for any function $g \in A$. Together with the functional calculus inequality we conclude

$$\|a(T)\| \le C \inf\{\|f\|_A \mid f = a + mg,\ g \in A\}.$$

We proceed with the lower estimate. We show that the inequality is achieved by the model operator M_S acting on $K_B = A/BA$. First observe that M_S is annihilated by m: Denoting by $\pi(f)$ the equivalence class in K_B of $f \in A$, we have

$$m(M_S)(\pi(f)) = \pi(m \cdot f) = 0.$$

Second M_S obeys an A functional calculus with constant 1, since it is a compression of the multiplication operator S whose norm is 1. Finally since A is a unital algebra,

$$\|a(M_S)\| \ge \|a(M_S)(\pi(1))\|_{K_B} = \|\pi(a)\|_{K_B} = \|a\|_{A/BA}\,.$$

□

Remark 6. *The lemma is limited to holomorphic functions but here we are interested in the inverse. The trick that extends the lemma to rational functions ψ was provided in* [Sz]. *Suppose ψ has a set of poles $\{\xi_i\}_{i=1}^p$ distinct from the zeros of m. One can apply Lemma 5 to the polynomial*

$$a(z) = \psi \prod_{i=1}^{p} \left(\frac{m(\xi_i) - m(z)}{m(\xi_i)} \right),$$

where all singularities are removed. By definition one has $a(T) = \psi(T)$.

This shows how the interpolation problem (3.1) is related to the model operator M_S. We choose $\psi = 1/z$, $A = W$ and we apply Lemma 5 to the above a. We get

$$\begin{aligned}\left\|M_S^{-1}\right\| = \|a\|_{W/BW} &= \inf\left\{\|f\|_W \mid f \in W,\ f(\lambda_i) = \lambda_i^{-1}\right\} \\ &= \inf\left\{\|h\|_W \mid h \in W,\ h(\lambda_i) = 0,\ h(0) = 1\right\} - 1,\end{aligned}$$

where we set $h(z) = zf(z) - 1$. Multiplying by $|\det(M_S)| = \prod_{i=1}^{|m|} |\lambda_i|$ and comparing to (3.1) we find the following lemma.

Lemma 7. *In the notation introduced above we have that*

$$\left\|\det(M_S) M_S^{-1}\right\| = \phi_{|m|}\left(\lambda_1, \ldots, \lambda_{|m|}\right).$$

This representation provides an explicit class of operators, which are optimal for the computation of $\phi_{|m|}\left(\lambda_1, \ldots, \lambda_{|m|}\right)$. This way the inner supremum in the representation of $\mathcal{S}$ is covered. The remaining supremum over sequences $(\lambda_1, ..., \lambda_n)$ corresponds to finding a suitable sequence of eigenvalues of M_S. The construction of such sequences is a crucial step in [GMP, Q2], where one might expect the hard work also in this article. However, we have already seen in Theorem 2 that the simple choice of the singleton spectrum $\{\lambda\}$ suffices to prove that $\phi(\lambda_1, ..., \lambda_n)$ grows like $\sqrt{n}$. From a theoretical point of view the proof of Lemma 7 is more interesting than its content. We wrote it out for $\psi = 1/z$ but the method works for any rational function ψ, which generalizes Lemma 5 to rational function. The theoretical consequence is that the Nagy-Foiaş commutant lifting approach to interpolation theory is not limited to holomorphic functions but is also suitable for interpolation with rational functions. See [Sz] for Lemma 5 written out for general rational functions.

To conclude, let us make Theorem 2 entirely explicit providing a sequence of Toeplitz matrices $T_\lambda \in \mathcal{M}_n$ that achieves the bound $\phi(\lambda, \ldots, \lambda)$. Given $\lambda \in \mathbb{D}\backslash\{0\}$ and $n \geq 1$ let E be the $n-$dimensional Banach space of rational functions of degree at most n whose poles are located at $1/\bar{\lambda}$, equipped with the norm $\|\cdot\|_{l^A_\infty}$. The space E coincides with the model space associated with the finite Blaschke product $\left(\frac{z-\lambda}{1-\bar{\lambda}z}\right)^n$. A natural orthonormal basis for E (with respect to the scalar product $\langle\cdot,\cdot\rangle$) is the Malmquist-Walsh basis $\mathcal{B} = \{e_j\}_{j=1,\ldots,n}$ given by ([N3, p. 117])

$$e_j(z) := \frac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda}z}\left(\frac{z-\lambda}{1-\bar{\lambda}z}\right)^{j-1}, \qquad j = 1 \ldots n.$$

Theorem 8. *For any fixed $\lambda \in \mathbb{D} \setminus \{0\}$ the $n \times n$ Toeplitz matrix*

$$T_\lambda = \begin{pmatrix} \lambda & 1-|\lambda|^2 & -\bar{\lambda}(1-|\lambda|^2) & \dots & (-\bar{\lambda})^{n-2}(1-|\lambda|^2) \\ 0 & \lambda & 1-|\lambda|^2 & \ddots & \vdots \\ 0 & \ddots & \lambda & \ddots & -\bar{\lambda}(1-|\lambda|^2) \\ \vdots & \ddots & \ddots & \ddots & 1-|\lambda|^2 \\ 0 & \dots & 0 & 0 & \lambda \end{pmatrix}$$

acting on $\left(E, \|\cdot\|_{l^A_\infty}\right)$ with respect to the basis $\mathcal{B}$ is an explicit counterexample to Schäffer's conjecture: It satisfies $\|T_\lambda\|_* \leq 1$,

$$\left\|\det(T_\lambda)T_\lambda^{-1}\right\|_* = \phi(\lambda, \dots, \lambda) \tag{4.2}$$

and

$$\mathcal{S} \geq \left\|\det(T_\lambda)T_\lambda^{-1}\right\|_* \geq c(\lambda)\sqrt{n},$$

where $\|\cdot\|_*$ is the operator norm induced by

$$\|R\|_{l^A_\infty} := \|\sum_{j=1}^n x_j e_j\|_{l^A_\infty}, \qquad R \in E,\ (x_j)_{j=1}^n \in \mathbb{C}^n.$$

Remark 9. *The norm $\|\cdot\|_{W/BW}$ is difficult to compute explicitly due the infimum in its definition. It is therefore not immediate how to obtain an explicit bound from the left hand side of Lemma 7. An exact computation is achieved in Theorem 8, where the optimization problem has been solved. Similar to Section 3, the idea to avoid the difficulty is based on the fact that the dual space of W/BW is identified with the annihilator of BW. We switch to the dual space using the transpose of M_S and a standard result* [R, Theorem 4.9], *see the proof below.*

The lower estimate of Theorem 2 can be viewed as a simple way to bound $\left\|\det(T_\lambda)T_\lambda^{-1}\right\|_*$ from below. The matrix $\det(T_\lambda)T_\lambda^{-1}$ is computed explicitly in [Sz, Theorem III.2]. Computing $\left\|\det(T_\lambda)T_\lambda^{-1}\right\|_*$ is equivalent to optimizing over possible test functions in Hölder's inequality, see Remark 10. In light of the tools developed in Section 6, it is easy to check that the simple test vector $X_0 = (0, \dots, 0, -1, 0, 1)$ (i.e. the rational function $R(z) = e_n(z) - e_{n-2}(z)$) already achieves the estimate in Theorem 2 (which can be viewed as an immediate consequence of Theorem 8):

$$\|T_\lambda^{-1} \cdot {}^T X_0\|_{l^A_\infty} \geq c'(\lambda) \cdot |\lambda|^{-n}\sqrt{n} \cdot \|{}^T X_0\|_{l^A_\infty}$$

where ${}^T X_0$ is the transpose of X_0 and $c'(\lambda) > 0$ depends only on λ. Let us finally mention that formula (4.2) for $\phi(\lambda, \dots, \lambda)$ can be generalized to $\phi(\lambda_1, \dots, \lambda_n)$ for any sequence $(\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$, see Remark 12 in Section 5.

4.3. **Proof of Theorem 8.** We prove Theorem 8 by introducing a scalar product and a natural orthonormal basis on K_B and by computing the matrix entries of M_S in this basis.

Proof of Theorem 8. Let $H^2 \subset L^2(\partial\mathbb{D})$ denote the standard Hardy space of the boundary $\partial\mathbb{D}$, see [N2] for details. A well-known orthonormal basis for the space $H^2/BH^2 \cong H^2 \ominus BH^2$ is the the Malmquist-Walsh basis $\{e_j\}_{j=1,\dots,|m|}$ given by ([N3, p. 117])

$$e_j(z) := \frac{(1-|\lambda_j|^2)^{1/2}}{1-\bar{\lambda}_j z}\prod_{i=1}^{j-1}\frac{z-\lambda_i}{1-\bar{\lambda}_i z}.$$

The empty product is defined to be 1 i.e. $e_1(z) = \frac{(1-|\lambda_1|^2)^{1/2}}{1-\bar{\lambda}_1 z}$. Making use of the fact that $W \subset H^2$ and that rational functions are contained in W it is not hard to see that the equivalence classes $[e_j] := \{e_j + Bg \,|\, g \in W\}$ with $j = 1, ..., |m|$ constitute an orthonormal basis of W/BW (with respect to the scalar product inherited from $L^2(\partial\mathbb{D})$). We introduce a norm $\|\cdot\|$ on $\mathbb{C}^{|m|} \cong W/BW$ by

$$\|x\| := \|\sum_{j=1}^{|m|} x_j e_j\|_{W/BW} = \inf\{\|\sum_{j=1}^{|m|} x_j e_j + Bg\|_W :\ g \in W\}$$

and we denote by $\|\cdot\|$ also the matrix norm induced by $\|\cdot\|$. The entries of M_S with respect to $\{e_j\}_{j=1,\dots,|m|}$ have been computed in [Sz, Proposition III.5]. We denote by $\hat{M}_S$ this matrix representation. Due to Lemma 7 we have

$$\phi_{|m|}\left(\lambda_1, \dots, \lambda_{|m|}\right) = \left\|\det(\hat{M}_S)(\hat{M}_S)^{-1}\right\|.$$

