Compatibility, coherence and the RIP

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Abstract We generalise the classical result on the compatibility of marginal, possible non-disjoint, assessments in terms of the running intersection property to the imprecise case, where our beliefs are modelled in terms of sets of desirable gambles. We consider the case where we have unconditional and conditional assessments, and show that the problem can be simplified via a tree decomposition.

1 Introduction

This paper deals with the *marginal problem*, that of the compatibility of a number of marginal assessments with a global model. This problem is trivial when the marginal models are defined on disjoint sets of variables: in that case, we could for instance determine a compatible joint model by considering the independent product of the marginal distributions. However, when the sets of variables where our marginal assessments are defined are not disjoint then the problem is not immediate anymore, and it has been received quite some attention in the literature [1,2,3].

A necessary condition for the compatibility of a number of marginal assessments is their pairwise compatibility. Using the theory of hypergraphs, Beeri et al. [4] established a necessary and sufficient condition for pairwise compatibility to imply global compatibility: the *running intersection property*, that means that the sets of indices $S_1, \ldots, S_r$ satisfy that $S_i \cap (\cup_{j<i} S_j)$ is included in some S_{j^*} for $j^* < i$, and this for every i.

Another way to find a compatible joint is the *Iterative Proportional Fitting Procedure* (IPFP) [5]; when there is a compatible joint, this procedure determines a sequence of probability measures that converges to the compatible joint that maximizes the Kullback-Leibler information [6].

The works above investigate the compatibility of probability measures; if the possibility spaces are infinite, they are assumed to be countably additive on a suitable σ-field. Nevertheless, there are situations where the available information does not allow us to model our knowledge by means of a precise probability measure. In those cases, we may consider a number of alternative models, which are sometimes gathered under the term *imprecise probabilities* [7]. The marginal problem has been investigated for some of these models by Studeny [8], Vejnarová [9] and Jirousek [10], using the IPFP.

In this paper, we study the problem of the compatibility of some partial assessments with a global one in as great a generality as possible: one the one hand, we assume that the assessments are expressed by means of an imprecise probability model that includes as particular cases all the models considered so far in the literature: *sets of desirable gambles*. In addition, we also investigate the case where the marginal assessments may be of a conditional nature.

After recalling some preliminary concepts in Section 2, in Section 3 we extend the classical result on the compatibility of marginal probability measures to the imprecise case. In Section 4 we deal with the case of conditional and unconditional information, and show that the compatibility problem can be simplified by means of a graphical procedure. Some additional comments are given in Section 5. Due to the space limitations, proofs have been omitted.

2 Preliminary concepts

2.1 Sets of desirable gambles and coherent lower previsions

Consider a possibility space $\mathcal{X}$. A *gamble* on $\mathcal{X}$ is a bounded real-valued function $f : \mathcal{X} \to \mathbb{R}$. We denote the set of all gambles on $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$, and denote by $\mathcal{L}^+(\mathcal{X}) := \{f \geq 0 : f \neq 0\}$, or simply $\mathcal{L}^+$ when no confusion is possible, the set of positive gambles. A subset $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$ is called *coherent* when $0 \notin \mathcal{D}$ and moreover $\mathcal{D} = \operatorname{posi}(\mathcal{D} \cup \mathcal{L}^+)$, where posi denotes the set of positive linear combinations. One trivial example is the *vacuous* (least-informative) set of gambles $\mathcal{L}^+$.

We say that $\mathcal{D}$ *avoids partial loss* when it is included in some coherent set of gambles. In that case, the smallest such set is called its *natural extension*, and it is given by $\mathcal{E} = \operatorname{posi}(\mathcal{L}^+ \cup \mathcal{D})$. Moreover, $\mathcal{D}$ avoids partial loss if and only if $0 \notin \mathcal{E}$.