Applying a standard result [R, Theorem 4.9] (with $X = W$ and $M = BW$) we identify via an isometric isomorphism the dual of the quotient space W/BW with the annihilator of BW. The latter is the subspace of W' that contains all functionals that annihilate BW and it can be identified with $H^2 \ominus BH^2 = \mathrm{span}\{e_j \,|\, j = 1, ..., |m|\}$. For $\lambda_1 = \lambda_2 = \cdots = \lambda_{|m|} = \lambda$ the Toeplitz matrix T_λ is the transpose of $\hat{M}_S$. A brief inspection of $\phi_{|m|}(\lambda, \dots, \lambda)$ shows that this quantity does not depend on the argument of λ, so it suffices to consider $\lambda \in (0, 1)$. Thus for the dual operator M_S^* – whose matrix representation is $\hat{M}_S^* = \overline{T_\lambda}$ – we have

$$\left\|\det(\hat{M}_S)(\hat{M}_S)^{-1}\right\| = \left\|\det(\overline{T}_\lambda)(\overline{T}_\lambda)^{-1}\right\|_* = \left\|\det(T_\lambda)(T_\lambda)^{-1}\right\|_*,$$

where $\|\cdot\|_*$ is the matrix norm induced by

$$\|x\|_* := \|\sum_{j=1}^{|m|} x_j e_j\|_{l^A_\infty}.$$

□

5. Additional Remarks

Remark 10. *The conceptual insight leading to our lower estimate in Theorem 2 lies in the recognition of Hölder duality between the norm on W and the one on l^A_∞. Applying*

Hölder's inequality to a test function t, $t(\lambda_i) = 0$ we conclude that any candidate function h satisfies

$$\|h\|_W \geq \frac{1}{\|t\|_{l^A_\infty}}.$$

We use $t = (1/z^2 - 1)B$ with $B(z) = \left(\frac{z-\lambda}{1-\lambda z}\right)^n$, but other choices are possible. The simpler test function $t = B$ leads to $\|h\|_W \geq 1/\|B\|_{l^A_\infty}$ but in the previous work [SZ1] *we have shown that if $\lambda_1 = ... = \lambda_n = \lambda \in (0,1)$ then $\|B\|_{l^A_\infty}$ decays as $\mathcal{O}(n^{-1/3})$ as $n \to \infty$. The present test function is a simple modification that yields an estimate in terms of $\|(1-z^2)B\|_{l^A_\infty}$, which decays as $n^{-1/2}$. As a direct consequence of the Hahn-Banach theorem the approach via Hölder duality is sharp.*

Remark 11. *The asymptotic decay rate of $\|B\|_{l^A_\infty}$ for $B(z) = \left(\frac{z-\lambda}{1-\lambda z}\right)^n$ is $\mathcal{O}(n^{-1/3})$. The k^{th}-Fourier coefficient of $(1-z^2)B$ is $\hat{B}(k) - \hat{B}(k-2)$ for $k \geq 2$ so that the factor $1-z^2$ potentially speeds up the decay of Fourier coefficients. In* [SZ1] *it is shown that the decay rate of $|\hat{B}(k)|$ is $\mathcal{O}(n^{-1/3})$ for sequences $k = k(n)$ such that k/n approaches the critical values $\frac{1-\lambda}{1+\lambda}$ or $\frac{1+\lambda}{1-\lambda}$. We prove in Section 6 that in this situation $\widehat{(1-z^2)B}(k)$ decays as $\mathcal{O}(n^{-2/3})$. Multiplying B by $1-z$ instead of $1-z^2$ would not lead to a quicker decay of the l^A_∞-norm as the factor $1-z$ speeds up the decay at the right boundary $\frac{1+\lambda}{1-\lambda}$ but not at the left one $\frac{1-\lambda}{1+\lambda}$.*

Remark 12. *Given an arbitrary sequence $(\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$ it follows from Lemma 7, the proof of Theorem 8 and* [Sz, Theorem III.2] *that*

$$\phi(\lambda_1, \ldots, \lambda_n) = \left\| \det(\hat{M}_{(\lambda_1,\ldots,\lambda_n)}) \hat{M}^{-1}_{(\lambda_1,\ldots,\lambda_n)} \right\|_*$$

where $\hat{M}_{(\lambda_1,\ldots,\lambda_n)}$ is the $n \times n$ upper-triangular matrix given entry-wise by

$$\left(\hat{M}_{(\lambda_1,\ldots,\lambda_n)}\right)_{ij} = \begin{cases} (1-|\lambda_i|^2)^{1/2}(1-|\lambda_j|^2)^{1/2} \prod_{\mu=i+1}^{j-1} \left(-\bar{\lambda}_\mu\right) & if\ i < j \\ \lambda_i & if\ i = j \\ \{0\} & if\ i > j, \end{cases}$$

and $\|\cdot\|_*$ is defined at the end of Section 4. This generalizes formula (4.2) *from Theorem 8.*

Remark 13. *In* [N1] *Nikolski introduced analytic capacities of finite sequences of $\mathbb{D}$ in function spaces. Given a finite sequence $\sigma = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$ Nikolski defines the W-capacity of σ as*

$$\begin{aligned} \mathrm{cap}_W(\sigma) &= \frac{1}{\prod_{i=1}^n |\lambda_i|} \phi(\lambda_1, \ldots, \lambda_n) \\ &= \inf\left\{\|f\|_W :\ f \in W,\ f(0) = 1,\ f(\lambda_i) = 0,\ i = 1 \ldots n\right\}, \end{aligned}$$

and given $\Delta > 1$ the quantities

$$\mathcal{K}_n(\Delta, W) = \sup\left\{\mathrm{cap}_W(\sigma) :\ \sigma \in \mathbb{D}^s,\ s \leq n,\ |\lambda| \geq \Delta^{-1}\ \forall \lambda \in \sigma\right\},$$

and

$$\underline{\mathcal{K}}_n(\Delta, W) = \sup\left\{\mathrm{cap}_W(\sigma) :\ \sigma \in \mathbb{D}^s,\ s \leq n,\ |\lambda| = \Delta^{-1}\ \forall \lambda \in \sigma\right\}.$$

As a consequence of [N1, Corollary 3.23] *he obtains* [N1, p. 665]

$$\Delta^n \le \underline{\mathcal{K}}_n(\Delta, W) \le \mathcal{K}_n(\Delta, W) \le \sqrt{en}\Delta^n$$

and asks for a better estimate. Choosing $s = n$ *and* $\lambda_1 = \cdots = \lambda_n = \frac{1}{\Delta}$ *Theorem 2 yields*

$$\underline{\mathcal{K}}_n(\Delta, W) \ge c(\lambda)\sqrt{n}\Delta^n,$$

which answers the question to the negative. The upper estimate on $\mathcal{K}_n$ *must grow at least like* $\sqrt{n}$*.*

6. Asymptotic analysis: On the l^A_∞−norm of $(1 - z^2)b^n_\lambda$

In this section we determine the asymptotic behavior of $\|(1 - z^2)b^n_\lambda\|_{l^A_\infty}$, where $b_\lambda = \frac{z-\lambda}{1-\lambda z}$ and $\lambda \in (0, 1)$. Recall the contour integral representation of Fourier coefficients

$$\widehat{(1 - z^2)b^n_\lambda}(k) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} (1 - z^2)b^n_\lambda(z)z^{-k}\frac{\mathrm{d}z}{z}.$$

From this representation it is immediate that one can split

$$\widehat{(1 - z^2)b^n_\lambda}(k) = \widehat{b^n_\lambda}(k) - \widehat{b^n_\lambda}(k - 2), \qquad k \ge 2.$$

In previous works [SZ1, SZ2] we developed the necessary tools from asymptotic analysis and we determined the asymptotic growth of the Taylor coefficients $\widehat{b^n_\lambda}(k)$ both with respect to k and n. Holomorphy of b^n_λ implies that for any fixed n the coefficients $\widehat{b^n_\lambda}(k)$ decay exponentially when k grows large. In a similar vein for any fixed value of k the coefficients $\widehat{b^n_\lambda}(k)$ decay exponentially in n. The interesting behavior, which is relevant for the l^A_∞-norm, therefore occurs when $k = k(n)$ is a sequence. More precisely, setting $\alpha_0 := \frac{1-\lambda}{1+\lambda}$, Proposition 2 of [SZ1] can be summarized as follows:

(1) if $\alpha \in (0, \alpha_0)$ and $k \notin [\alpha n,\ \alpha^{-1}n]$ then $\widehat{b^n_\lambda}(k)$ decays exponentially in n,
(2) if $\beta \in (\alpha_0, 1)$ and $k \in [\beta n,\ \beta^{-1}n]$ then $\widehat{b^n_\lambda}(k) = \mathcal{O}\left(n^{-1/2}\right)$,
(3) if $k = \lfloor \alpha_0 n \rfloor$ or $k = \lfloor \alpha_0^{-1} n \rfloor$ then $\widehat{b^n_\lambda}(k) = \mathcal{O}\left(n^{-1/3}\right)$.

Point (1) says that $\left[\alpha_0 n,\ \alpha_0^{-1} n\right]$ is a "critical n-dependent interval" for the coefficients k: as long as k remains separate from this interval, $\widehat{b^n_\lambda}(k)$ decays exponentially in n. Point (2) says that strictly inside the "critical interval" $\widehat{b^n_\lambda}(k)$ decays as $\mathcal{O}\left(n^{-1/2}\right)$. The slowest decay of order $\mathcal{O}(n^{-1/3})$ occurs on the boundary, see point (3). An upper bound on the asymptotic growth of $\widehat{(1 - z^2)b^n_\lambda}(k)$ can be obtained by an application of the triangular inequality and points (1) - (3) above. This approach shows that the l^A_∞−norm of $(1-z^2)b^n_\lambda$ is bounded above by $\mathcal{O}(n^{-1/3})$ but fails to provide $\|(1 - z^2)b^n_\lambda\|_{l^A_\infty} = \mathcal{O}(n^{-1/2})$. Building on the methods developed in [SZ1, SZ2] the goal of this section is to determine more precisely the asymptotics of $\widehat{(1 - z^2)b^n_\lambda}(k)$. To this aim we will rely on the *method of stationary phase* [E] for $k \in [\beta n,\ \beta^{-1}n]$ and $\beta \in (\alpha_0, 1)$. When k/n gets close to one of the boundaries $\alpha_0,\ \alpha_0^{-1}$ the situation turns out to be much more delicate: The asymptotic behavior of $\widehat{(1 - z^2)b^n_\lambda}(k)$ is described in terms of the *Airy function* and we rely on a *uniform* version of the method of stationary phase/ steepest descents as is introduced in [CFU].

Our findings are contained in the following proposition and summarized in Table 6.1 below. To formulate our results we introduce some standard notation from asymptotic analysis. For two positive functions $f, g : \mathbb{C} \to \mathbb{R}^+$ we say that f is dominated by g, denoted by $f \lesssim g$, if there is a constant $c > 0$ such that $f \leq cg$.

Proposition 14. *Let $\lambda \in (0,1)$, $b_\lambda = \frac{z-\lambda}{1-\lambda z}$ and $n \geq 1$. Set $\alpha_0 := \frac{1-\lambda}{1+\lambda}$ and choose fixed $\alpha \in (0, \alpha_0)$ and $\beta \in (\alpha_0, 1)$. In the following we consider sequences $k = k(n)$ and all assertions are meant to hold for large enough n.*

(1) *If $k/n \leq \alpha$ then $|\widehat{(1-z^2)b_\lambda^n}(k)|$ decays exponentially and uniformly over k as n tends to ∞. Similarly if $k/n \geq \alpha^{-1}$ then $|\widehat{(1-z^2)b_\lambda^n}(k)|$ decays exponentially and uniformly over k as n tends to ∞.*

(2) *If $k/n \in (\alpha, \alpha_0 - n^{-2/3}] \cup [\alpha_0^{-1} + n^{-2/3}, \alpha^{-1})$ then we have the following asymptotic growth estimate*

$$\left|\widehat{(1-z^2)b_\lambda^n}(k)\right| \lesssim \frac{\left(\min\left((\alpha_0 - k/n),\, (k/n - \alpha_0^{-1})\right)\right)^{1/4}}{n^{1/2}} \cdot \exp\left(-\frac{2}{3}n\left(\min\left((\alpha_0 - k/n),\, (k/n - \alpha_0^{-1})\right)\right)^{3/2}\right) + \mathcal{O}\left(\frac{1}{n}\right).$$

(3) *If $k/n \in [\alpha_0 - n^{-2/3}, \alpha_0 + n^{-2/3}] \cup [\alpha_0^{-1} - n^{-2/3}, \alpha_0^{-1} + n^{-2/3}]$ then*

$$|\widehat{(1-z^2)b_\lambda^n}(k)| \lesssim \frac{1}{n^{2/3}}.$$

(4) *If $k/n \in [\alpha_0 + n^{-2/3}, \beta) \cup (\beta^{-1}, \alpha_0^{-1} - n^{-2/3}]$ then*

$$\left|\widehat{(1-z^2)b_\lambda^n}(k)\right| \lesssim \frac{\left(\frac{k}{n} - \alpha_0\right)^{1/4}\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/4}}{n^{1/2}}.$$

(5) *If $k/n \in [\beta, \beta^{-1}]$ then*

$$\left|\widehat{(1-z^2)b_\lambda^n}(k)\right| \leq (1-\lambda^2)\sqrt{\frac{2}{\pi n}}\frac{(k/n - \alpha_0)^{1/4}(\alpha_0^{-1} - k/n)^{1/4}}{\lambda (k/n)^{3/2}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Before presenting the proof we review the consequences of Proposition 14 for the l^A_∞−norm of $(1-z^2)b_\lambda^n$. We fix α and β and consider sufficiently large n. Due to point (1) of Proposition 14 we have that

$$\sup_{k\geq 0}|\widehat{(1-z^2)b_\lambda^n}(k)| = \sup_{k\in[\alpha n,\, \alpha^{-1}n]}|\widehat{(1-z^2)b_\lambda^n}(k)|.$$

We know from points (2), (3), (4), (5) that the supremum can be bounded from above for large n only by values $k/n \in (\beta, \beta^{-1})$, i.e.

$$\begin{aligned}\left\|(1-z^2)b_\lambda^n\right\|_{l^A_\infty} &\leq \sup_{k\in(\beta n,\, \beta^{-1}n)}|\widehat{(1-z^2)b_\lambda^n}(k)| \\ &\leq (1-\lambda^2)\sqrt{\frac{2}{\pi n}}\frac{(k/n - \alpha_0)^{1/4}(\alpha_0^{-1} - k/n)^{1/4}}{\lambda (k/n)^{3/2}}\left(1 + \mathcal{O}(n^{-1})\right).\end{aligned}$$

Direct computation shows that

$$x^* := \arg\max\left([\beta, \beta^{-1}] \ni x \mapsto \frac{(x-\alpha_0)^{1/4}(\alpha_0^{-1}-x)^{1/4}}{\lambda x^{3/2}}\right)$$
$$= \frac{5\,(1+\lambda^2) - \sqrt{\lambda^4 + 98\,\lambda^2 + 1}}{4(1-\lambda^2)}.$$

It is elementary to check that

$$\max\left([0,1] \ni \lambda \mapsto (1-\lambda^2)\sqrt{\frac{2}{\pi}}\frac{(x^*-\alpha_0)^{1/4}(\alpha_0^{-1}-x^*)^{1/4}}{\lambda {x^*}^{3/2}}\right) \approx 2.800200360.$$

This shows that for sufficiently large n one has $\|(1-z^2)b_\lambda^n\|_{l_\infty^A} \leq K/\sqrt{n}$ with $K < 2.81$, which proves Lemma 1. Consequently

$$\frac{\mathcal{S}}{\sqrt{n}} \geq 0.35\left(1 + \mathcal{O}\left(n^{-1}\right)\right).$$

Remark 15. *The lower estimate $\mathcal{S}/\sqrt{n} \geq 0.35$ is weaker than $\mathcal{S}/\sqrt{n} \geq \sqrt{1/2e}$ of* [Q2]. *We did not attempt to improve the known numerical estimate to $\mathcal{S}/\sqrt{n}$. A reader interested in this topic might rely on Theorem 8: The matrix $\det(T_\lambda){T_\lambda}^{-1}$ is computed explicitly in* [Sz, Theorem III.2]. *For fixed λ the quantity $\left\|\det(T_\lambda){T_\lambda}^{-1}\right\|_*/\sqrt{n}$ can be plotted as a function of n with the support of appropriate software. This provides numerical estimates on $\mathcal{S}/\sqrt{n}$, which are potentially stronger than the one obtained through* (3.2) *and the asymptotic analysis of the l_∞^A-norm of $(1-z^2)b_\lambda^n$.*

The mathematical concepts involved in the asymptotic analysis of the various regions of Proposition 14 turn out to be rather differing. We split the proof of the proposition accordingly and begin with the conceptually most simple region of exponential decay.

Proof of Proposition 14, Point 1). For $z, w \in \mathbb{D}$ the magnitude of the elementary Blaschke factor $\frac{z-w}{1-\bar{w}z}$ can be estimated, see [G], by

$$\frac{|z|-|w|}{1-|z||w|} \leq \left|\frac{z-w}{1-\bar{w}z}\right| \leq \frac{|z|+|w|}{1+|z||w|}.$$

We write $\widehat{b_\lambda^n}(k)$ as a contour integral over a circle of radius $s \in (0, 1/\lambda)$,

$$\widehat{b_\lambda^n}(k) = \frac{1}{2\pi i}\oint_{|z|=s} b_\lambda^n(z) z^{-k-1}\mathrm{d}z.$$

Using above estimates we find for the magnitude of the integral that

$$|\widehat{b_\lambda^n}(k)| \leq \max_{|z|=s}\frac{|b_\lambda^n(z)|}{|z|^k} \leq \begin{cases} \frac{1}{s^k}b_\lambda^n(s), & s \in (1, 1/\lambda) \\ \frac{1}{s^k}\left(\frac{s+\lambda}{1+\lambda s}\right)^n, & s \in (0, 1] \end{cases}. \tag{6.1}$$

These are the key estimates to conduct the asymptotic analysis in the exponential region.

- If $k/n \leq \alpha$ then there exists $s^* \in (\lambda, 1)$ such that

$$\frac{\left(\frac{s^*+\lambda}{1+\lambda s^*}\right)}{{s^*}^{k/n}} \leq \frac{\left(\frac{s^*+\lambda}{1+\lambda s^*}\right)}{{s^*}^{\alpha}} < 1.$$

Values of $k(n)$ in interval	Decay of $\widehat{(1-z^2)b_\lambda^n}(k)$	Region
$[0,\, \alpha n]$	exponential	I
$(\alpha n,\, \alpha_0 n - n^{1/3}]$	$\frac{(\alpha_0-k/n)^{1/4}}{n^{1/2}} \exp\left(-\frac{2}{3}n\left(\alpha_0-k/n\right)^{3/2}\right)$	II
$[\alpha_0 n - n^{1/3},\, \alpha_0 n + n^{1/3}]$	$\frac{1}{n^{2/3}}$	III
$[\alpha_0 n + n^{1/3},\, \alpha_0^{-1} n - n^{1/3}]$	$\frac{\left(\frac{k}{n}-\alpha_0\right)^{1/4}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/4}}{n^{1/2}}$	IV
$[\alpha_0^{-1} n - n^{1/3},\, \alpha_0^{-1} n + n^{1/3}]$	$\frac{1}{n^{2/3}}$	V
$[\alpha_0^{-1} n + n^{1/3},\, \alpha^{-1} n)$	$\frac{\left(\frac{k}{n}-\alpha_0^{-1}\right)^{1/4}}{n^{1/2}} \exp\left(-\frac{2}{3}n\left(\frac{k}{n}-\alpha_0^{-1}\right)^{3/2}\right)$	VI
$[\alpha^{-1} n,\, \infty)$	exponential	VII

TABLE 6.1. Illustration of asymptotic bounds for $|\widehat{(1-z^2)b_\lambda^n}(k)|$ as a function of $k = k(n)$ (up to a multiplicative constant). Here $\alpha_0 = \frac{1-\lambda}{1+\lambda}$ and $\alpha \in (0, \alpha_0)$ arbitrary but fixed.

s^* can be computed explicitly as a function of α and λ and does not depend on k:

$$s^* = \frac{\alpha(1+\lambda^2)-(1-\lambda^2)}{2\lambda\alpha} + \sqrt{\left(\frac{\alpha(1+\lambda^2)-(1-\lambda^2)}{2\lambda\alpha}\right)^2 - 1}.$$

It follows that if $k/n \le \alpha$ then $|\widehat{b_\lambda^n}(k)|$ decays exponentially in n and uniformly over k by the second estimate in (6.1).