Given possibility spaces $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and a subset S of $\{1, \ldots, n\}$, we shall let $\mathcal{X}_S := \times_{j \in S} \mathcal{X}_j$. In order to simplify the notation, we shall use $\mathcal{X}^n := \mathcal{X}_{\{1,\ldots,n\}}$. Let π_S be the *projection operator*, given by

$$\begin{aligned} \pi_S : \mathcal{X}^n &\to \mathcal{X}_S \\ x &\hookrightarrow (x_j)_{j \in S}. \end{aligned}$$

We shall say that a gamble f on $\mathcal{X}^n$ is S-measurable if and only if $f(x) = f(y)$ for every $x, y \in \mathcal{X}^n$ such that $\pi_S(x) = \pi_S(y)$, and we shall denote by $\mathcal{K}_S$ the subset of $\mathcal{L}(\mathcal{X}^n)$ given by the $\mathcal{X}_S$-measurable gambles. There exists a one-to-one correspondence between $\mathcal{L}(\mathcal{X}_S)$ and $\mathcal{K}_S$, and we will sometimes abuse the notation by writing $\mathcal{D} \cap \mathcal{L}(\mathcal{X}_S)$ when we mean $\mathcal{D} \cap \mathcal{K}_S$ for a given set of gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X}^n)$. In this sense, we shall say that a set $\mathcal{D} \subseteq \mathcal{K}_S$ is *coherent relative to* $\mathcal{K}_S$ when the set $\mathcal{D}' \subseteq \mathcal{L}(\mathcal{X}_S)$ that we can make a one-to-one correspondence with, is coherent.

In addition, we shall also consider in this paper *separately coherent* sets of desirable gambles. If we consider two disjoint subsets S_1, S_2 of $\{1, \ldots, n\}$, a separately coherent set of desirable gambles $\mathcal{D}_{S_1|S_2}$ will be given by

$$\mathcal{D}_{S_1|S_2} := \cup_{x \in \mathcal{X}_{S_2}} \mathcal{D}_{\cdot|x},$$

where $\mathcal{D}_{|x}$ is a coherent set of desirable gambles relative to $\mathcal{K}_{S_1}$ for every $x \in \mathcal{X}_{S_2}$. Formally, $\mathcal{D}_{S_1|S_2}$ is a subset of $\mathcal{K}_{S_1 \cup S_2}$, but it need not be coherent relative to it: it is only coherent once we focus on each particular element $x \in \mathcal{X}_{S_2}$.

A slightly more restrictive imprecise probability model is that of coherent lower previsions. A functional $\underline{P} : \mathcal{L}(\mathcal{X}) \to \mathbb{R}$ is called a *coherent lower prevision* when it satisfies $\underline{P}(f) \geq \inf f$, $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ and $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g)$ for every

$f, g \in \mathcal{L}(\mathcal{X})$ and every $\lambda > 0$. When the third of these conditions is satisfied with equality for every f, g, then $\underline{P}$, denoted by P in the special case, is called a coherent (linear) prevision, and it corresponds to the expectation operator with respect to the finitely additive probability on $\mathcal{P}(\mathcal{X})$ that is its restriction to events.

A coherent set of desirable gambles $\mathcal{D}$ induces a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X})$ by means of the formula

$$\underline{P}(f) = \sup\{\mu : f - \mu \in \mathcal{D}\}; \tag{1}$$

however, there may be different coherent sets of desirable gambles $\mathcal{D}_1, \mathcal{D}_2$ that induce the same coherent lower prevision $\underline{P}$ by means of Eq. (1), so in this sense coherent sets of desirable gambles are more general than coherent lower previsions.

2.2 Compatibility, coherence and RIP

Consider now subsets $S_1, \dots, S_r$ of $\{1, \dots, n\}$, and let $P_1, \dots, P_r$ be (finitely additive) probability measures, so that P_i is defined on the power set $\mathcal{P}(\mathcal{X}_{S_i})$. Then each P_i has a unique extension as a coherent (linear) prevision P_i' on $\mathcal{L}(\mathcal{X}_{S_i})$ or, using the one-to-one correspondence mentioned above, to $\mathcal{K}_{S_i}$. The compatibility problem studies if it is possible to find a joint probability measure on $\mathcal{P}(\mathcal{X}^n)$ with marginals $P_1, \dots, P_r$. Taking into account the one-to-one correspondence between previsions and probability measures in the precise case, this is equivalent to the existence of a coherent prevision P' on $\mathcal{L}(\mathcal{X}^n)$ such that $P'(f) = P_i'(f)$ for every $i = 1, \dots, r$ and every $f \in \mathcal{K}_{S_i}$.