- If $k/n \ge \alpha^{-1}$ then there exists $s^* \in (1, 1/\lambda)$ such that

$$\frac{b_\lambda(s^*)}{s^{*k/n}} \le \frac{b_\lambda(s^*)}{s^{*\alpha^{-1}}} < 1.$$

s^* can be computed explicitly as a function of α and λ and does not depend on k:

$$s^* = \frac{\alpha^{-1}(1+\lambda^2)-(1-\lambda^2)}{2\lambda\alpha^{-1}} - \sqrt{\left(\frac{\alpha^{-1}(1+\lambda^2)-(1-\lambda^2)}{2\lambda\alpha^{-1}}\right)^2 - 1}.$$

It follows that if $k/n \ge \alpha^{-1}$ then $|\widehat{b_\lambda^n}(k)|$ decays exponentially in n and uniformly over k by the first estimate in (6.1).

By the triangular inequality it follows that if $k/n \le \alpha$ or $k/n \ge \alpha^{-1}$ then $|\widehat{(1-z^2)b_\lambda^n}(k)|$ decays exponentially and uniformly over k as n tends to ∞. $\square$

We proceed with the asymptotic analysis in the remaining regions. Before providing the reader with proofs a preliminary discussion of the asymptotic behavior of integrals of the form

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} g(z)\exp\left(nf_a(z)\right)\frac{\mathrm{d}z}{z} \tag{6.2}$$

is in order. Notice that with the choices $g(z) = (1 - z^{-2})$ and

$$f_a(z) := \log\left(\frac{z^a(1-\lambda z)}{z-\lambda}\right), \quad a \in \mathbb{R}^+,$$

where log denotes the principal branch of the complex logarithm and $a = k/n$, the integral in (6.2) equals $(\widehat{1 - z^2)b^n_\lambda}(k)$. Determining the asymptotic behavior of such integrals as $n \to +\infty$ is a relatively standard task when f is fixed, see e.g. [W, BH]. For us the situation is slightly more complicated as f_a depends on k and n but still we can rely on existing methodology. It is common that the dominant contribution to integrals of the form (6.2) comes from a small neighborhood around the stationary points of f_a. We begin by identifying those points.

Lemma 16. *Let $f_a(z)$ be as defined above. We have the following assertions.*

1. *If $a \in (\alpha_0, \alpha_0^{-1})$ then $f_a(\cdot)$ has two distinct stationary points $z_\pm \in \partial\mathbb{D}$ of order one, i.e. $\frac{\partial f_a}{\partial z}(z_\pm) = 0$ but $\frac{\partial^2 f_a}{\partial z^2}(z_\pm) \neq 0$, satisfying $z_- = \overline{z_+}$.*
2. *If $a \in \{\alpha_0, \alpha_0^{-1}\}$ then f_a has one stationary point $z_0 \in \{-1, 1\}$ of order two, i.e. $\frac{\partial f_a}{\partial z}(z_0) = \frac{\partial^2 f_a}{\partial z^2}(z_0) = 0$ but $\frac{\partial^3 f_a}{\partial z^3}(z_0) \neq 0$.*
3. *If $a \notin [\alpha_0, \alpha_0^{-1}]$ then f_a has two stationary points $z_\pm \in \mathbb{R}$ of order one, i.e. $\frac{\partial f_a}{\partial z}(z_\pm) = 0$ but $\frac{\partial^2 f_a}{\partial z^2}(z_\pm) \neq 0$, satisfying $z_- = z_+^{-1}$.*

The stationary points z_+ and z_- are given explicitly by

$$z_\pm = \frac{a(1+\lambda^2) - (1-\lambda^2)}{2\lambda a} \pm \sqrt{\left(\frac{a(1+\lambda^2) - (1-\lambda^2)}{2\lambda a}\right)^2 - 1}. \tag{6.3}$$

Proof of Lemma 16. Computing derivatives we confirm

$$\frac{\partial f_a}{\partial z} = -\frac{1}{z-\lambda} + \frac{a}{z} - \frac{\lambda}{1-\lambda z},$$
$$\frac{\partial^2 f_a}{\partial z^2} = \frac{1}{(z-\lambda)^2} - \frac{a}{z^2} - \frac{\lambda^2}{(1-\lambda z)^2},$$
$$\frac{\partial^3 f_a}{\partial z^3} = -\frac{2}{(z-\lambda)^3} + \frac{2a}{z^3} - \frac{2\lambda^3}{(1-\lambda z)^3}.$$

The function $f_a(z)$ has a stationary point if and only if $\partial f_a/\partial z = 0$, i.e. iff

$$a = 1 + \frac{\lambda}{z-\lambda} + \frac{\lambda z}{1-\lambda z}.$$

Solving the latter for z yields the representation (6.3) for the roots $z_\pm$ of $\frac{\partial f_a}{\partial z}$. If $a \notin \{\alpha_0, \alpha_0^{-1}\}$ then z_+ and z_- are distinct. If $a \in (\alpha_0, \alpha_0^{-1})$ then $z_\pm \in \partial\mathbb{D} - \{-1, 1\}$ and if

$a \notin [\alpha_0,\, \alpha_0^{-1}]$ then $z_\pm \in \mathbb{R} - \{-1, 1\}$. Plugging in we see that

$$\left.\frac{\partial^2 f_a}{\partial z^2}\right|_{z=z_\pm} = \frac{(1-\lambda^2)(1-z_\pm^2)\lambda}{z_\pm(z_\pm-\lambda)^2(1-\lambda z_\pm)^2}. \tag{6.4}$$

If $a \in \left\{\alpha_0,\, \alpha_0^{-1}\right\}$ then $\frac{\partial f_a}{\partial z}$ has a unique zero. If $a = \alpha_0^{-1}$ then $z_+ = z_- = 1 = \bar{z}_0$ and

$$f_{\alpha_0^{-1}}(1) = \frac{\partial f_{\alpha_0^{-1}}}{\partial z}(1) = \frac{\partial^2 f_{\alpha_0^{-1}}}{\partial z^2}(1) = 0,$$

with

$$\frac{\partial^3 f_{\alpha_0^{-1}}}{\partial z^3}(1) = -\frac{2\lambda(1+\lambda)}{(1-\lambda)^3} \neq 0.$$

If $a = \alpha_0$ then $z_+ = z_- = -1 = z_0$ and

$$\frac{\partial f_{\alpha_0}}{\partial z}(-1) = \frac{\partial^2 f_{\alpha_0}}{\partial z^2}(-1) = 0, \qquad \frac{\partial^3 f_{\alpha_0}}{\partial z^3}(-1) = -\frac{2\lambda(1-\lambda)}{(1+\lambda)^3} \neq 0.$$

□

The lemma shows that the location of stationary points of f_a in $\mathbb{C}$ is determined by the location of a relative to the critical interval $[\alpha_0, \alpha_0^{-1}]$. As a approaches the boundary the stationary points degenerate. Thus we treat the situations, where a is separate from the boundary and where a approaches the boundary separately. The former scenario corresponds to point *5)* in Proposition 14, i.e. there is a $\beta \in (\alpha_0, 1)$ that separates a from the boundary, $a \in (\beta, \beta^{-1})$. In the second scenario, where a approaches the boundary, the asymptotic behavior depends on the speed at which a approaches the boundary. This is reflected in the points *2), 3), 4)* of Proposition 14. Speaking roughly we employ the following methods to determine the asymptotics.

- If a is separate from the boundary then the stationary points $z_\pm$ of f_a belong to the contour of integration $\partial\mathbb{D}$. Since $\left|z^{k/n}\frac{1-\lambda z}{z-\lambda}\right| = 1$ for any $z \in \partial\mathbb{D}$ we can introduce the real function

  $$h(\varphi) = h_a(\varphi) := -if_a(e^{i\varphi}) \qquad \varphi \in [0, \pi],$$

  to write the integral as a generalized Fourier integral,

  $$\overline{(1-\widehat{z^2)b_\lambda^n}(k)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\varphi})e^{nf_a(e^{i\varphi})}\mathrm{d}\varphi = \frac{1}{\pi}\Re\left\{\int_0^{\pi} g(e^{i\varphi})e^{inh_a(\varphi)}\mathrm{d}\varphi\right\}$$

  To determine the asymptotic behavior of this integral we will rely on the *method of stationary phase* [E].
- When a approaches the critical interval we are faced with coalescing saddle points. To capture the asymptotic behavior we will also take account of the speed at which a approaches the boundary. To achieve this we employ a *uniform extension* of the method of steepest descents as is introduced in [CFU].

Proof of Proposition 14, Point 5). Suppose $a \in [\beta, \beta^{-1}]$. The stationary points of $h = h_a$ are given by (see Lemma 16 point 1))

$$z_{+,-} = \frac{a(1+\lambda^2) - (1-\lambda^2)}{2\lambda a} \pm i\sqrt{1 - \left(\frac{a(1+\lambda^2) - (1-\lambda^2)}{2\lambda a}\right)^2} \in \partial\mathbb{D}$$

and we write $z_{+,-} = e^{i\varphi_{+,-}}$ with $\varphi_+ \in [0,\pi]$ and $\varphi_- \in (-\pi, 0]$. Only z_+ is relevant because we integrate over $[0,\pi]$. For the second derivative we have that

$$ih''(\varphi) = \frac{\partial}{\partial\varphi}\left(\frac{\partial f}{\partial z}\frac{dz}{d\varphi}\right) = \frac{\partial^2 f}{\partial z^2}\left(\frac{dz}{d\varphi}\right)^2 + \frac{\partial f}{\partial z}\frac{d^2 z}{(d\varphi)^2}.$$

It follows from (6.4) that

$$i\frac{\partial^2 h}{\partial\varphi^2}\bigg|_{\varphi=\varphi_+} = -z_+^2 \frac{(1-\lambda^2)(1-z_+^2)\lambda}{z_+(z_+-\lambda)^2(1-\lambda z_+)^2}$$

so that $h''(\varphi_+) > 0$. To determine the asymptotic behavior we apply a standard result of A. Erdélyi [E, Theorem 4], which however requires that the stationary point is an endpoint of the interval of integration. Hence we begin by splitting our generalized Fourier integral

$$\int_0^\pi g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = \int_0^{\varphi_+} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi + \int_{\varphi_+}^\pi g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi,$$

where we slightly overload the notation writing briefly $g(\varphi)$ for $g(e^{i\varphi})$. We study the two integrals individually. For the second integral Theorem 4 of [E] yields

$$\begin{aligned}\int_{\varphi_+}^\pi g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi &= \frac{1}{2}\Gamma(1/2)\kappa_1(0)e^{i\frac{\pi}{4}}n^{-1/2}e^{inh(\varphi_+)} + \frac{1}{2}\Gamma(1)\kappa_1'(0)e^{i\frac{\pi}{2}}n^{-1}e^{inh(\varphi_+)}\\ &\quad - \frac{i}{n}e^{inh(\pi)}\frac{g(\pi)}{h'(\pi)} + \mathcal{O}\left(n^{-3/2}\right),\end{aligned}$$

with

$$\begin{aligned}\kappa_1(0) &= 2^{1/2}g(\varphi_+)\left(h''(\varphi_+)\right)^{-1/2},\\ \kappa_1'(0) &= \frac{2}{h''(\varphi_+)}g'(\varphi_+) - \frac{2}{h''(\varphi_+)}\frac{h^{(3)}(\varphi_+)}{3h''(\varphi_+)}g(\varphi_+).\end{aligned}$$

The analysis of the first integral $\int_0^{\varphi_+} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi$ is essentially the same but we change the variable of integration $\varphi \mapsto -\varphi$ as suggested in [E, p. 23]. We get

$$\int_0^{\varphi_+} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = \int_{-\varphi_+}^0 g(-\varphi)e^{inh(-\varphi)}\mathrm{d}\varphi.$$

Applying Theorem 4 of [E] yields

$$\begin{aligned}\int_{-\varphi_+}^0 g(-\varphi)e^{inh(-\varphi)}\mathrm{d}\varphi &= \frac{1}{2}\Gamma(1/2)\tilde{\kappa}_1(0)e^{i\frac{\pi}{4}}n^{-1/2}e^{inh(\varphi_+)} + \frac{1}{2}\Gamma(1)\tilde{\kappa}_1'(0)e^{i\frac{\pi}{2}}n^{-1}e^{inh(\varphi_+)}\\ &\quad - \frac{i}{n}e^{inh(0)}\frac{g(0)}{h'(0)} + \mathcal{O}\left(n^{-3/2}\right)\end{aligned}$$

with

$$\tilde{\kappa}_1(0) = 2^{1/2} g(\varphi_+) \left(h''(\varphi_+)\right)^{-1/2},$$

$$\tilde{\kappa}_1'(0) = -\frac{2}{h''(\varphi_+)} g'(\varphi_+) + \frac{2}{h''(\varphi_+)} \frac{h^{(3)}(\varphi_+)}{3h''(\varphi_+)} g(\varphi_+).$$

Observing that $h(0) = 0$ while $h(\pi) = (a-1)\pi$, $g(0) = 0$ while $g(\pi) = 0$, and $h'(0) = \frac{(a-1)(1-\lambda)-2\lambda}{1-\lambda}$ while $h'(\pi) = -\frac{(a-1)(1+\lambda)+2\lambda}{1+\lambda}$ we compute

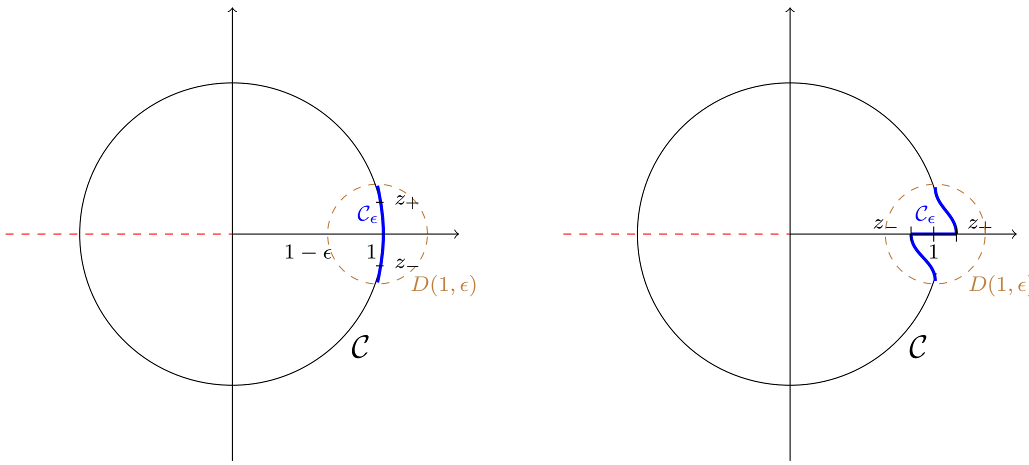
$$\begin{aligned}
\int_0^\pi g(\varphi) e^{inh(\varphi)} \mathrm{d}\varphi &= \Gamma(1/2) \left(2^{1/2} g(\varphi_+) \left(h''(\varphi_+)\right)^{-1/2}\right) e^{i\frac{\pi}{4}} n^{-1/2} e^{inh(\varphi_+)} + \mathcal{O}\left(n^{-3/2}\right) \\
&= e^{inh(\varphi_+) + i\frac{\pi}{4}} (1 - z_-^2) \left(\frac{2|z_+ - \lambda|^4}{n\lambda(1-\lambda^2)|1-z_+^2|}\right)^{1/2} \Gamma(1/2) + \mathcal{O}\left(n^{-3/2}\right) \\
&= e^{inh(\varphi_+) - i\varphi_+ + i\frac{3\pi}{4}} \frac{(1-\lambda^2)(a-\alpha_0)^{1/4}(\alpha_0^{-1} - a)^{1/4}}{\lambda a^{3/2}} \left(\frac{2\pi}{n}\right)^{1/2} + \mathcal{O}\left(n^{-3/2}\right),
\end{aligned}$$

where we made use of $|z_+ - \lambda| = \sqrt{\frac{1-\lambda^2}{a}}$ and $2\Im(z_+) = |z_+^2 - 1| = \frac{(1-\lambda^2)\sqrt{(a-\alpha_0)(\alpha_0^{-1}-a)}}{\lambda a}$. We conclude that

$$\begin{aligned}
&\frac{1}{\pi} \Re \left\{ \int_0^\pi g(\varphi) e^{inh(\varphi)} \mathrm{d}\varphi \right\} \\
&= (1-\lambda^2) \sqrt{\frac{2}{\pi n}} \frac{(a-\alpha_0)^{1/4}(\alpha_0^{-1} - a)^{1/4}}{\lambda a^{3/2}} \cos\left(nh(\varphi_+) - \varphi_+ + \frac{3\pi}{4}\right) \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).
\end{aligned}$$

A key point of this derivation is that in both integrals the $\mathcal{O}$-terms are automatically uniform over $a \in [\beta, \beta^{-1}]$ by the choice of the interval since $\beta > \alpha_0$. This well-known fact is studied in detail e.g. in [AD1, Theorem 1.3] and [F2, Theorem 2.4]. For the readers convenience we shall reproduce the argumentation in detail in the appendix. □

The situation is more complicated when k approaches the boundary of $[\alpha_0 n, \alpha_0^{-1} n]$. When a varies in a domain of the complex plane the saddle points $z_\pm$ vary with a and coalesce when a approaches the critical boundary. If a was fixed the method of stationary phase / steepest descents would apply but if a approaches the critical boundary the radius of convergence of the resulting asymptotic expansion goes to 0. The so-called *uniform method of steepest decent* [CFU] was developed to provide asymptotic expansions that are uniform in a. According to Lemma 16 f_a has two simple saddle points when k is separate from the boundary. In what follows we assume that k approaches the right boundary, $\lim_{n\to\infty} k/n = \alpha_0^{-1}$, the reasoning for the left one being similar. In this situation the saddle points $z_\pm$ coalesce to $z_+ = z_- = 1$. If a approaches the boundary from the inside the two saddle points $z_\pm$ remain on $\partial\mathbb{D}$. However, when a approaches the boundary from the outside the saddle points $z_\pm$ move along the real line. While in the former situation $z_\pm$ lie on $\partial\mathbb{D}$, in the latter case we will first deform the contour of integration such that it passes through $z_\pm$. This is ensured by a potential deformation of the circle $\partial\mathbb{D}$ such that the new contour $\mathcal{C}$ always passes through the saddle points $z_\pm$ and 1, as depicted in Figure 6.2 below.



(A) The contour $\mathcal{C}$ when $a < \alpha_0^{-1}$. (B) The contour $\mathcal{C}$ when $a > \alpha_0^{-1}$.

TABLE 6.2. Deformation of $\partial\mathbb{D}$ in case of coalescing saddle points. The new contour of integration $\mathcal{C}$ is chosen such that it passes through $z_\pm$ and 1.

We begin by proving that the main contribution to the integral comes from a small piece $\mathcal{C}_\epsilon$, depicted in blue in Figure 6.2, of $\mathcal{C}$ containing 1 and $z_\pm$. Let $\bar{D}(1,\epsilon)$ be the closed disk of center 1 and radius ϵ, which is chosen such that $z_\pm \in D(1,\epsilon)$ are contained in the interior of $\bar{D}(1,\epsilon)$.