Of course, one necessary condition for the existence of P' is the *pairwise compatibility* of $P_1', \dots, P_r'$, that means that for every $i \neq j$ it holds that $P_i'(f) = P_j'(f)$ for every $f \in \mathcal{K}_{S_i} \cap \mathcal{K}_{S_j}$. If that is the case, then $P_1', \dots, P_r'$ allow us to define a lower prevision Q' on $\mathcal{K} := \cup_{i=1}^r \mathcal{K}_{S_i}$ by $Q'(f) = P_i'(f)$ if $f \in \mathcal{K}_{S_i}$. Pairwise compatibility simply means that Q' is well-defined. Compatibility then means that there exists a coherent prevision on $\mathcal{L}(\mathcal{X}^n)$ (or equivalently, a finitely additive probability on $\mathcal{P}(\mathcal{X}^n)$) that coincides with Q' on $\mathcal{K}$, and this in turn is equivalent to the coherence of Q' on its domain, in the sense considered in [11].

What the classical result tells us then is that, given $P_1, \dots, P_r$ on $\mathcal{K}_{S_1}, \dots, \mathcal{K}_{S_r}$, their pairwise compatibility guarantees the coherence of Q' if and only if the sets of variables $S_1, \dots, S_r$ satisfy the *running intersection property (RIP)*: for every $i = 2, \dots, r$ it holds that

$$S_i \cap (\cup_{j<i} S_j) = S_i \cap S_{j^*} \text{ for some } j^* < i.$$

Now, the compatibility problem can be expressed in terms of coherence in the imprecise case: if we have marginal assessments given by coherent lower previsions $\underline{P}_1, \dots, \underline{P}_r$ on $\mathcal{K}_{S_1}, \dots, \mathcal{K}_{S_r}$, then the existence of a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}^n)$ with these marginals is equivalent to the coherence of the lower prevision $\underline{Q}'$ that we can define on $\mathcal{K} = \cup_{i=1}^r \mathcal{K}_{S_i}$ by means of $\underline{P}_1, \dots, \underline{P}_r$, provided these are pairwise compatible. In Section 3 we shall prove that the classical result can be extended to the imprecise case.

3 Compatibility of sets of desirable gambles

In this section, we shall study the compatibility problem of a number of partial assessments, when these assessments are modelled by coherent sets of desirable gambles; this includes as particular cases those of coherent lower previsions and finitely additive probability measures.

We consider therefore subsets $S_1, \dots, S_r$ of $\{1, \dots, n\}$, and for every $j = 1, \dots, r$ let $\mathcal{D}_j$ be a subset of $\mathcal{L}(\mathcal{X}^n)$ that is coherent with respect to the set $\mathcal{K}_{S_j}$ of $\mathcal{X}_{S_j}$-measurable gambles. Our goal is to find conditions that guarantee the existence of a coherent set of desirable gambles $\mathcal{D}$ that is compatible with $\mathcal{D}_1, \dots, \mathcal{D}_r$. In order to alleviate the notation, we shall use $\mathcal{K}_j := \mathcal{K}_{S_j}$.

Let us clarify what we mean by pairwise and global compatibility, in terms of sets of desirable gambles. On the one hand, if we consider $i \neq j$ in $\{1, \dots, r\}$, we say that $\mathcal{D}_i, \mathcal{D}_j$ are pairwise compatible if and only if

$$\mathcal{D}_i \cap \mathcal{K}_j = \mathcal{D}_j \cap \mathcal{K}_i.$$

In other words, those gambles on $\mathcal{D}_i$ that are S_j-measurable belong to $\mathcal{D}_j$, and viceversa. On the other hand, we shall say that a set of gambles $\mathcal{D}$ is (globally) compatible with $\mathcal{D}_1, \dots, \mathcal{D}_r$ if and only if it is pairwise compatible with each of them, in the sense that $\mathcal{D} \cap \mathcal{K}_j = \mathcal{D}_j$ for every $j = 1, \dots, r$.