Lemma 17. *Let $\mathcal{C}$ and $D(1,\epsilon)$ be as described above and let $\mathcal{C}_\epsilon = \mathcal{C} \cap D(1,\epsilon)$. Suppose g is analytic on $D(1,\epsilon)$ and $\varphi \mapsto g(e^{i\varphi})$ is continuously differentiable on $[-\pi,\pi]$. Then we have*

$$\oint_{\partial\mathbb{D}} g(z)\exp\left(nf_a(z)\right)\frac{\mathrm{d}z}{z} = \int_{\mathcal{C}_\epsilon} g(z)\exp\left(nf_a(z)\right)\frac{\mathrm{d}z}{z} + \mathcal{O}\left(\frac{1}{n}\right),$$

where the $\mathcal{O}(1/n)$-term is uniform over a in a neighborhood of α_0^{-1}.

Proof. This is a well-known consequence of the van der Corput lemma [St, p. 334]. We first split the integral

$$\oint_{\partial\mathbb{D}} g(z)\exp\left(nf_a(z)\right)\frac{\mathrm{d}z}{z} = \int_{\mathcal{C}\setminus\mathcal{C}_\epsilon} g(z)\exp\left(nf_a(z)\right)\frac{\mathrm{d}z}{z} + \int_{\mathcal{C}_\epsilon} g(z)\exp\left(nf_a(z)\right)\frac{\mathrm{d}z}{z}.$$

$\mathcal{C} \setminus \mathcal{C}_\epsilon$ is an arc of a circle that covers angles between $-\varphi_\epsilon$ and φ_ϵ. To estimate the first summand we set $h(\varphi) = -if_a(e^{i\varphi})$, $g(\varphi) = 1 - e^{-2i\varphi}$. Computing derivatives we find that h and h' are monotone and the van der Corput lemma yields

$$\left|\int_{\varphi_\epsilon}^{\pi} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi\right| \le (2\pi+4)\left(\frac{1}{n|h'(\varphi_\epsilon)|} + \frac{1}{n|h'(\pi)|}\right).$$

By assumption a is located in a small neighborhood of α_0^{-1}. We write this assumption as $|a-\alpha_0^{-1}| \leq \frac{2\lambda}{1+\lambda}\sin^2\left(\frac{\varphi_\epsilon}{2}\right)$. On the one hand we have $h'(\pi) = a - \alpha_0$ and therefore

$$\begin{aligned} |h'(\pi)| &\geq (\alpha_0^{-1} - \alpha_0) - |\alpha_0^{-1} - a| \\ &\geq \frac{4\lambda}{1-\lambda^2} - \frac{2\lambda}{1+\lambda} = \frac{2\lambda}{1-\lambda} > 0. \end{aligned}$$

On the other hand

$$\begin{aligned} |h'(\varphi_\epsilon)| = \left| a - \frac{1-\lambda^2}{1+\lambda^2 - 2\lambda\cos\varphi_\epsilon} \right| &\geq \frac{2\lambda(1-\cos\varphi_\epsilon)}{(1+\lambda)(1-\lambda)} - |a - \alpha_0^{-1}| \\ \geq \frac{4\lambda\sin^2\varphi_\epsilon}{1+\lambda} - \frac{2\lambda\sin^2\varphi_\epsilon}{1+\lambda} &= \frac{2\lambda\sin^2\varphi_\epsilon}{1+\lambda}. \end{aligned}$$

We repeat the same argument to estimate the integral $\int_{-\pi}^{-\varphi_\epsilon} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi$ from above and the result follows. □

From Lemma 17 we conclude that for the proof of the points (2), (3), (4) it is sufficient to study the integral over the short piece $\mathcal{C}_\epsilon$ of the whole contour. In the sequel we follow the description of [BH, W] to apply the result of [CFU] to analyse the remaining integral. To simplify the dependence of $z_\pm$ on a we change the variable of integration via a locally one-to-one transformation. This transformation is implicitly given by $t = t_a(z)$ solving the equation

$$f_a(z) = -\frac{t^3}{3} + \gamma^2 t + \rho.$$

The parameters γ and ρ are determined such that $t = 0$ is mapped to $z = 1$ and the saddle points $z_\pm$ are mapped symmetrically to $t = \pm\gamma$. More precisely, the following proposition from [CFU] describes this local transformation and is stated in the formulation of [W, Theorem 1 p. 368], see also [BH, Theorem 9.2.1 p. 371].

Proposition 18. *For a in a small neighborhood of α_0^{-1} the cubic transformation*

$$f_a(z) = -\frac{t^3}{3} + \gamma^2 t$$

with

$$\gamma^2 = \frac{(a-\alpha_0^{-1})(1-\lambda)}{(\lambda(1+\lambda))^{1/3}} + o(a - \alpha_0^{-1})$$

has exactly one branch $t = t(z,a)$ that can be expanded into a power series in z with coefficients that are continuous in a. On this branch the points $z = z_\pm$ correspond to $t = \pm\gamma$. The mapping of z to t is one-to-one on a small neighborhood of $z = 1$ containing z_+ and z_-.

This is an immediate corollary of [CFU]. A proof can be found in [SZ1, Appendix] for a in a neighborhood of α_0^{-1} and $a < \alpha_0^{-1}$. To determine the asymptotics of the integral over $\mathcal{C}_\epsilon$ we apply the transformation of Proposition 18. This yields a uniform expansion of the integral in terms of the Airy function Ai. For real arguments the latter can be defined as an improper Riemann integral

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3}+xt\right)\mathrm{d}t.$$

For large negative arguments the Airy function shows oscillatory behavior

$$Ai(-x) \sim \frac{1}{x^{1/4}\sqrt{\pi}}\cos\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right),\ Ai'(-x) \sim \frac{x^{1/4}}{\sqrt{\pi}}\sin\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right),\ x\to+\infty,$$

and exponential behavior for large positive arguments

$$Ai(x) \sim \frac{1}{2x^{1/4}\sqrt{\pi}}\exp\left(-\frac{2}{3}x^{3/2}\right),\ Ai'(x) \sim -\frac{x^{1/4}}{2\sqrt{\pi}}\exp\left(-\frac{2}{3}x^{3/2}\right),\ x\to+\infty.$$

This leads to the following result which contains the regions of points (2), (3), (4) in Proposition 14.

Proposition 19. *Let $\lambda\in(0,1)$, $b_\lambda = \frac{z-\lambda}{1-\lambda z}$ and $n\geq 1$. Set $\alpha_0 := \frac{1-\lambda}{1+\lambda}$ and choose fixed $\alpha\in(0,\alpha_0)$ and $\beta\in(\alpha_0,1)$. Suppose α and β are close enough to α_0. If $a\in[\alpha,\beta]\cup[\beta^{-1},\alpha^{-1}]$ and $n\to\infty$ we have*

$$\begin{aligned}&\left|\int_{\mathcal{C}_\epsilon} g(z)\exp\left(nf_a(z)\right)\frac{\mathrm{d}z}{z}\right|\\ &\leq \delta\left|\mathrm{sgn}(\Delta)(1-\lambda^2)\sqrt{|\Delta||\gamma|}Ai(n^{2/3}\gamma^2)+\frac{\lambda^2(a+1)+a-1}{n^{1/3}\sqrt{|\gamma|}}Ai'(n^{2/3}\gamma^2)\right|+\mathcal{O}\left(\frac{1}{n}\right),\end{aligned}$$

where $\delta = \frac{(1-\lambda^2)|\Delta|^{1/4}}{2n^{1/3}\lambda^2 a^{5/2}}$ and $\Delta = (a-\alpha_0)(\alpha_0^{-1}-a)$. If $a\in[\beta^{-1},\alpha^{-1}]$ then

$$\gamma^2 = \frac{(a-\alpha_0^{-1})(1-\lambda)}{(\lambda(1+\lambda))^{1/3}}+o(a-\alpha_0^{-1}),$$

see Proposition 18. If $a\in[\alpha,\beta]$ then

$$\gamma^2 = \frac{(1+\lambda)\,(\alpha_0-a)}{(\lambda(1-\lambda))^{1/3}}+o(\alpha_0-a).$$

In either case the $\mathcal{O}\left(1/n\right)$-term is uniform over a.

Proof of Proposition 19. We assume that $a\in[\beta^{-1},\alpha^{-1}]$ the proof for $a\in[\alpha,\beta]$ being almost identical. We adopt the notation $\hat{\mathcal{C}}_\epsilon$ from [BH, Chapter 9] for the image of $\mathcal{C}_\epsilon$ under the transformation $z\mapsto t(z)$. With Proposition 18 and following [BH, Section 9.2] we get

$$\frac{1}{2i\pi}\int_{\mathcal{C}_\epsilon} g(z)\exp\left(nf_a(z)\right)\frac{\mathrm{d}z}{z} = \frac{1}{2i\pi}\int_{\hat{\mathcal{C}}_\epsilon} G_0(t)\exp\left(n\left(-\frac{t^3}{3}+\gamma^2 t\right)\right)\mathrm{d}t$$

with

$$G_0(t) = \frac{1-z(t)^{-2}}{z(t)}\frac{dz}{dt},$$

which is regular on the image $\hat{D}(1,\epsilon)$ of $D(1,\epsilon)$ under the transformation $z \mapsto t(z)$. We exploit the fact that if the integrand vanishes near a critical point then its contribution to the asymptotic expansion is diminished. Thus we expand

$$G_0(t) = A_0 + A_1 t + (t^2 - \gamma^2) H_0(t).$$

As long as H_0 is regular in $\hat{D}(1,\epsilon)$ the last term of the above identity vanishes at the two saddle points $t = \pm\gamma$. We can then determine A_0, A_1 by setting $t = \pm\gamma$ in the above to get

$$A_0 = \frac{G_0(\gamma) + G_0(-\gamma)}{2}, \; A_1 = \frac{G_0(\gamma) - G_0(-\gamma)}{2\gamma}. \tag{6.5}$$

With A_0, A_1 so defined it is shown in [BH, p. 373] that $H_0 = \frac{G_0(t) - A_0 - A_1 t}{t^2 - \gamma^2}$ is regular in $\hat{D}(1,\epsilon)$ as desired. We conclude

$$\int_{\mathcal{C}_\epsilon} g(z) \exp\left(n f_a(z)\right) \frac{\mathrm{d}z}{z} \;=\; \int_{\hat{\mathcal{C}}_\epsilon} (A_0 \;+\; A_1 t) \exp\left(n\left(-\frac{t^3}{3} + \gamma^2 t\right)\right) \mathrm{d}t \;+\; R_0(n) \tag{6.6}$$

with $R_0(n) = \int_{\hat{\mathcal{C}}_\epsilon} (t^2 - \gamma^2) H_0(t) \exp\left(n\left(-\frac{t^3}{3} + \gamma^2 t\right)\right) \mathrm{d}t$. In the sequel we follow the procedure described in [BH, p. 371–375] and consider a contour C_1 which is asymptotically equivalent to $\hat{\mathcal{C}}_\epsilon$. This means that the contribution of C_1 near the critical points coincides with that of $\hat{\mathcal{C}}_\epsilon$. But C_1 "continues to ∞" as a contour of steepest descent. C_1 "starts at infinity" with points of argument $-2\pi/3$ and "ends at infinity" with points of argument $2\pi/3$, see Figure 6.3 below. See [BH, Section 7.2] for a detailed description of such contours.

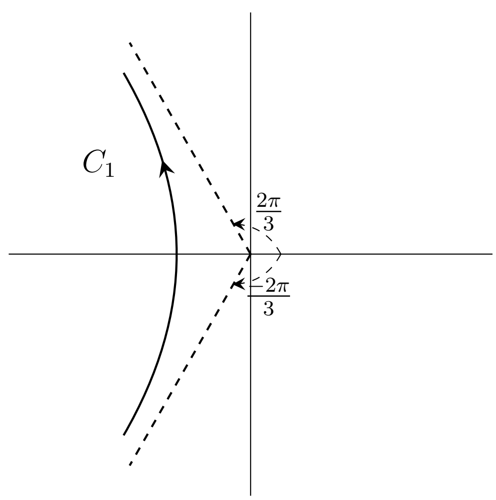


TABLE 6.3. Introduction of the asymptotically equivalent contour C_1.

Replacing $\hat{\mathcal{C}}_\epsilon$ by C_1 in (6.6) the introduced error is negligible – i. e. the integral of $(A_0 + A_1 t) \exp\left(n\left(-\frac{t^3}{3} + \gamma^2 t\right)\right)$ over $C_1 \setminus \hat{D}(1,\epsilon)$ – is asymptotically smaller than the integral over $\hat{\mathcal{C}}_\epsilon$, see [BH, p. 372] for details. The Airy function can be represented as an integral over C_1. By a change of variable $\tau \mapsto i\tau$ and a deformation of the contour of integration one obtains

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos\left(\frac{\tau^3}{3} + \tau x\right) \mathrm{d}\tau = \frac{1}{2i\pi}\int_{C_1} \exp\left(-\frac{s^3}{3} + sx\right) \mathrm{d}s.$$

The upper estimate $R_0(n) = \mathcal{O}(1/n)$ is shown in detail in [BH, p. 373–375] and leads to

$$\frac{1}{2i\pi}\int_{\mathcal{C}_\epsilon} g(z) \exp\left(nf_a(z)\right) \frac{\mathrm{d}z}{z} = \frac{A_0}{n^{1/3}} Ai(n^{2/3}\gamma^2) + \frac{A_1}{n^{2/3}} Ai'(n^{2/3}\gamma^2) + \mathcal{O}\left(\frac{1}{n}\right), \qquad n \to \infty,$$

where A_0, A_1 are defined in (6.5). To compute A_0, A_1 we write

$$G_0(\pm\gamma) = G_0(t_\pm) = \frac{1 - z_\pm^{-2}}{z_\pm} z'(t_\pm).$$

Equation [BH, (9.2.11)] yields $(z'(t_\pm))^2 = \mp\frac{\gamma}{f_a''(z_\pm)}$ while equation [BH, (9.2.12)] yields $z'(0)^3 = -2/f^{(3)}_{\alpha_0^{-1}}(1) = \frac{(1-\lambda)^3}{\lambda(1+\lambda)} > 0$. Furthermore if $a \le \alpha_0^{-1}$ then $\gamma \in i\mathbb{R}_+$ and if $a > \alpha_0^{-1}$ then $\gamma > 0$. Making use of Equation (6.4) computation shows that $f_a''(z_\pm) = \mp a z_\mp^2 \sqrt{-\Delta}$ if $a > \alpha_0^{-1}$ and $f_a''(z_\pm) = \mp a z_\mp^2 i\sqrt{\Delta}$ if $a \le \alpha_0^{-1}$ and that

$$z'(t_\pm) = z_\pm \sqrt{\frac{|\gamma|}{a}} \frac{1}{|\Delta|^{1/4}}, \quad \text{with } a = k/n, \quad \text{and } \Delta = (a - \alpha_0)\left(\alpha_0^{-1} - a\right),$$

and therefore

$$G_0(\pm\gamma) = \frac{1 - z_\pm^{-2}}{z_\pm} z'(t_\pm) = \sqrt{\frac{|\gamma|}{a}} \frac{1}{|\Delta|^{1/4}} (1 - z_\mp^2).$$

Plugging into (6.5) it follows that

$$A_0 = \operatorname{sgn}(\Delta) \frac{(1-\lambda^2)^2\sqrt{|\gamma|}}{2\lambda^2 a^{5/2}} |\Delta|^{3/4},$$

and

$$A_1 = \frac{|\Delta|^{1/4}}{2\sqrt{|\gamma|}} \frac{(1-\lambda^2)(\lambda^2(a+1) + a - 1)}{a^{5/2}\lambda^2}.$$

□

We are now ready to complete the proof of Proposition 14.

Proof of Proposition 14, points 2), 3), 4). Applying first Lemma 17 and then Proposition 19 (assuming without loss of generality that $a = k/n$ lies in a small neighbourhood of α_0^{-1}) leads to the following conclusions:

(2) If $k/n \ge \alpha_0^{-1} + n^{-2/3}$ then $n^{2/3}\gamma^2$ is strictly separate from 0 as n tends to ∞.

(a) If $n^{2/3}\gamma^2$ is bounded then the convergence of k/n to α_0^{-1} – and therefore the one of $|\Delta|$ and $|\gamma|^2$ – is of order $n^{-2/3}$. Thus the first term on the right-hand side of Proposition 19 is bounded from above (up to a multiplicative constant depending only on λ) by $\frac{(n^{-2/3})^{3/4} \cdot (n^{-2/3})^{1/4}}{n^{1/3}} = n^{-1}$. The second term on the right-hand side of Proposition 19 is bounded from above (up to a multiplicative constant depending only on λ) by $\frac{(n^{-2/3})^{1/4}}{n^{2/3}\cdot(n^{-2/3})^{1/4}} = n^{-2/3}$.

(b) If $n^{2/3}\gamma^2 \to +\infty$ as n tends to ∞, since $Ai(x) \sim \frac{1}{2x^{1/4}\sqrt{\pi}} \exp\left(-\frac{2}{3}x^{3/2}\right)$ as $x \to +\infty$, we find on the one hand that

$$\left|A_0 \frac{Ai(n^{2/3}\gamma^2)}{n^{1/3}}\right| \lesssim \frac{\left(\frac{k}{n} - \alpha_0^{-1}\right)^{3/4}}{n^{1/2}} \exp\left(-\frac{2}{3}n\left(\frac{k}{n} - \alpha_0^{-1}\right)^{3/2}\right)$$

and since $Ai'(x) \sim -\frac{x^{1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}x^{3/2}\right)$ as $x \to +\infty$ we find on the other hand that

$$\left|A_1 \frac{Ai'(n^{2/3}\gamma^2)}{n^{2/3}}\right| \lesssim \frac{n^{1/6}(\frac{k}{n} - \alpha_0^{-1})^{1/4}}{n^{2/3}} \exp\left(-\frac{2}{3}n\left(\frac{k}{n} - \alpha_0^{-1}\right)^{3/2}\right).$$

(3) If $k/n \in [\alpha_0^{-1} - n^{-2/3}, \alpha_0^{-1} + n^{-2/3}]$ then $n^{2/3}\gamma^2$ is bounded in n. Since $|A_0| = \mathcal{O}\left(|\alpha_0^{-1} - \frac{k}{n}|\right)$, $|A_1| = \mathcal{O}(1)$ we get $\frac{Ai(n^{2/3}\gamma^2)}{n^{1/3}} A_0 = \mathcal{O}(n^{-2/3-1/3}) = \mathcal{O}(n^{-1})$ and $\frac{Ai'(n^{2/3}\gamma^2)}{n^{2/3}} A_1 = \mathcal{O}(n^{-2/3})$.

(4) Assume finally that $k/n \leq \alpha_0^{-1} - n^{-2/3}$.

(a) If $n^{2/3}\gamma^2$ is bounded in n, then we refer to point (3).

(b) If $n^{2/3}\gamma^2 \to -\infty$ as n tends to ∞, then recalling that $Ai(-x) \sim \frac{1}{x^{1/4}\sqrt{\pi}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)$ as $x \to +\infty$ we find on the one hand that

$$\left|\frac{Ai(n^{2/3}\gamma^2)}{n^{1/3}} A_0\right| \lesssim \frac{1}{n^{1/6}} \frac{1}{(\alpha_0^{-1} - \frac{k}{n})^{1/4}} \frac{\alpha_0^{-1} - \frac{k}{n}}{n^{1/3}}$$

because $\alpha_0^{-1} - \frac{k}{n} \to 0$ as n tends to ∞, and on the other hand, since $Ai'(-x) \sim \frac{x^{1/4}}{\sqrt{\pi}} \sin\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)$ as $x \to +\infty$, that

$$\left|\frac{Ai'(n^{2/3}\gamma^2)}{n^{2/3}} A_1\right| \lesssim \frac{n^{1/6}(\alpha_0^{-1} - \frac{k}{n})^{1/4}}{n^{2/3}}.$$

□

Appendix

In this appendix we provide some established details regarding the uniformity of the asymptotic formulas of Erdélyi [E, Theorem 4], which are used implicitly in the proof of Proposition 14, Point 5).