As mentioned in Section 2.2, the running intersection property is the key for pairwise compatibility to imply global compatibility in the precise case. We next extend this result to the imprecise case:

Proposition 1. *If $S_1, \dots, S_r$ satisfy RIP and the sets $\mathcal{D}_1, \dots, \mathcal{D}_r$ are pairwise compatible, then there exists a coherent set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X}^n)$ that is globally compatible with $\mathcal{D}_1, \dots, \mathcal{D}_r$.*

As a corollary, we can establish a similar result in terms of coherent lower previsions:

Corollary 1. *Consider subsets $S_1, \dots, S_r$ of $\{1, \dots, r\}$ satisfying RIP and for every j let $\underline{P}_j$ be a coherent lower prevision on $\mathcal{X}_{S_j}$. Then there exists a coherent lower prevision $\underline{P}$ on $\mathcal{X}^n$ such that $\underline{P}(f) = \underline{P}_j(f)$ for every $f \in \mathcal{K}_j$, $j = 1, \dots, r$ if and only if $\underline{P}_i(f) = \underline{P}_j(f)\ \forall f \in \mathcal{K}_i \cap \mathcal{K}_j$, and for every $i \neq j \in \{1, \dots, r\}$.*

As a particular case of Corollary 1, we obtain the result for linear previsions, that is, expectation operators with respect to finitely additive probabilities, and as a consequence also for countably additive probabilities.

It is not difficult to see that without RIP, pairwise compatibility does not imply global compatibility. This is actually not surprising: we deduce from [4, Theorem 3.4] that, if the sets of variables $S_1, \dots, S_r$ do not satisfy RIP, then it is possible to find marginal probability measures $P_1, \dots, P_r$ that are pairwise compatible but not globally compatible. Using the correspondence with sets of desirable gambles, it is possible to express the result in terms of sets of desirable gambles, too.

More generally, we may have unconditional and conditional information, and we should study whether they can be encompassed into a joint model. However, the meaning of compatibility is not as clear as in our previous results, in the sense that such a joint may necessarily induce additional assessments that are not immediately present in the original ones.

Taking this into account, given a number of sets of desirable gambles $\mathcal{D}_1, \dots, \mathcal{D}_r$ that gather the information on different sets of variables $S_1, \dots, S_r$, we shall investigate to which extent these sets avoid partial loss, meaning that they have a joint coherent superset; but we are not requiring anymore that $\mathcal{D} \cap \mathcal{K}_j = \mathcal{D}_j$ for every $j = 1, \dots, r$.

Our first result tells us that if a variable appears only in one of these sets, then our assessments on this variable are not relevant for the problem:

Proposition 2. *Consider subsets $S_1, \dots, S_r$ of $\{1, \dots, n\}$ and coherent sets of desirable gambles $\mathcal{D}_1, \dots, \mathcal{D}_r$, where $\mathcal{D}_j$ is coherent relative to the set $\mathcal{K}_j$ of $\mathcal{X}_{S_j}$-measurable gambles. For every $i = 1, \dots, r$, let $\mathcal{D}_i^*$ be the restriction of $\mathcal{D}_i$ to the class of $\mathcal{X}_{S_i \cap (\cup_{j \neq i} S_j)}$-measurable gambles. Then $\cup_{i=1}^r \mathcal{D}_i$ avoids partial loss if and only if $\cup_{i=1}^r \mathcal{D}_i^*$ avoids partial loss.*

We may think that the problem considered above can be simplified into the study of pairwise compatibility, in the sense that, given $i \neq j$, we can let $\mathcal{D}_i^j$ be the restriction of $\mathcal{D}_i$ to the class of $\mathcal{X}_{S_i \cap S_j}$-measurable gambles. Then $\cup_{i \neq j} \mathcal{D}_i^j \subseteq \cup_i \mathcal{D}_i^*$, and as a consequence we have that

$$\cup_{i=1}^r \mathcal{D}_i \text{ avoid partial loss } \Rightarrow \cup_{i \neq j} \mathcal{D}_i^j \text{ avoid partial loss.}$$

However, the converse is not true, as the following example shows:

Example 1. Consider $\mathcal{X}_1 = \dots = \mathcal{X}_6 = \{0, 1\}$, and the sets $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2, 4\}$, $S_3 = \{1, 3, 5\}$, $S_4 = \{2, 3, 6\}$. Take the following sets of desirable gambles:

$$\mathcal{D}_1 := \{f : fI_A \geq 0, \text{ where } A = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \in \{1, 2\}\} \setminus \{0\}$$
$$\mathcal{D}_2 := \{f : fI_B \geq 0, \text{ where } B = \{(x_1, x_2, x_4) : x_1 + x_2 \in \{0, 2\}\} \setminus \{0\}$$
$$\mathcal{D}_3 := \{f : fI_C \geq 0, \text{ where } C = \{(x_1, x_3, x_5) : x_1 + x_3 \in \{0, 2\}\} \setminus \{0\}$$
$$\mathcal{D}_4 := \{f : fI_D \geq 0, \text{ where } D = \{(x_2, x_3, x_6) : x_2 + x_3 \in \{0, 2\}\} \setminus \{0\}.$$

To see that $\cup_{i \neq j} \mathcal{D}_i^j$ avoids partial loss, note that $\mathcal{D}_i^j$ is vacuous for every $i \neq j$. However, $\mathcal{D}_1, \dots, \mathcal{D}_4$ do not avoid partial loss, as we can see by considering the gambles $f_1, \dots, f_4$ given in the following table:

(X_1, X_2, X_3)	(1,1,1)	(1,1,0)	(1,0,1)	(1,0,0)	(0,1,1)	(0,1,0)	(0,0,1)	(0,0,0)
f_1	-4	1	1	1	1	1	1	-4
f_2	1	1	-4	-4	-4	-4	1	1
f_3	1	-4	1	-4	-4	1	-4	1
f_4	1	-4	-4	1	1	-4	-4	1

Then $f_i \in \mathcal{D}_i$ for $i = 1, \dots, 4$, but $f_1 + \dots + f_4 < 0$. ♦

4 Graphical representation and compatibility

When some of the assessments are of a conditional type, the notion of coherence of lower previsions can be extended in two different manners: *weak* and *(strong) coherence*. In [12], we showed that the verification of these conditions can be simplified by means of a graphical representation known as *coherence graphs*, which partition the set of assessments by means of the so-called *superblocks*. We proved that it suffices to verify the (weak or strong) coherence of those assessments that belong to the same superblock to automatically deduce the global coherence of all of them together. This allows to make a first simplification of the compatibility problem.

Consider thus a number of conditional templates $O_1|I_1, \ldots, O_r|I_r$, and assume that we have a belief assessment for the variable X_{O_j} conditional on X_{I_j}, and this for $j = 1, \ldots, r$. We represent these templates in a coherence graph. Taking into account the results from [12] mentioned above, we shall assume that this coherence graph consists of only one superblock; otherwise we treat each superblock separately.

Next we make a graphical representation of these templates so that we put the variables $O_j \cup I_j$ in one node, for $j = 1, \ldots, r$, and connect two nodes when their associated sets of variables have non-empty intersection. From this graphical representation, it is always possible to make a tree of cliques called *join tree*, so that the sets of variables present in the different cliques satisfy the running intersection property condition (see [15] for more details). RIP guarantees that if a variable k is in two cliques in the same path, then it also belongs to all the other cliques in the same path.

We assume that on each of the cliques of the join tree we have a coherent set of desirable gambles $\mathcal{D}_j$ on the corresponding set of variables. This set can be obtained by aggregating the information of the different nodes from the initial graph, that in turn is modelled by means of separately coherent sets of desirable gambles. The set $\mathcal{D}_j$ is coherent relative to the set $\mathcal{K}_j$ of $\mathcal{X}_{S_j}$-measurable gambles, and we assume moreover that the sets are pairwise compatible. The RIP condition guarantees that the intersection $S_i \cap (\cup_{j \neq i} S_j)$ can be obtained as the union of the intersections $S_i \cap S_j$, where the node j is adjacent to i. Since the different sets of variables satisfy the running intersection property, we can deduce from Proposition 1 that there exists a coherent set of desirable gambles that includes all these assessments as soon as the sets of desirable gambles we are considering avoid partial loss. The smallest such set can be obtained by means of the procedure of natural extension.

We next give an iterative procedure for determining this natural extension (it is junction tree propagation adapted to dealing with sets of desirable gambles):

- We pick any node as a root. Since the tree is undirected and totally connected, we can make a partition of its set of nodes $\{1, \ldots, r\}$ into sets $A_0, A_1, \ldots, A_k$, $k < r$, where A_i includes those nodes that are at a distance i from the root. Thus, A_0 includes only the root.
- Step 1. We consider the nodes in A_k. For each of them, we take its associated set of desirable gambles. Note that no pair of nodes in A_k can be adjacent, because of the tree structure.
- Step 2. We consider the nodes in A_{k-1}. For each node j of them, we have two possibilities:

- If it has no adjacent nodes in A_k, we define $\mathcal{D}'_j$ as its set $\mathcal{D}_j$ of desirable gambles.
- If it has adjacent nodes in A_k, we take the set A of adjacent nodes, and define $\mathcal{D}'_j$ as the natural extension of $\mathcal{D}_j \cup \bigcup_{l\in A} \mathcal{D}'_{l|S_j\cap S_l}$

∘ We proceed in this manner until step $k+1$, that produces a set of desirable gambles $\mathcal{D}'_0$ on the root node.

We have proven that the procedure above provides us with the restriction of the natural extension of $\mathcal{D}_1,\dots,\mathcal{D}_r$ to those gambles that depend on the variables in the root node:

Proposition 3. *The set $\mathcal{D}'_0$ constructed in the manner described above is the restriction of the natural extension $\mathcal{E}$ of $\mathcal{D}_1,\dots,\mathcal{D}_r$ to $\mathcal{K}_{S_0}$. It follows that $\mathcal{D}_1,\dots,\mathcal{D}_r$ avoid partial loss if and only if $\mathcal{D}'_0$ is coherent.*

In order to obtain the natural extension of $\mathcal{D}_1,\dots,\mathcal{D}_r$ on $\mathcal{L}(\mathcal{X}^n)$, we can consider the reverse procedure: given the same root node as before and the sets of desirable gambles $\mathcal{D}'_0,\dots,\mathcal{D}'_{r-1}$ we generated above, we define iteratively sets $\mathcal{D}''_0,\dots,\mathcal{D}''_{r-1}$ as follows:

∘ We make $\mathcal{D}''_0 := \mathcal{D}'_0$.
∘ Step 1: if a node i belongs to A_1, we define $\mathcal{D}''_i := \operatorname{posi}(\mathcal{D}'_i \cup \mathcal{D}'_{0|S_i\cap S_0} \cup \mathcal{L}^+(\mathcal{X}_{S_i}))$.
∘ Step 2: for any $i \in A_2$, we let B_i denote its neighbours in A_1, and let $\mathcal{D}''_i := \operatorname{posi}(\mathcal{D}'_i \cup \bigcup_{j\in B_i} \mathcal{D}''_{j|S_j\cap S_i} \cup \mathcal{L}^+(\mathcal{X}_{S_i}))$.

We proceed iteratively in this manner until we get to the nodes in A_k. Then the set $\mathcal{D}''_i$ we obtain with this procedure is the natural extension of the natural extension of $\mathcal{D}_1,\dots,\mathcal{D}_r$ to the class of $\mathcal{X}_{S_i}$-measurable gambles. Note that this holds in particular for the root node, taking into account Proposition 3.

Proposition 4. *Let $\mathcal{E}$ be the natural extension of $\mathcal{D}_1,\dots,\mathcal{D}_r$. If we follow the procedure above, then $\mathcal{D}''_i = \mathcal{E}\cap\mathcal{K}_i$ for every $i=1,\dots,r$.*

It is worth noting that the procedure above cannot be simplified, in the sense that, for a given index $i=1,\dots,r$, it does not hold that

$$\operatorname{posi}(\cup_{j=1}^r \mathcal{D}_j \cup \mathcal{L}^+) \cap \mathcal{K}_i = \operatorname{posi}(\cup_{S_j\cap S_i\neq\emptyset} \mathcal{D}_j \cup \mathcal{L}^+) \cap \mathcal{K}_i;$$

that is, even if a set of desirable gambles does not involve any variable in the set S_i, it could be that it has behavioural implications on S_i when we propagate through the tree:

Example 2. Let $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3$ be binary variables, and consider the conditional assessments $X_2|X_1$ and $X_3|X_2$ given by $X_1=0 \Rightarrow X_2=1; X_1=1 \Rightarrow X_2=1; X_2=0 \Rightarrow X_3=0; X_2=1 \Rightarrow X_3=1$. These produce the sets

$$\mathcal{D}_{12} = \{f \in \mathcal{L}(\mathcal{X}_1\times\mathcal{X}_2) : f(0,1)\geq 0, f(1,1)\geq 0, \max\{f(0,1),f(1,1)\}>0\};$$
$$\mathcal{D}_{23} = \{f \in \mathcal{L}(\mathcal{X}_2\times\mathcal{X}_3) : f(0,0)\geq 0, f(1,1)\geq 0, \max\{f(0,0),f(1,1)\}>0\}.$$

Their natural extension includes the gamble $g = I_{X_3=1} - 2I_{X_3=0}$: to see this, note that $g \geq f_1+f_2$, for $f_1 = 0.5I_{X_2=1} - 3I_{X_2=0} \in \mathcal{D}_{12}$ and $f_2 = 0.5I_{X_2=X_3} - 3I_{X_2\neq X_3} \in \mathcal{D}_{23}$. However, g does not belong to the restriction of $\mathcal{D}_{23}$ to $\mathcal{L}(\mathcal{X}_3)$. ⧫

5 Conclusions

We have showed that the classical compatibility result based on the running intersection property can be generalised in a number of ways: first, to finitely additive probabilities, getting rid of measurability constraints; secondly, to coherent lower previsions, which are equivalent to sets of probability measures; and finally to sets of desirable gambles, which include coherent lower previsions as a particular case and are more suited for dealing with the problem of conditioning on sets of probability zero.

In addition, we have combined this result with our earlier work in [12] to simplify the study of the coherence of unconditional and conditional assessments, going beyond coherence graphs and using a tree decomposition to give a necessary and sufficient condition for avoiding partial loss. A study of the computational complexity, the expression of our desirability results in terms of conditional lower previsions [13], the relation to conglomerable extensions [14] and to other works are future lines of research.

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References

1. Kellerer, H.: Verteilungsfunktionen mit gegebenen marginalverteilungen. Z. Wahrscheinlichkeitstheorie **3** (1964) 247–270
2. Skala, H.J.: The existence of probability measures with given marginals. The Annals of Probability **21**(1) (1993) 136–142
3. Fritz, T., Chaves, R.: Entropic inequalities and marginal problems. IEEE Transactions on Information Theory **59**(3) (2013) 803–817
4. Beeri, C., Fagin, R., Maier, D., Yannakis, M.: On the desirability of acyclic database schemes. Journal of the ACM **30** (1983) 479–513
5. Deming, W., Stephan, F.: On a least square adjustment of a sampled frequency table when the expected marginal totals are known. Annals of Mathematical Statistics **11** (1940) 427–444
6. Csiszár, I.: I-divergence geometry of probability distributions and minimization problems. The Annals of Probability **3**(1) (1975) 146–158
7. Augustin, T., Coolen, F., de Cooman, G., Troffaes, M., eds.: Introduction to Imprecise Probabilities. Wiley (2014)
8. Studeny, M.: Marginal problem in different calculi of AI. In: Advances in Intelligent Computing- IPMU'1994, Springer (1994) 348–359
9. Vejnarová, J.: A note on the interval-valued marginal problem and its maximum entropy solution. Kybernetika **34**(1) (1998) 19–26
10. Jirousek, R.: Solution of the marginal problem and decomposable distributions. Kybernetika **27**(5) (1991) 403–412
11. Walley, P.: Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London (1991)
12. Miranda, E., Zaffalon, M.: Coherence graphs. Artificial Intelligence **173**(1) (2009) 104–144
13. Miranda, E., Zaffalon, M.: Notes on desirability and conditional lower previsions. Annals of Mathematics and Artificial Intelligence **60**(3–4) (2010) 251–309
14. Miranda, E., Zaffalon, M.: Conglomerable natural extension. International Journal of Approximate Reasoning **53**(8) (2012) 1200–1227
15. Jensen, F., Nielsen, T.: Bayesian networks and decision graphs. Springer (2007)