We begin by demonstrating uniformity for the asymptotic expansion of the integral $\int_{\varphi_+}^{\pi} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi$, see the proof of Proposition 14 Point 5), within the methods of [AD1, AD2]. An application of [AD1, Theorem 1.3] yields the *identity*

$$\int_{\varphi_+}^{\pi} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = \frac{1}{2}\Gamma(1/2)\kappa_1(0)e^{i\frac{\pi}{4}}n^{-1/2}e^{inh(\varphi_+)} + \frac{1}{2}\Gamma(1)\kappa_1'(0)e^{i\frac{\pi}{2}}n^{-1}e^{inh(\varphi_+)} + \frac{1}{2}\Gamma(3/2)\kappa_1''(0)e^{i\frac{3\pi}{4}}n^{-3/2}e^{inh(\varphi_+)} + e^{inh(\pi)}\left(\frac{i}{n}\frac{g(\pi)}{h'(\pi)} + \frac{1}{n^2}\frac{g'(\pi)}{(h'(\pi))^2}\right) + R^{(1)}(n) + R^{(2)}(n),$$

where

$$\kappa_1''(0) = \frac{2^{5/2}}{(h''(\varphi_+))^{3/2}} \cdot \left(\frac{g''(\varphi_+)}{2} - \frac{g'(\varphi_+)h^{(3)}(\varphi_+)}{4h''(\varphi_+)} + \left(\frac{5}{36}\left(h^{(3)}(\varphi_+)\right)^2 - \frac{h''(\varphi_+)h^{(4)}(\varphi_+)}{12}\right)\frac{3g(\varphi_+)}{4h''(\varphi_+)^2}\right)$$

and $R^{(1)}(n)$, $R^{(2)}(n)$ denote remainder terms. Observe that for $k \in [\beta n, \beta^{-1}n]$, since $\beta > \alpha_0$, we have

$$h''(\varphi_+) = \frac{k}{n}\sqrt{\left(\frac{k}{n} - \alpha_0\right)\left(\alpha_0^{-1} - \frac{k}{n}\right)} \geq \min_{a\in[\beta,\beta^{-1}]} a\sqrt{(a-\alpha_0)(\alpha_0^{-1}-a)} =: C(\beta,\lambda) > 0.$$

This shows that

$$\frac{1}{2}\Gamma(3/2)\kappa_1''(0)e^{i\frac{3\pi}{4}}n^{-3/2}e^{inh(\varphi_+)} = \mathcal{O}\left(n^{-3/2}\right)$$

and the $\mathcal{O}$-term is automatically uniform over $a = \frac{k}{n} \in [\beta, \beta^{-1}]$. Making use of the fact that $h'(\pi) = \frac{k}{n} - \alpha_0$ we also find that for $a = \frac{k}{n} \in [\beta, \beta^{-1}]$

$$e^{inh(\pi)}\left(\frac{i}{n}\frac{g(\pi)}{h'(\pi)} + \frac{1}{n^2}\frac{g'(\pi)}{(h'(\pi))^2}\right) = e^{inh(\pi)}\frac{1}{n^2}\frac{g'(\pi)}{(h'(\pi))^2} = \mathcal{O}\left(n^{-2}\right),$$

where the $\mathcal{O}$-term is again uniform over $a = \frac{k}{n} \in [\beta, \beta^{-1}]$. It remains to show that the remainder terms satisfy

$$R^{(j)}(n) = \mathcal{O}\left(n^{-3/2}\right), \qquad j = 1, 2$$

uniformly over $a = \frac{k}{n} \in [\beta, \beta^{-1}]$. For $k \in [\beta n, \beta^{-1}n]$ the unique critical point φ_+ of h satisfies $x \leq \varphi_+ \leq \pi - x$ for some $x = x(\beta, \lambda) > 0$ because

$$\left|e^{i\varphi_+} - 1\right| \geq (1-\lambda)\sqrt{\frac{\beta}{\lambda}}\sqrt{\alpha_0^{-1} - \beta^{-1}}, \qquad \left|e^{i\varphi_+} + 1\right| \geq (1+\lambda)\sqrt{\frac{\beta}{\lambda}}\sqrt{\beta - \alpha_0}.$$

To obtain upper estimates on $R^{(j)}$ we follow the lines of [AD1, Section 1]. We introduce the functions $\psi_j : I_j \longrightarrow \mathbb{R}$ on $I_1 = [\varphi_+, \pi - \eta]$, $I_2 = [\varphi_+ + \eta, \pi]$, $\eta = \frac{x}{4} \in \left(0, \frac{\pi - \varphi_+}{2}\right)$, by

$$\psi_1(\varphi) = (h(\varphi) - h(\varphi_+))^{\frac{1}{2}}, \quad \psi_2(\varphi) = h(\pi) - h(\varphi),$$

which are used to change variables in the respective integrals. Let furthermore $s_1 = \psi_1(\pi - \eta)$, $s_2 = \psi_2(\varphi_+ + \eta)$, then ψ_j is known to be a diffeomorphism between I_j and $[0, s_j]$, see [AD1, Proposition 3.2]. For $j = 1, 2$, let $\kappa_j : (0, s_j] \to \mathbb{C}$ be the functions defined by

$$\kappa_j(s) := g\left(\psi_j^{-1}(s)\right)\left(\psi_j^{-1}\right)'(s).$$

It is shown in [AD1, Proposition 3.3] that κ_j can be continuously extended to $[0, s_j]$ and that $\kappa_j \in \mathcal{C}^3([0, s_j])$. Let $\nu : [\varphi_+, \pi] \longrightarrow \mathbb{R}$ be a smooth function (often called a neutralizer) such that $\nu = 1$ on $[\varphi_+, \varphi_+ + \eta]$, $\nu = 0$ on $[\pi - \eta, \pi]$ and $0 \leqslant \nu \leqslant 1$, where η is defined above. For $j = 1, 2$, let $\nu_j = [0, s_j] \to \mathbb{R}$ be the functions defined by

$$\nu_1(s) = \nu \circ \psi_1^{-1}(s), \quad \nu_2(s) = (1 - \nu) \circ \psi_2^{-1}(s).$$

It is shown in [AD1, Theorem 1.3] (see also [AD2, Theorem 1.2]) that

$$\left|R^{(j)}(n)\right| \leqslant \frac{1}{4}\Gamma\left(\frac{3}{2}\right) n^{-\frac{3}{2}} \int_0^{s_j} \left|\frac{d^3}{ds^3}\left[\nu_j\kappa_j\right](s)\right| \mathrm{d}s, \qquad j = 1, 2.$$

To prove that $R^{(j)}(n) = \mathcal{O}\left(n^{-3/2}\right)$ uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$ we write

$$\int_0^{s_j} \left|\frac{d^3}{ds^3}\left[\nu_j\kappa_j\right](s)\right| \mathrm{d}s \leq s_j \max_{s\in[0,s_j]} \left|\frac{d^3}{ds^3}\left[\nu_j\kappa_j\right](s)\right|, \qquad j = 1, 2.$$

It remains to show that for $j = 1, 2$ the third derivatives $\frac{d^3}{ds^3}\left[\nu_j\kappa_j\right](s)$ are uniformly bounded. For $j = 2$ this follows from the facts that

$$\kappa_2(s) = -\frac{g(\psi_2^{-1}(s))}{h'(\psi_2^{-1}(s))},$$

and $h''(\varphi) > 0$, $h'(\varphi_+) = 0$, $h'(\pi) = \frac{k}{n} - \alpha_0 > 0$ so that h' does not vanish on $(\varphi_+ + \eta, \pi]$. By construction of the neutralizer the derivatives of ν_2 are uniformly bounded. We conclude that $R^{(2)}(n) = \mathcal{O}\left(n^{-3/2}\right)$ uniformly for $\frac{k}{n} \in [\beta, \beta^{-1}]$. This reasoning applies similarly for $j = 1$. In this case

$$\kappa_1(s) = \frac{2sg(\psi_1^{-1}(s))}{h'(\psi_1^{-1}(s))}$$

and h' does not vanish on (φ_+, π) and κ_1 as well as its derivatives are uniformly bounded in a neighborhood of 0 since $h''(\varphi_+) \geq C(\beta, \lambda) > 0$. The discussion of the uniformity of the $\mathcal{O}$-term in the integral $\int_{-\varphi_+}^0 g(-\varphi)e^{inh(-\varphi)}\mathrm{d}\varphi$ follows along the same lines.

A different and more direct approach to the asymptotic analysis of integrals of the form

$$I(\xi, a) = \int g(\varphi, a, \xi)e^{ixh(\varphi,a)}\mathrm{d}\varphi$$

has been proposed by M.V. Fedoryuk: Proposition 14, point (5) can also be obtained as a consequence of [F2, Theorem 2.4 p. 80] (see [F1, Theorem 1.6 p.107] for a simple version in one dimension). To apply Fedoryuk's result to

$$I(n, a) = \int_0^{\pi} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi$$

where $h = h_a$ and $a \in [\beta, \beta^{-1}]$ we verify the assumptions of [F2, Theorem 2.4 p. 80]. The function $g : \varphi \mapsto 1 - e^{-2i\varphi}$ neither depends on a nor on n. For $a = \frac{k}{n} \in [\beta, \beta^{-1}]$ the unique critical point $\varphi_+ = \varphi_+(a)$ of $\varphi \mapsto h_a(\varphi)$ satisfies (as discussed above)

$$h''(\varphi_+) \geq C(\beta, \lambda) > 0,$$

which allows us to apply Fedoryuk's asymptotic formula with asymptotic order 1. We find

$$\begin{aligned} I(n,a) &= \sqrt{2\pi}|h''(\varphi_+)|^{-1/2} e^{i\frac{\pi}{4}} e^{inh(\varphi_+)} g(\varphi_+) n^{-1/2} + \mathcal{O}\left(n^{-3/2}\right) \\ &= e^{inh(\varphi_+) - i\varphi_+ + i\frac{3\pi}{4}} \frac{(1-\lambda^2)(a-\alpha_0)^{1/4}(\alpha_0^{-1}-a)^{1/4}}{\lambda a^{3/2}} \left(\frac{2\pi}{n}\right)^{1/2} + \mathcal{O}\left(n^{-3/2}\right), \end{aligned}$$

where the $\mathcal{O}$-term is uniform over $a \in [\beta, \beta^{-1}]$.

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